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# JOURNAL

OF THE

## INSTITUTE OF ACTUARIES.

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"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

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*[The Council of the Institute of Actuaries wish it to be understood that while they consider it their duty to give, from time to time, publicity to certain of the papers presented to the Institute, and to abstracts of the discussions at the Sessional Meetings, they are not responsible for the opinions put forward therein.]*

[ENTERED AT STATIONERS' HALL.]

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# JOURNAL

## OF THE

# INSTITUTE OF ACTUARIES.

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*Opening Address by the President, S. G. WARNER, ESQ.*

[Delivered 4 December 1917.]

GENTLEMEN,—We meet to-night to open another War Session. The hopes, probably entertained by many of us, that the twelve months which began for us at our similar gathering of November 1916, would see the end of the struggle, have not been fulfilled. Still our main activities are suspended, our numbers depleted, our anxieties prolonged, under the shadow which overhangs the world. We are at all events one year nearer to our goal, as a people; but how distant its attainment still may be we are as unable as ever to say. At least, however, this may be asserted: that we, in common with our fellow countrymen, our kindred oversea, and our Allies, are as resolute as ever that attained that goal shall be: that we, in common also with them, are prepared determinedly to endure such privation, suffering or sacrifice as may be necessary to that end; resolved, in the words of our great poet and patriot of a past day, that we will not

“bate a jot

“Of heart or hope, but still bear up and steer

“Right onward.”

You will be interested to hear the numbers, brought down to the present date, of members and probationers of the Institute who have joined His Majesty's forces on land or sea; the total is now 390, and of these 44 have fallen. Since the last announcement of this kind, at our Annual Meeting in June, ten more names are included in that Roll of Honour, devoted to the memory of those whom we shall see no more.

They have laid down their lives for their country, and no comment could do other than weaken the force of that simple statement of fact. In each case, on your behalf, an expression of sympathy has been sent to those from whose homes they have gone. They have left us, but they are not forgotten.

Two other losses by death have to be recorded. Mr. Andrew Hugh Turnbull, who died last month, had been a Fellow of the Institute for nearly forty years. His name is naturally associated with the great Life Assurance Office in the service of which so much of his life was spent and to the success of which his management so substantially contributed. For four years he served on our Council; he was a member of the joint Mortality Investigation Committee; for two years he held the office of President of the Faculty of Actuaries; and he was the author of two useful books of Actuarial Tables. Those who knew him intimately bear witness to the wisdom of his counsel and the charm of his character. As Fellow of both Institute and Faculty he rendered good service to each, and by each his name will be held in honoured remembrance. The death of Mr. Edward Algernon Newton, who became a Fellow in 1865, leaves on our Roll only four names which have a longer record in that respect. As he died in his ninetieth year, it follows that his active business career belonged to rather a bygone day: and as throughout his retirement he had few opportunities of meeting us he is probably to many of us little more than a name; but his work was well and faithfully done, and in losing him we break one of the few remaining links with a distant past.

Among the events of the war during the past twelve months there is one of outstanding importance which has an aspect of special interest to British Actuaries. I refer to the entry into the contest, on that which we hold to be the side of righteousness and freedom, of the United States of America. I suppose there is no one of our race who did not feel a stirring of the blood and an uplifting of the spirit at this act of the one great English-speaking nation outside the British Empire. Beyond this, however, we have our own keen interest in and warm friendship for our brother actuaries across the Atlantic. The Actuarial Society of America has been long and honourably known to us for its great services to our science, and for the freshness, originality and ability with which it has handled many actuarial problems. In these

circumstances there occurred during the summer an interchange of messages, which are doubtless familiar to you since their publication in our Insurance Press, but which I venture to reproduce here so as to secure for them permanent record in the pages of our *Journal*. The first was as follows :

“The members of the Actuaries’ Club, London, send greetings to the members of the Actuarial Society of America. They hail with joy the entrance of your great country into the struggle for justice and freedom among the nations.”

This elicited the following reply :

“The Actuarial Society of America in annual meeting assembled acknowledges with deep appreciation the message of friendship from the Actuaries’ Club, and unanimously sends most cordial greetings to the Actuaries’ Club and to the members of the Institute of Actuaries. American and Canadian members have worked side by side in this Society for many years, and we all now rejoice that America and the British Empire will hereafter stand side by side in the great struggle for freedom and civilization.”

It will be observed that the American message includes in its scope not only the senders of that to which it is a response, but our Institute as a body. It seems appropriate therefore that on this, our first opportunity since its receipt, we should send a reply reciprocating the feelings expressed. It is my intention accordingly, before our proceedings close, to propose for your acceptance a resolution which will answer that purpose.\*

On the last two occasions of an inaugural address in this Hall, the war and its problems naturally formed the central theme. To-day the situation, developed during the last twelve months along broad lines of strategy and statemanship, hardly seems to present material making it necessary for us as a professional body to devote another entire evening to its consideration. The great conflict has had its chances and changes, its encouragements and disappointments. Steadfast effort and endurance are more than ever necessary, but perseveringly continued they will not fail of their reward. The financial burdens of the nations are heavier, but there is no weakening

\* [The resolution, which was unanimously adopted at the close of the proceedings, was in the following terms: “The Institute of Actuaries in General Meeting assembled, acknowledges the greeting sent through the Actuaries’ Club by the Actuarial Society of America, cordially reciprocates the friendship that inspired it and rejoices in the fellowship, as comrades in arms, of the British and American peoples.”—Eds. *J.I.A.*].

of the will to bear them rather than, on any excuse, to throw the fruits of past sacrifice away. We have all been hard at work, under the double strain of depleted help and the new problems which constantly arise. As instances of the latter, some have had to undertake a study of the intricacies of new foreign and colonial Income Tax Acts, which is in itself a liberal education; others have been facing questions of pensions to combatants and dependants which are increasingly complicated and difficult; while the help of some of our number has been sought, and effectively given, in the arduous statistical labour necessary in connection with our shipping. The national finance involves issues exceedingly complex, and sometimes keenly controversial; but to-night, when controversy is a thing to be avoided, it is well to be assured that all actuaries are united as regards the one supreme financial duty, the continual enforcement upon our people of the need for an unflinching and untiring economy. The voluntary expert help which members of our Institute have in such large measure given to War Savings Committees, and to important Funds created by and connected with the war, we record not with any claim to have done more than patriotic citizens' duty, but with satisfaction that the call to such duty has not failed to find a willing response.

We learn with great satisfaction that Mr. T. B. Macaulay, a Fellow of this Institute by examination, has been appointed Chairman of the National Committee of Food Resources in Canada. We congratulate him on his appointment, and the Canadian Government on its selection; and record the event as an honour to our body and our profession.

Leaving now this battle-shaken ground of the present, I invite you to follow me in a line of thought more directly concerned with our own profession. It may seem to lead us far away from the practical actualities of our time, but in the end we may find that it is not without real bearing on the great questions which are likely to be our close companions in a not very distant future.

When we last gathered together in this Hall, in June of the present year, it was to take part in the seventieth Annual General Meeting of the Institute of Actuaries. We belong therefore to a body which has completed the term of three-score years and ten. I hope we may say that its eye is not dim, nor its natural force abated. Yet the attainment of such



an age may perhaps not unfitly be regarded as marking a kind of epoch ; especially when it happens to coincide with a world convulsion that seems likely to lead to a social reconstruction of a character which we can just now only dimly surmise. We have some excuse for looking back into the past, an exercise which may give us some help in forecasting the future, and which is at all events a much surer and I think too a more profitable process.

In November, 1853, just over sixty-four years ago, a paper was read to the Institute by Mr. H. W. Porter "On some points connected with the education of an Actuary." The ideal there sketched by the author is a high one. He stipulates for "a good general education—such a liberal education as a university course provides." This is to include "a good useful knowledge" of Latin and Greek. These languages are, among other uses, to assist him should his Directors require him "to prepare a nosological table of the causes of the deaths that have occurred in the office." In the same connection he is to have "some knowledge of the physiology of the human frame." Modern languages are also recommended, and at least an acquaintance with French. A scientific knowledge of statistics is required, and a study of schemes of social improvement as affecting longevity. Enough law to become "a valuable coadjutor of the legal adviser of his Company" is thrown in. He should also be "a good accountant", "a ready correspondent", and "well read in all the literature appertaining to his profession." A knowledge of finance is also essential. All this is added of course to a thorough masterly knowledge of arithmetic and algebra and of the science of life contingencies ; with a warning that geometry should not be neglected. "A perfect actuary", says Mr. Porter, "should be a kind of admirable Crichton." There seems indeed to be little of the admirable Crichton left out, except athletic accomplishment : and this might have been included, had golf been a national pastime in 1853.

To criticize fairly such an utterance as this we must try to place ourselves, in imagination, back at the date of its delivery. The Institute had been in existence for six years only. Examinations were in their infancy. Text-books, as we now understand them, were unknown. The outstanding feature of the paper is its obvious motive of impressing on

the intending actuary a sense of the honour and dignity of his profession. There could be no worthier object. It is as true now as it was then, and as it will always remain, that in proportion as that sense is felt our work will be done well. From the nature of the case such an essay, when it was written, partook largely of the nature of a forecast. The great critic and illuminator of all forecast is experience. Sixty-four years of development lie between us and this one. It would be strange if by the aid of their light we were not able to-day to revise some of its conclusions; to place its main motive in a setting more adapted to the conditions and demands of to-day.

What chiefly occurs to us, then, when we make this attempt, is that while the author's purpose is finely conceived, there is something lacking. To say that the perfect actuary should be an admirable Crichton is not, after all, to say very much. In a sense it is true; but in that sense it might as truly be said of the perfect banker, the perfect accountant, the perfect auctioneer. In the comprehensiveness of the ideal all distinctiveness is lost; and distinctiveness is vital. Distinctiveness then we must find. It is not to be expressed in a sentence, or a formula; but I think it will grow upon us with sufficient precision by a consideration, even in the brief and inadequate fashion which alone is possible this evening, of some features of the history of our science; while at the same time the consciousness of its dignity and honour should be intensified.

The beginning of the story, familiar enough to most of us, takes us back far beyond the existence of the Institute, to the middle of the seventeenth century; when Latin was still the "Lingua franca" for learned men of all nations, when such men were still comparatively few in number, and when fresh ideas, once formulated, spread with a ready freemasonry amongst them and were eagerly received and developed. There is no intention, as there is no need, to retrace in detail ground familiar here to us all, but it is surely something to be proud of that the germs of the investigations which made our science possible are to be traced to the brain of a thinker of such world-wide eminence as Blaise Pascal. The possibilities latent in the application of the principles of probability to questions dependent on the duration of human life were readily perceived and diligently pursued by the



mathematicians of northern Europe; England was not found wanting, and we find ourselves in the line of succession with the great name of Edward Halley. Thus set flowing, the current of the new thought gradually set towards and finally settled in our own country. There the speculations of the philosophers consolidated themselves into practical form. This was natural, for the British intellect is on the whole rather practical than speculative; but it furnishes another reason why we, as Britons and actuaries, should honour our craft.

It is possible that a purist might take objection to the words "actuarial science", pointing out that what is meant is not a science, but a particular form of applied mathematics. Whether this be strictly so or not it will be convenient for our purpose to use, for conciseness' sake, the word rather than the phrase, as there can be no misunderstanding what is meant.

Taking this liberty, then, if I may, it is to be observed that a new science, born into a world of active thought, goes through, in the earlier period of its development, what may be called a romantic stage. It attracts attention from, and excites interest in, a wide spread circle of thinkers and investigators. This is a historical aspect of actuarial research which we perhaps do not think of very often. In these strenuous days especially we have little leisure for contemplating the past. I do not know that I can plead in defence of such an exercise that it is likely to serve any immediate practical end, though indirectly perhaps it may. But anyhow, in the attainment of our seventieth birthday we are entitled to spend a few minutes on looking back at our past; and at the earlier years, before our corporate existence began, when the forces which led up to our existence were shaping themselves in the life of the country.

To begin with, is it not significant that Dr. Richard Price, that redoubtable figure of the later eighteenth century, classical scholar, Unitarian preacher, Radical politician, and adviser (with questionable judgment) of William Pitt on the National Debt, should have found time among all his other affairs to construct the Northampton Table of Mortality? That is typical of the rising flood of interest surrounding the childhood of the new science. But there are other names more immediately appropriate to our purpose. One thinks

for instance of George Barrett, the Surrey farmer's son, who in the scanty leisure afforded by hard daily work fought his way unhelped through the mysteries of mathematics; toiled during twenty-five years, while by other labour supporting himself and dependent relatives, at the compilation of a mass of assurance and annuity tables, and arrived by independent discovery at the Commutation Method. In obscurity and isolation, his only help for the arithmetical work required being that of his home circle, he elaborated the invention which was to prove so powerful an instrument in the actuary's hands. He got the position of actuary to an assurance company, but held it for only two years. His failure to obtain public recognition for his work shortened his days. It seemed, he said, as if "thou shalt not prosper" was written on all his undertakings. One wishes there could have been granted him a prophetic glimpse of what his invention would mean to those who followed him.

A considerable figure of a very different kind, who also for a time held the post of actuary to an assurance company, was Charles Babbage, who combined mathematical genius with a good deal of rather ungovernable eccentricity. He was a man of wit, for when he along with some Cambridge associates was trying to get the notation of Leibnitz for the Calculus established in place of Newton's he described himself and his friends as "promoting the principles of pure *D*-ism in opposition to the *Dot*-age of the University." He had a European reputation, and has left to actuaries one conspicuous gift in his well known Table of Logarithms. His versatile intellect ranged over many fields. We know his little book of 1826 on the life assurance offices of that day. Even now it may be read with profit. But his mental activity was endless. Co-operative workshops, the interpretation of ciphers, the prevention of railway accidents, systems of signalling by lights, the construction of automatic chess players, are only a few of the subjects which engaged his attention. But his ruling passion was the art of mechanical calculation, for which he had the honour of receiving the first Gold Medal awarded by the Royal Astronomical Society, and in which he deserves from all actuaries the tribute due to a pioneer. His devotion to it forms a pathetic story, for it was there that what he deemed lack of proper recognition made him a permanently embittered and disappointed man. At

South Kensington Museum are still to be seen parts of the giant machine on which, out of his own fortune and the public money, some £23,000 were spent; and although it remains there, in a sense a failure, we who enjoy the benefit of later developments, and may yet see more powerful developments still, may not unfitly recall the words "other men laboured and ye are entered in to their labours."

Considering these cases, we are led to what seems to me one of the most remarkable and attractive features of this early period of our science: the number of men of first rate ability, not interested in it primarily for any professional or commercial reason, who were drawn to it, and worked at it, as a new mathematical development full of powerful possibilities. Francis Baily, for instance, we know actuarially as the author of a treatise on Annuities and Assurances which was for long the standard work on the subject. But Baily was far more than this. A successful stockbroker and exceptionally able man of business, his outstanding reputation was won in astronomy. A double gold medallist of the Royal Astronomical Society, one of its fourteen original members, and at his death its President, his labour on star-catalogues, and his success, after five years' arduous labour, with the "Cavendish experiment" for measuring the earth's density, give him a monumental place in astronomical research. He was conspicuous in trying to obtain for George Barrett that recognition of his work by the Royal Society, the failure to receive which was one of that luckless man's heaviest disappointments. Baily died three years before our Institute was founded, but all actuaries who can appreciate a single-minded devotion to the pursuit of knowledge must feel proud of the association of such a name with our science.

The mention of Francis Baily's name suggests another, and I venture to think a greater one, that of his intimate friend for many years, Augustus De Morgan. It is probably a matter of personal taste and temperament, but I confess that to me De Morgan has always seemed one of the outstandingly attractive figures of the Victorian age. His versatility, his wit and humour, his simplicity of character and inflexible devotion to principle, combined with his strength and acuteness of intellect and the width and variety of his learning, constitute a combination which never fails to fascinate me. He comes closer too to our own day. All

his best characteristics are shown in his inimitable "Budget of Paradoxes", which I think it one of our greatest honours to possess in serial form in our *Journal*. As for his humour, it is of the kind that makes the whole world kin. Who could feel anything but kindness for a man who having in a letter to a friend emphatically delivered his mind on a serious subject, instead of winding up with the familiar Latin tag "Dixi", writes "Richard's optic"? Further, as well as being an outstanding man he belongs to a remarkable period: a period typified by that Association for which so much of his best work was done, and which in its twenty years' existence accomplished so much good, "The Society for the Diffusion of Useful Knowledge." That time was a remarkable one in our intellectual history. The spirit of new knowledge was in the air. It pointed out possibilities in many directions, and gave birth to many enthusiasms. Some of them are by this time chilled and forgotten, but we are none the better for that as a people, and some of us envy the veterans, now so few surviving, who can look back on that period in the spirit of Wordsworth's great phrase "Bliss was it in that dawn to be alive." For along with this joy in the new learning went a limitless faith in the enterprise of imparting it to the people at a cheap price: so we had the fruitful labours of Charles Knight, the Chambers Brothers, and John Cassell. The Society I have mentioned did good work for our own subject on this path. Through its press issued our familiar friend "Jones on Annuities", De Morgan's great treatise on the Calculus, and, better than all, the "Penny Cyclopædia", containing several actuarial articles by him among the 711 contributions which witness to his exhaustless energy. Our own *Journal* also is enriched by similar work from his pen, and we have his separate treatise on Probability as applied to Life Contingencies.

Two associated names of remarkable men who have contributed to our science, if we may so say, from without, are those of Peter Gray and William Orchard. Gray had a long life, Orchard died in youth. Separately and in concert they are responsible for some of our most useful books of reference. Gray's application of the Gaussian logarithms to continuous table construction was a triumph of ingenuity; while Orchard's development of a fundamental relationship of two functions into conversion tables produced a work

which became perhaps more intimately the actuary's daily companion than any other.

Peter Gray was an Aberdonian. In mentioning his name we, for the first time on our list, "crossed the border." I would do so yet again. No actuary who honours his profession should forget Edward Sang. Fond of the study from boyhood, Dr. Sang developed early into a brilliant and versatile mathematician. Engineering was his original profession, but the greater part of his life was spent as a teacher of mathematics in Edinburgh. His work in mechanics, the science of naval architecture, hydrostatics, and astronomy was original and valuable; while the story of how he impressed the population of Constantinople by the prediction of a solar eclipse reads like the famous chapter describing a similar exploit of Allan Quatermain in Rider Haggard's well-known novel. He was responsible for a table of Logarithms very extensively used by our profession; while another, to 15 places, in 47 volumes, the result of forty years' labour, lies in state in the rooms of the Royal Society of Edinburgh, a monument of scientific devotion. He too felt the attraction of actuarial science, compiled extensive annuity tables, wrote essays on life assurance, and a treatise for students on life contingencies, which, notwithstanding a rather unfamiliar notation and a severely compressed style, is of great value.

I close this catalogue, which I trust has not become wearisome, with the name of Dr. William Farr. To him, as his biographer fitly says, the science of vital statistics may be said to owe its foundation. He is interesting as contributing a new element to actuarial research. He approached the subject as a medical man and a statistician. To him we owe the origin of that co-operation of the population registrar and the actuary, which for that great branch of actuarial science which is concerned with national vitality we may call the breath of life. In his hands the Registrar-General's report became instinct with scientific interest, and he created the English Life Tables. Without such work, enterprises in the interest of national health and well-being, which have done much and may yet do much more, would have been impossible.

Now I have not brought these names and reminiscences together for the purpose of giving new information, for I



suppose little, if anything, has been said which is not familiar. The object is of another kind. Here we have a number of eminent men, varying widely in experience, training and surroundings. They are all attracted from their individual paths of mental activity to study something which lies outside them, and that the same thing. When we view the subject thus historically, we can be in no doubt about what that thing is. In the application of the science of probabilities to the facts relating to human mortality, the statistical ascertainment of those facts and the application to them of that science, combined when the purpose was financial with the laws of compound interest, they found something new and absorbing. Here then we have arrived at the distinctiveness we sought, and can thus, while doing I trust full justice to the spirit of Porter's ideal, make its comparative formlessness symmetrical by giving due place to its centre of gravity. For in that same study which in the days of its youth drew to itself the attention and the help of these distinguished workers in other fields, and the possibilities of which are unexhausted, is, and always must be, the essence and the distinguishing mark of the actuary's vocation. As regards the accessories indicated in that utterance of 1853, it would be idle to deny their value: but as accessories, if we would get the true perspective of our subject, they must be considered. As knowledge on all sides multiplies, and specialization is more and more forced on us as a necessity, it will be for individuals to select from these subsidiary paths as temperament and circumstances dictate; but the philosophy of the matter so far as they are concerned will be best expressed by the spirit of Dr. Johnson's famous saying about Greek: "Greek, sir, is like lace. A man gets as much of it as he can."

I have spoken of the romantic stage of actuarial science. There is one chapter of its history, very different from the rest, to which that description seems to me to apply in some ways more deservedly than to any other; and which no attempt at retrospect, however inadequate, can afford to leave unnoticed. The early years of the nineteenth century saw the beginning of an attempt among the English working class to stimulate, and organize into definite form, the Friendly Society movement, which in scattered and spasmodic fashion had existed for more than a hundred years. The

story of its subsequent development to its present stage cannot of course on an occasion like this be sketched in the merest outline, even were the knowledge and the power for such a task mine, which they are not. All I should like to attempt is a very brief mention of how actuarial principles made their way into Friendly Society finance. We are fortunately able to trace this from the full records we possess of the rise and progress of one great typical organization, the Manchester Unity of Oddfellows, through which that work was done. The story is told in Mr. R. W. Moffrey's most interesting book "*A Century of Oddfellowship*", to which I gratefully acknowledge my indebtedness. During the first half of the last century, while energetic leaders were successfully enlarging the organization, and building it up into a coherent whole, the more farsighted among them were made more and more keenly alive to the fact that among the units which they were thus binding together there was, and most naturally, a profound ignorance of the elements of the science of adjusting contributions to benefits, an observance of which was essential to avert disaster. The occasion was grave, and was met with courage. We can form no adequate idea of the difficulties of suspicion, inertia, and disbelief, which from the nature of the case would lie in the path of the reformers. The leader for the emergency was found in Henry Ratcliffe, appointed corresponding secretary of the Unity in 1848. A self educated man, he grasped the necessity of the situation and devoted his life to its demands. In 1850 he published his famous "*Observations on the Rate of Mortality and Sickness existing among Friendly Societies*", with appropriate assurance and annuity tables. We do no injustice to the previous labours of Finlaison and Neison in saying that this work, in all the circumstances, was epoch-making. At the mass of material there presented he had laboured long, for from fifteen to seventeen hours a day. His salary for the whole duties of the Secretaryship was £150 per annum, and for the extra work involved he received a personal remuneration of £50. He lived to repeat this investigation twice, in 1862 and 1872, each time in the light of later experience. His personality and energy had won their battle. On his death in 1877, while his full ideal had not been attained, he had seen and done enough to feel well assured that the light would spread till throughout the

great organization for which he had lived actuarial solvency and security should prevail. That great possibility was due to his own untiring labour. Mr. Moffrey well says it justly entitles him "to be remembered as the father of friendly society actuarial science." His work was taken up by Reuben Watson, long his close colleague in the effort, and himself also a self-educated actuary. He too has passed away, but the influence of the achievement of these men and their like has saved one of the greatest developments of national thrift from failure. When we look back as actuaries on the development of our science, let us pay them the honour that is their due.

There is an interesting reference in Mr. Porter's essay to the problems which come before the actuaries of Reversionary Investment Companies, which, he says, "are of a far higher order than the ordinary Assurance Office calculations." What was probably meant was that such questions presented greater variety, called for more individual initiative, and were less amenable to treatment by stereotyped formulas. The subject is an interesting one, and contributes a feature of its own to the history of actuarial development. This was certainly a case in which our science opened up a distinct path of public service. It is difficult to form any definite conception of what happened to reversionary securities before scientific ways of dealing with them began to be established. There are no accessible materials available for such a purpose. We can only arrive at a general suspicion that transactions of this kind must have been carried on in obscurity, that they were probably associated with speculative dealings of rather a rapacious and discreditable kind, and that the atmosphere of the whole business was unattractive. When actuarial science showed how to attach to such prospective interests fair and legitimate present values, the air was cleared. Securities of this character took their proper place in the financial market, as proper material for straightforward bargains in which buyer and seller, or borrower and lender, might deal on equal terms. So far as the actuary himself was concerned, the effect was educational in a valuable way. He was brought into relation with a widely varying body of investments, which he had to value in a way that involved forming an estimate, not of their present worth but of that which they might be expected to possess in a more or less distant future.



This meant a forecast widely different from those inherent in his other work. The probabilities of sickness and death in the case of a Stock Exchange security are a good deal more complex, and difficult of appraisal, than those of human life. To deal with them needed knowledge, both actual and historical, of the money market, and the judgment which from the past and present could make some practical estimate of the future. Gradually, as technical methods increased in power and exactitude, contingencies of the most intricate kind were brought within the scope of ready calculation, and a large and successful business has been developed. Actuarial science may justly be credited with the achievement of having purified and organized on an equitable basis a branch of finance which, by releasing for present use capital not otherwise immediately accessible, has doubtless often played a useful part in the development of national resources. The conventional notion that it is the spendthrift only who raises money on his expectations is as frequently wrong as most conventional notions are. It may just as easily be the man who can put immediate capital to judicious use, and to help him is, as I have said, a public service.

Reversionary transactions, in early days, before we had Institute or *Journal*, must have been the material for many ingenious devices, and elegant approximations, among the older actuaries, of which no record remains. That reflection suggests a much larger one, about the whole growth of our science during that now distant time. When we consider what scanty materials, compared with those we now possess, were in the hands of our predecessors, we realize more fully what considerable men they must have been. The conditions were favourable to the development of individual capacity, independent views, original thinking. Before the Carlisle Table saw the light, or such a thing as an Offices' Experience Table was thought of, some of the older actuaries worked on Tables of their own construction from their own records. Their wisdom was contained in note books long vanished, conversations long forgotten. "There were great men before Agamemnon", but all these memorials are faded and fled. We shall not recover them. We possess some of their pamphlets (notably William Morgan's addresses to the Equitable Society), and some of their Tables; but the setting of the whole, the personal atmosphere that took the place of

our present organized publicity, all that, which was of the essence of the matter, is gone beyond recall, with the entire social order of which it formed a part.

When the romantic formative period of a science is over, it gradually settles down into one of ordered systematic progress, which while supremely useful has less about it to attract the imagination. I suppose, broadly speaking, we may say that the original Actuaries' Club represented the old order, and the Institute the new. That I know is by no means a perfect division, but only a very rough one. We possess, of course, on the Institute roll the names of many illustrious men, active in its inception and throughout its earlier years, Finlaison (its first President), Jellicoe, Brown, and others, who were links between the old world and the modern day. Even down to our very recent history, we have had with us the honoured presence of some whose recollections and whose spirit united the present and the past: one in particular, Ralph Price Hardy, who seemed to carry with him a charm from the days of the fathers of the profession. It was evident in the predominance of his personality over his formal written contributions to our proceedings—substantial as their value was; in the seeds of suggestiveness and inspiration sown by him in the minds of the younger men around him, and bearing fruit in their lives and work. Gradually, however, the influences which had shaped the beginnings of our progress consolidated themselves into a middle period with characteristics of a different kind. The time of the pioneers was over. First principles were established, and the lines of further advance laid down. The attention attracted from distinguished mathematicians outside the ranks of the profession inevitably slackened; and the economic conditions of the time associated its members more and more closely in one practical interest, the propagation and building up of the business of life assurance throughout the nation. This atmosphere is evident, for instance, throughout the whole of Mr. Porter's essay, and obviously controls his ideal. It would be the greatest possible mistake, however, to suppose that this development coincided with any diminution of powerful mental activity. The intensive work showed certainly no sign of such a tendency, but was carried on with keenness and vigour. Looking back upon that time, and endeavouring to view it in its true perspective, we see running through it as

its central outstanding feature what we should naturally expect; the endeavour to perfect the statistical basis on which the work was to be founded. If the actuary's main sphere for the period was to be that of life assurance, the existing tables of mortality, constructed from general population returns, were not the most suitable. The tools must be more closely fitted to the task, and so we have the great repeated efforts of the British actuaries to obtain tables representing the experience of assured lives. It has been a long-continued and persevering labour, taking form in three successive bodies of results; each in volume of content and elaboration of treatment an advance on its predecessor. Into it a mass of the best ability of our profession has gone, while the attendant processes of graduation, and of appraising the influence of medical selection, have called forth mathematical skill of a high order. When we were dealing with the farther past, it was not very difficult to select for mention a few outstanding names. As the time approaches nearer to our own, the number increases; and it is not easy, without some omission, to make a choice. I can only ask forgiveness if I am guilty of that, but I do not think there will be any dissent when I recall as honourably associated with the great enterprise just described in one or other of its branches, the names of J. A. Higham, William Sutton, W. M. Makeham, James Meikle, T. G. Ackland, W. S. B. Woolhouse, and George Francis Hardy, who have passed away, and, still with us, Dr. T. B. Sprague and Mr. George King.

Grouped around this central undertaking, the period was fertile in much other research. The quest for a law of mortality, leading to the brilliant generalization of Gompertz, which with its modifying development by Makeham has proved so powerful an actuarial instrument, one would rank next in importance. It is rather curious that Mr. Porter's theory of actuarial education lays comparatively little stress on the higher mathematics. The history of our science in its subsequent development has been distinctly to emphasize their value. It is not uninteresting to trace the process. Actuarial study could not be long pursued without recognition of the fact that it was primarily concerned with a branch of the doctrine of series: and further that from the nature of the case it was proper to assume continuity. This led directly to the Differential and Integral Calculus. For some time

there was hesitation about including an elementary knowledge of these in the examination syllabus. Ingenious demonstrations of formulas based on assumptions of uniform annual distribution of deaths were the result; but the clumsiness of these was evident, and an epochal step was that of Woolhouse, when in his essays on the "Continuous Method" he boldly applied the Calculus both to mortality and interest, thus obtaining expressions far superior in elegance and simplicity and more closely in accord with the real conditions. The Gompertz-Makeham formula already mentioned, with its reliance on transcendental functions, further emphasized the value of a free use of methods in advance of those of ordinary algebra. On all hands and in unexpected ways this value revealed itself. The brilliant use of summation formulas in dealing with complicated survivorship contingencies, which had previously eluded any expeditious and exact method of calculation, may be given as an instance. The cumulative evidence from all sides of the power and benefit thus derived could not be resisted; and, first optionally, then compulsorily, the elements of the calculus were introduced into the examination tests required of our students. No such experiment has ever been better vindicated by the result. The intending actuary of our day is in the position, enviable by some of us whose training belongs to an earlier period, of being, at a time when the mind is plastic to new impressions, familiarized with the use of a mathematical tool of first-rate power, the mastery of which at a later age is very much more difficult to attain.

If it is a correct generalization to say that there have thus been two periods in British actuarial history, distinguished from each other by some recognizable characteristics, the same reasoning will I think lead to the conclusion that there is yet a third, of which we may say that it is still opening before us and that we as yet stand on its threshold. When looking at the subject in this light, we are not to expect any sharp dividing lines. The distinctive features of change merge into one another. Many of those which become evident have long before been latent, and it is only when a general view over a sufficiently wide reach of time is taken that the differences are seen. The principle at work is really the reaction of the material to the conditions surrounding it. It would, for instance, be a mistake to suppose that none of

the earlier actuaries were conversant with such ideas as those of framing a Life Table from assurance experience or analyzing the effect of selection. The growth and consolidation of life assurance business made such problems, as years advanced, both practical and practicable.

So in the development to which I now refer, we have to think not of a transfer of the actuary's activity from one sphere to another, but of the opening to that activity of a wider field, while preserving that of the past; and, in that process, of bringing into concrete, systematized form, much that has long existed in a more or less fragmentary and fluid state. A good instance may be found in the work so admirably done for Pension Fund Tables by the late Mr. Manly. Such Funds had existed, and been subject to actuarial advice, long before. It was when the subject came to occupy wider public attention; when consequently the demand for guidance upon it, in the varied and complex forms it was assuming, grew more insistent, that its codification took material shape. Now this is but one minor ease of a general tendency, of which I feel assured we have as yet seen only the beginning. We may measure its effect in another and quite different way. Speaking to us in 1881, one of our most experienced Presidents could say:—"Men have occasionally attempted to "practice as pure actuaries, and have put a brass plate on "the door; but whether they have ever taken sufficient fees "to pay the rent of their offices is doubtful." We need no more striking evidence of what is taking place than to compare those words with our experience to-day. There is the widening field, and it is one of general public service.

Let me, however, emphasize what was said just now, that this is a widening, not a transfer, of activity. The actuary's place in the life assurance sphere is as central, as essential, as ever. Much as the labours of the past have clarified and codified, individual judgment and initiative will always be indispensable. As the mass of observed experience increases, fresh revision of it for tabular purposes will be necessary, and its growing volume will make possible many useful and interesting investigations which have not yet been taken up. The actuary to an assurance company, moreover, has unfailing opportunities of usefulness in the practical study of that vitally important subject, the investment of its funds. This, in the days that lie ahead of us, will demand more



than ever watchful and well-trained handling. Of all such considerations one is not forgetful, when calling attention to the developing possibilities for actuarial science outside their borders.

But those possibilities are insistent, and important. Here again they are the result of surrounding conditions; but conditions in this case greater in their appeal than those of any particular business, conditions of the national life as a whole. Movements of thrift throughout the mass of our population have been growing for more than a century, and have been gradually coming more and more within the shelter of actuarial guidance, but the intervention of the State itself as an active participator has within the memory of us all transformed the entire situation; and the convulsion through which we are now passing as a people will have the result of intensifying the sense of collective responsibility and effort for the common good. Stronger than ever in such circumstances is the need for the help of the actuary.

In this connection we have to take account of the modern scientific study of statistics, associated primarily with such names as those of Sir Francis Galton and Professor Karl Pearson. Its rationale, and general principles, may be grasped by anyone possessing the usual mathematical training of an actuary, without exceptional difficulty; but its more elaborate developments call for some rather severe and special study. It is more or less, for the present, a science in the making; and presents an ample field to the trained investigator. Abstruse as it may seem, and in some respect is, its object, and the results it seeks, are eminently practical. The enormous importance of statistical inquiry is being more fully realized every day. Statistics lie at the base of every material aspect of our national existence. The crude methods by which they were often treated in the past, the pitfalls of which their study is full and into which so many have fallen, have made them the subject of popular suspicion, and of some very uncomplimentary proverbs. A moment's reflection, however, will show to common sense that from their accurate compilation and correct interpretation, if these can be accomplished, must follow results of the utmost human value. Thus much should certainly be clear to the actuary. The new methods for their study are on the path of true progress, and there is nothing

fuller of future potency; for as questions relevant to the condition of the nation grow in importance, so will statistical investigations increase and multiply. Here is surely a supreme opportunity for actuarial science to place itself in the main current of public usefulness. Contemplating that field of great possibility, one feels instinctively that the actuary should be there. The work that some of our number have so done and are doing we gladly acknowledge; and those of us to whom advancing years deny the mental elasticity sufficient for the enterprise look across to the promised land and say to those more fortunately circumstanced, "Go up and possess it." It is curious, and an illustration of what has been said as to tendencies being latent before changing conditions make them evident and dominant, to find Dr. Farr writing, more than sixty years ago: "The whole commerce of the country turns on contingencies which demand the application of scientific observation and calculation; and as English agriculture has its chemists, English commerce must, to keep pace with it, ultimately employ actuaries to calculate the risks which are now only roughly guessed at, and thus extend the useful sphere of an important scientific class of men." The result of subsequent experience, while it might alter in some respects the phrasing of that statement, but confirms its principle.

To come back to our starting-point: in every profession, trade, or business there is for a man no surer source of inspiration during work which must often be hard, and—what is worse—often monotonous, than a sense of the honour and dignity of his calling. That sense, for its complete attainment, depends on two things: a worthy history in retrospect, an increasing opportunity of public service in prospect. As evidence of the first I have tried to bring together a few facts familiar to us all, the excuse for gathering them into one focus being the circumstance that our Institute's existence happens just now to have completed a rounded term of years. As evidence of the second, I point to the conditions surrounding us and the future they are likely to bring.

The actuary occupies, it seems to me, a peculiarly favourable position. On the one hand he is the exponent of a science, concerned with exact knowledge. That gives a consciousness of natural law, of first principles, of boundaries which cannot

be overstepped. This consciousness has a calming and regulating influence upon the mind. It holds in check all tendencies to fantastic speculation, to unbalanced activity, to the treacherous impulse towards seeing things as one wishes them to be, instead of as they are. It is a stronghold within which to take refuge when confronted with error or fallacy. Then, on the other hand, he has to deal with men—their multitudinous affairs, their divergent personalities. There is no more wholesome mental exercise than that. It keeps the faculties alert, corrects any tendency to pedantry, applies to any fancy for theoretical extravagance the saving touch of common sense. The philosophy evolved in the study has to be brought into the market place; to be made intelligible and convincing to the plain man. In this combination of the pursuit of scientific accuracy and the sense of public utility there is surely an ideal training. We are missionaries of warfare against the hoary and deep-rooted delusion that theory and practice are enemies; and our weapon is the translation of scientific truth into the dialect of daily intercourse. It is one thing to prepare some scheme of communal benefit which is the best that the financial conditions will allow, and another to vindicate its merits to the satisfaction of a crowd of eager participants who are profoundly convinced that it should have been much better. A differential equation and a discontented policyholder are extremely diverse difficulties. The man who can successfully negotiate both has attained something which neither alone could give him, and which is of no small value. This is what our education and experience taken together can do for us, and it is in the just balance of the academic and the popular elements that the problem of successful achievement lies. More, I venture to say, than successful achievement in any limited sense; for the man who can be equally at home in the cloistered mental seclusion of an exact thinker, and the companionship and understanding of his fellow men, has not only made a professional success: he has widened and enriched his entire life.

I should like to say just one other thing before I have done. It is, I suppose, something depending largely on individual temperament, and therefore entitled to a limited value only, as such utterances, like rivers, can rise no higher than their source; and on such terms alone it is offered.



Expressions of that kind on such occasions as these, whether they be right or wrong, may perhaps equally serve some useful end, as indications of personality. I have a very strong conviction of the value to us, as to all professional men, of sufficient touch with the general sum of what is said and thought in the world as will give us a chance of viewing our profession not only as a thing in itself, but as something which is a part of an infinitely greater whole. This is not the mere familiar theory that every man should have a hobby, though that in itself is good; and when we hear that the first actuary of one of the oldest British life offices was a brilliantly famous chess player, or that Joshua Milne had an unusually minute knowledge of natural history, and one of the best botanical libraries in London, I think we should reckon it to them for righteousness. What I mean will be better conveyed by another illustration. One of the greatest pulpit orators of the nineteenth century, when speaking on one occasion to students preparing for his profession, advised them throughout their lives assiduously to read, adding with emphasis that he did not mean exclusively theological reading. Reading, he told them, was essential for keeping the mind fresh; and although what they read might seem entirely remote from their subject, it would in unexpected ways be of constant help to them. I do not think anything much wiser has ever been said. However well a man knows his own business, I venture to say that to have the power of viewing it from without as well as from within, of regarding it as part only of the inexhaustible and unfathomable variety of the world in which we live, and considering it in its relations to that whole, is an invaluable help to his knowledge of the thing itself. It would be foolish to say that this can or should come from reading only. Men and minds differ. What some get in that way others may get from art, from travel, from the discourse of friendship. But in some form to have it, in some degree to enter into the spirit of the saying, "*Homo sum; humani nihil a me alienum puto*," will help us as nothing else can to that largeness of conception and breadth of view in dealing with affairs, which mere technical absorption, however excellent, tends if uncorrected to cramp or even to paralyze. For which reason I prefer to close with words not my own, the simple directness of which cannot be improved upon; the words of one long preserved to us through a benign old age, whose

presence now removed we have not yet ceased to miss from our gatherings: the words of Arthur Huteson Bailey, when, speaking from this chair thirty-six years ago he said,

“An actuary should be a man of general culture, with a knowledge both of books and men, and the more he has of both the better.”

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*Yields on Redeemable Securities.*

ALTHOUGH the investment yield with allowance for income-tax has long been familiar to students of the theory of compound interest, it is practically unknown to the ordinary investor, and has not, probably, been generally adopted even by actuaries.

Several causes have contributed to the yield of theory not having become the standard yield of practice. One is that many investors—a majority, perhaps, in number, although not in amount—are more interested in the immediate return on their invested capital than in the true yield, and are, consequently not much concerned as to a factor which affects only the latter. This would be a good reason for quoting yields without allowance for either profit on redemption or income-tax, but not for quoting with allowance for the one and not for the other. The fact that all investors are not liable for tax at the full rate would introduce a little difficulty, but the difficulty could be met by a simple approximate adjustment, and even without such adjustment the yield with allowance for tax at the full rate would, in most cases, be nearer the true yield than that obtained by neglecting tax altogether. In any case, the point is not one that affects the more important classes of investors such as insurance companies and trust companies.

Another contributing cause has, no doubt, been that until quite recently the allowance for income-tax did not, as a rule, make much difference in the yield, and that since it has become material it has been rather overshadowed by more important financial factors. Even now the average financial man, although he may go so far as to recognize that the fact of the profit on redemption not being subject to income-tax is a “good feature” of an investment, is inclined to regard its quantitative measurement as an “actuarial refinement.”

A third cause may have been the influence of the American bond-value tables. The question of income-tax, as a factor in

the yield on securities, did not, of course, arise in the United States until quite recently, and it will not arise now—as affecting in any special way the yield on redeemable securities—if it should be held that profits on redemption are subject to tax under the U.S. law. It is not unlikely, therefore, that the American tables will be continued in their present form, and so far as they are used in the United Kingdom they will, in that case, tend to maintain the yield without allowance for tax. Incidentally, an interesting problem arises as to the yield on a redeemable security subject to both American and British tax, supposing the former to be charged both on income and increase of capital-value, and the latter on income only.

A more effective cause, perhaps, (at any rate so far as actuaries are concerned) than any of those mentioned, has been a doubt as to the theoretical basis of the yield with allowance for tax. As ordinarily calculated the yield with allowance for tax is the yield that would be realized (by an investor subject to tax at the full rate) in the event of the rate of tax remaining constant at the rate in force at the time of calculation, and of no change taking place in the mode of assessment. This is obviously unlikely in general to be the correct yield, for it is improbable that the rate of tax would remain constant throughout the term of any security except one maturing in a very short time. But it may be expected to be more nearly correct than the yield obtained on the basis of no tax at all. This will be apparent from a comparison of yields on any reasonable assumptions as to the future course of the rate of income-tax. Suppose, for example, that during the term of a security redeemable at par in  $n$  years, and bearing interest at rate  $g$  yearly, the rate of tax were to increase or decrease from  $t_1$  to  $t_n$  in such a way that  $\frac{1}{1-t_1}, \frac{1}{1-t_2}, \&c.$ , were in arithmetic progression. The gross yield, at a price of  $1-k$ , would be roughly

$$\left[ g + \frac{k}{n} \frac{1 - \frac{1}{2}(t_1 + t_n)}{(1-t_1)(1-t_n)} \right] \left[ 1 - \frac{n+1}{2n} k \right]$$

Or suppose that with the same data the rate of tax were to increase or decrease in such a way that the net dividends were in geometric progression with a common ratio of  $(1+\lambda)$ . The gross yield in that case would be roughly

$$\left[ g + \frac{k}{n} \cdot \frac{1}{1-t_1} \cdot \frac{\mathbf{a}'_n}{n} \right] \left[ 1 - \frac{n+1}{2n} k \right],$$

where  $a'_n$  is calculated at rate  $\lambda$ . For practical values of  $t_n$  or  $\lambda$  either of these expressions would be much more nearly equal to

$$\left[ g + \frac{k}{n} \cdot \frac{1}{1-t_1} \right] / \left[ 1 - \frac{n+1}{2n} k \right]$$

the approximate gross yield on the basis of a constant rate of tax  $t_1$  than to

$$(g + k/n) / \left( 1 - \frac{n+1}{2n} k \right)$$

the corresponding gross yield on the basis of no tax.

The possibility of a change in the mode of assessment is more important and cannot be entirely excluded. It is possible, for example, that the basis on which life assurance companies are assessed might be fundamentally altered so that calculations made on the assumed continuance of the present practice of taxing interest-earnings would no longer hold good. But any such alteration would, probably, bring compensating advantages, and the possibility of taxation being extended to profits on redemption, while interest-earnings continue to be taxed, may perhaps be disregarded. Apart from the practical difficulties which such a change would entail, it would clearly be inequitable in view of the large number of investments that have been made on the basis of the present law. Moreover, it is a reasonable assumption that if at any time profits on redemption were made liable to income-tax existing investments would be exempt—on the analogy of purchases of reversions subject to old rates of duty. This might not be entirely satisfactory—nor equitable as regards trustees and others who may be compelled to realize investments—because market-values would still be affected adversely, but it would, at any rate, enable an investor who had purchased on the basis of the existing income-tax law to realize his anticipated yield by holding his investment until maturity.

While each of the foregoing causes may have accounted, in some degree, for the yield with allowance for tax having remained more or less an academic yield, a determining cause has almost certainly been the fact that no table of yields on this basis has been published. Sufficiently simple formulas for calculating the yield with allowance for tax in individual cases have been given (*see* Text-Book, Part I, Revised Edition, pp. 118, 121, and *J.I.A.*, vol. xlix, pp. 366-369), but a formula, however simple, does not, in practice, answer quite the same purpose as a table.

It seems worth while, therefore, to consider the practicability of constructing a convenient table.

Since (with the usual notation) the net yield with allowance for tax is approximately

$$[g(1-t) + k/n] \bigg/ \left[ 1 - \frac{n+1}{2n} k \right]$$

it follows that the difference between the net yields,  $i_1$  and  $i_2$ , corresponding to two rates of tax  $t_1$  and  $t_2$  would be very nearly

$$i_1 \times g(t_2 - t_1) / [g(1 - t_1) + k/n]$$

which—so far as  $k$  and  $n$  are concerned—depends only on the ratio  $k/n$ . It would appear therefore that a table of net yields for a standard rate of tax—say at the present time 5s. in the £—with a supplementary column of the values of  $g(t_2 - t_1) / [g(1 - t_1) + k/n]$  for  $t_2 - t_1 = \text{say } \frac{1}{20}$  (corresponding to a difference of 1s. in the £ in the rate of tax) would give the net yield with approximate accuracy on the basis of any rate of tax that may be anticipated within the next few years. A simpler, but somewhat rougher, adjustment for a difference in the rate of tax would be

$$g(t_2 - t_1) \bigg/ \left( 1 - \frac{n+1}{2n} k \right)$$

which is practically independent of  $n$  except when  $n$  is very small and may be taken for practical purposes as equivalent to

$$g(t_2 - t_1) \bigg/ \left( 1 - \frac{1}{2} k \right).$$

As regards the table itself, the most convenient arguments in practical use would obviously be the *price* and the *unexpired term*. A table for these arguments would, however, be open to the objections that the useful values would run across the page in a comparatively narrow diagonal band, and that it would not lend itself well to interpolation. If the table were sufficiently extensive these objections might be outweighed by the great practical advantage of entering the table directly with the data. But for a table of limited extent a different arrangement would be more suitable. The approximate formula

$$[g(1-t) + k/n] \bigg/ \left[ 1 - \frac{n+1}{2n} k \right]$$

again suggests a solution. For since the yield corresponding to



a given value of  $g$  depends mainly on the value of  $k/n$ , it would appear that the arguments  $k/n$  and  $n$  would give a compact table and one in which first difference interpolation should be nearly accurate. Tables are, accordingly appended on this basis. It will be seen that the differences in both directions vary slowly, and that an interpolated value for  $n+m$ ,  $k/n+\kappa$ , may be taken as approximately  $i_{n, k/n} + \frac{m}{5} \Delta_n + 10\kappa \Delta_{k/n}$ . The interpolation, being of a simple nature, may be performed by inspection.

For example, the net yield allowing for tax at 5s. on a 3 per-cent security redeemable at par in 33 half-years, and bought at 75 (so that  $k/n=.7575$ ), will be approximately  $4.14 + .6 \times .11 + .575 \times .32 = 4.39$ ; and the yield allowing for tax at 6s. would be approximately  $4.39 - 4.0 \times .0439 = 4.21$ . The correct results to two places would be 4.40 and 4.22.

A table is added showing the adjustments for a difference in tax by the alternative method mentioned above. In the foregoing example the adjustment by this method for an extra 1s. tax would be  $-.17$ , making the net yield 4.22.

*Approximate net yield (convertible Half-yearly) with Allowance for Income-tax on a security redeemable at par in  $n$  half-years, bearing interest at rate  $g$ , payable half-yearly, and bought at a discount of  $k$  per-cent*

$$g = .025.$$

$n$	NET YIELD WITH ALLOWANCE FOR TAX AT 5s.									
	$k/n$									
	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	
5	3.13	3.35	3.56	3.78	4.00	4.22	4.44	4.66	4.88	
10	3.18	3.41	3.64	3.87	4.11	4.35	4.59	4.83	...	
15	3.24	3.48	3.72	3.97	4.23	4.49	4.76	...	...	
20	3.29	3.55	3.81	4.09	4.37	4.65	4.95	...	...	
25	3.35	3.63	3.91	4.21	4.51	4.83	...	...	...	
30	3.42	3.71	4.02	4.34	4.68	...	...	...	...	
35	3.49	3.81	4.14	4.49	4.87	...	...	...	...	
40	3.56	3.91	4.27	4.66	...	...	...	...	...	
45	3.64	4.01	4.42	4.85	...	...	...	...	...	
50	3.73	4.14	4.58	...	...	...	...	...	...	
55	3.83	4.27	4.76	...	...	...	...	...	...	
60	3.93	4.42	4.97	...	...	...	...	...	...	
Add or deduct % for difference of 1s. in tax	4.1	3.8	3.6	3.4	3.2	3.1	2.9	2.8	2.7	

$$g = .03.$$

<i>n</i>	NET YIELD WITH ALLOWANCE FOR TAX AT 5s.							
	<i>k/n</i>							
	.5	.6	.7	.8	.9	1.0	1.1	1.2
5	3.30	3.51	3.73	3.95	4.16	4.38	4.60	4.83
10	3.34	3.57	3.80	4.03	4.27	4.51	4.75	4.99
15	3.39	3.63	3.88	4.13	4.38	4.64	4.91	...
20	3.44	3.70	3.96	4.23	4.51	4.79	...	...
25	3.49	3.77	4.05	4.34	4.64	4.96	...	...
30	3.55	3.84	4.14	4.46	4.80	5.15	...	...
35	3.61	3.92	4.25	4.60	4.97	...	...	...
40	3.67	4.01	4.36	4.74	...	...	...	...
45	3.74	4.10	4.49	4.91	...	...	...	...
50	3.81	4.20	4.63	...	...	...	...	...
55	3.89	4.31	4.78	...	...	...	...	...
60	3.97	4.43	4.95	...	...	...	...	...
Add or deduct % for difference of 1s. in tax	4.6	4.4	4.1	3.9	3.7	3.5	3.4	3.2

$$g = .035.$$

<i>n</i>	NET YIELD WITH ALLOWANCE FOR TAX AT 5s.								
	<i>k/n</i>								
	.3	.4	.5	.6	.7	.8	.9	1.0	1.1
5	3.25	3.47	3.68	3.90	4.11	4.33	4.55	4.77	4.99
10	3.28	3.50	3.73	3.96	4.19	4.43	4.66	4.90	...
15	3.31	3.54	3.78	4.03	4.28	4.53	4.79	...	...
20	3.34	3.58	3.84	4.11	4.37	4.64	4.93	...	...
25	3.37	3.63	3.90	4.18	4.47	4.77	...	...	...
30	3.40	3.67	3.96	4.26	4.58	4.90	...	...	...
35	3.43	3.72	4.03	4.35	4.69	...	...	...	...
40	3.46	3.77	4.10	4.45	4.82	...	...	...	...
45	3.50	3.83	4.18	4.56	4.97	...	...	...	...
50	3.53	3.88	4.26	4.67	...	...	...	...	...
55	3.57	3.94	4.35	4.80	...	...	...	...	...
60	3.61	4.01	4.45	4.94	...	...	...	...	...
Add or deduct % for difference of 1s. in tax	5.4	5.1	4.8	4.6	4.3	4.1	4.0	3.8	3.6

$$g = \cdot 04.$$

$n$	NET YIELD WITH ALLOWANCE FOR TAX AT 5s.							
	$k/n$							
	$\cdot 2$	$\cdot 3$	$\cdot 4$	$\cdot 5$	$\cdot 6$	$\cdot 7$	$\cdot 8$	$\cdot 9$
5	3.42	3.63	3.85	4.06	4.28	4.50	4.72	4.94
10	3.44	3.66	3.89	4.12	4.35	4.59	4.82	...
15	3.46	3.69	3.93	4.18	4.42	4.68	4.93	...
20	3.48	3.72	3.98	4.24	4.50	4.78	...	...
25	3.50	3.76	4.03	4.30	4.59	4.89	...	...
30	3.52	3.79	4.08	4.37	4.68	...	...	...
35	3.54	3.83	4.13	4.45	4.79	...	...	...
40	3.56	3.87	4.19	4.53	4.90	...	...	...
45	3.59	3.91	4.25	4.62	...	...	...	...
50	3.61	3.95	4.32	4.71	...	...	...	...
55	3.64	3.99	4.39	4.81	...	...	...	...
60	3.66	4.04	4.46	4.93	...	...	...	...
Add or deduct % for difference of 1s. in tax	5.9	5.6	5.3	5.0	4.8	4.6	4.4	4.2

$$g = \cdot 045.$$

$n$	NET YIELD WITH ALLOWANCE FOR TAX AT 5s.						
	$k/n$						
	$\cdot 1$	$\cdot 2$	$\cdot 3$	$\cdot 4$	$\cdot 5$	$\cdot 6$	$\cdot 7$
5	3.59	3.80	4.01	4.23	4.44	4.66	4.88
10	3.60	3.82	4.04	4.27	4.50	4.74	4.97
15	3.61	3.84	4.08	4.32	4.57	4.82	...
20	3.62	3.86	4.11	4.37	4.64	4.91	...
25	3.63	3.88	4.15	4.43	4.71	...	...
30	3.64	3.91	4.19	4.48	4.79	...	...
35	3.65	3.93	4.23	4.54	4.87	...	...
40	3.66	3.96	4.27	4.61	4.96	...	...
45	3.67	3.98	4.32	4.68	...	...	...
50	3.68	4.01	4.37	4.75	...	...	...
55	3.70	4.04	4.42	4.83	...	...	...
60	3.71	4.07	4.47	4.91	...	...	...
Add or deduct % for difference of 1s. in tax	6.3	6.0	5.7	5.4	5.1	4.9	4.7





*Investments a Hundred Years Ago.* By W. PALIN ELDERTON,  
F.I.A.

"It is a favourite maxim of mine that history, while it should be scientific in its method, should pursue a practical object. That is, it should not merely gratify the reader's curiosity about the past, but modify his view of the present and his forecast of the future."—J. R. SEELEY, "Expansion of England."

IT is, perhaps, worth while at the present time to go back to the earlier records of insurance with the hope of finding out how the old masters of insurance management faced the problems which arose out of fluctuations in the price of securities due to international political events. Every insurance generation has its fashion in investments, but the main principles will always be much the same, so that even at the present time it may be possible to get some help, or at any rate some interest, from the study of the investments of life assurance offices made many years ago. It has become the fashion to invest in various classes of securities instead of to restrict investments to British Government funds and mortgages: but owing to the wish to help to finance the country in the present difficulties, and partly owing to the anxiety of the Government itself in this connection which caused it to ask insurance companies to make somewhat substantial investments in War Loan, insurance companies at the present time are far more nearly in the position of having their funds mainly in Government securities than they have been for more than fifty years. In looking back into past records it is difficult to get facts which are suitable for investigation. Few life insurance companies had funds sufficiently large a hundred years ago to make their investments the matter of extreme importance which they now appear to us to be. There was, however, one exception to this general rule, namely the "Society for the Equitable Assurances on Lives and Survivorships," and the present notes relate merely to an examination of the investment policy of that Society: this is, perhaps, especially interesting because its affairs were controlled during the difficult times at the end of the eighteenth and beginning of the nineteenth centuries by a man who might be described as the father of practical actuarial work from the standpoint of life assurance.

As the funds of the Old Equitable Society in 1810 were over three millions and were increasing by £300,000 a year, the problem of investment had assumed considerable proportions. Under its original deed of settlement the Society had taken what seem

to be fairly wide powers of investment, for the clause governing its investments reads as follows :

“ . . . . . the premiums of assurance paid by the  
“ Members of the said Society (except so much of the said  
“ premiums as shall, by a General Court of the said Society,  
“ be judged necessary for the current expenses of the said  
“ Society) shall be laid out in Government or other good and  
“ sufficient securities.”

Although one would have expected from the wording of this clause that the Society could have invested in various classes of securities, it confined its investments during the early years entirely to Government securities, and as it published this fact in the short account of the Society which served as a prospectus, it seems to have been felt necessary in 1778 to put the matter before the members, and they passed a bye-law which begins by reciting that it has been “ the usage of the Directors to invest “ the monies of the Society altogether in the Public Funds, and “ this is accordingly published in the succinct account of the “ Plan and Basis of the Society as the usual practice ”, and it goes on to say that nevertheless the Directors have full and free liberty to invest in accordance with the clause in the original Deed, and adds that “ it may be desirable that some of the “ monies should be laid out upon different securities, presumably “ of equal validity, rather than that the whole should rest upon “ one and the same security.” Here the matter seems to have rested until 1858, when a bye-law was passed giving the Directors power to lend on policies (in this connection it may be mentioned that they had received legal opinion from Edward B. Sugden, J. Campbell, and A. R. Sidebottom in 1831 that it was impossible for them to do this). In 1865, in 1876, and on subsequent occasions more definite powers were given, but we need not dwell upon these in the present connection. It is probable that the Society limited its investments to Government Annuities in the first instance because the Directors, as trustees, did not feel that they had power to invest in anything else, and even after the somewhat indefinite bye-law of 1778 they may have been influenced by similar considerations.

We may now turn to what happened in the actual practice of investment, and from the earliest accounts it seems that the Society had, in 1767, only £6,000 in the 3 per-cent Consolidated Annuities, and £4,000 in 3 per-cent Anno 1726 : £2,000 of the former having been purchased in that year at 86 $\frac{3}{4}$  and £1,000 of

the latter at 88½. At these times the funds were growing very slowly because more than half the business was for a term of one year or for a short term. It may be of interest in passing to mention that the Society borrowed £650 at 5 per-cent from Charles Gould (afterwards Sir Charles Morgan), who was then Vice-President of the Society, to enable it to pay a claim for that amount "without selling out some of the stocks still remaining "in the Old Trustees' Names."

From 1770 onwards the funds of the Society seem to have grown steadily, and in 1772 the various Government securities amounted to just under £30,000, and were made up of old South Sea Annuities, Consolidated Annuities, Anno 1726 3 per-cent and a small amount of new South Sea holdings at 3 per-cent. All the holdings were valued at 86½, which seems to be about one point lower than the amount paid for stock purchased during the year in question : but it is impossible at this distance of time to estimate the extent to which accrued interest may have appeared in the prices paid for securities as stated in the accounts. In 1778 the Society sold £20,000 of stock out of its holding of 3 per-cent Consolidated Bank Annuity "in order to purchase other stock" : a point worth mentioning if only because it shows, as, in fact, do many subsequent dealings, that the idea that an insurance company may refrain from looking for opportunities of making profitable sales, was not the view of these pioneers of life insurance.

In 1779 mortgages appear in the accounts for the first time, and it may be of interest to mention that one of the mortgages now on the books of the Society was granted originally in June, 1798, although it has, of course, in the meantime had to pass through the vicissitudes of many re-arrangements. From 1779 onwards the new investments were divided between mortgages and investments in the Public Funds, and at the beginning of the nineteenth century about £400,000 was invested in mortgages and the holdings in the 3 per-cent and 4 per-cent Annuities amounted to £1,290,000 and £400,000 respectively.

It was during the closing years of the eighteenth century and the early part of the nineteenth century that the funds of the Society were growing most rapidly, and evidence of this is found in the large new investments which were made by the office during these years. It frequently appears that several hundred thousand pounds of new investments were made, and as the Society was granting new assurances to the extent of a million of

sum assured, which was far more than had been obtained in previous years, it was clear that the funds of the Society would be expected to increase for a considerable period. At the end of 1816 however, the Society adopted a bye-law which, by limiting to 5,000 the number of policies participating in profits, had the effect of discouraging new insurances, and although the funds continued to increase for several years, the position gradually underwent a change, and after the funds had been over nine millions in 1840, they gradually dropped until they were about four millions in 1870.

The following table shows the investments of the Society on various occasions, and may give some idea of the changes that took place in the investments that had been made.

*Table showing Investments (in £1,000's) of Old Equitable.*

Year	GOVERNMENT STOCKS			Exchequer Bills	Mortgages
	3 %	4 %	3½ %		
1773	34	...	...	...	...
1793	357	120	...	...	197
1803	1,290	400	...	...	418
1805	1,905	400	...	...	412
1810	3,540	400	...	19	402
1815	5,520	400	...	...	562
1820	7,800	400	...	...	729
1825	9,005	...	380	...	1,157
1830	8,130	...	400	...	2,181
1835	7,750	...	400	60	2,812

In the fifteen years from 1779, when mortgages were first granted, until 1793, the sum laid out on mortgage reached £197,000, and by 1799 it reached £400,000, at which figure it remained for many years. Stocks were high during the early period covered by our table, but from 1793 onwards prices fell and between that date and 1805 the prices of 3 per-cent Government stocks were sometimes below 50, nearly always below 60, and rarely above 70. During the following ten years prices were a little higher, but still low, and while they were never below 55 they were never above 73. After 1815 prices improved, and from 1828 onwards 3 per-cent stocks were almost always above 80 and frequently above 90.

If these variations in the value of the Government stocks are



compared with the changes in the investments indicated by our table, it will be seen that the amount invested in mortgages was not increased during the time when Government stocks were so low that profitable investments could be made, but in the early period before the low prices became established, and in the later period, substantial investments were made in mortgages. Large amounts of securities were sold between 1825 and 1840 in order to increase the amount advanced on mortgage. It will also be observed that investments were made in Exchequer Bills as well as in the more permanent type of security, and these seem to have been bought and sold from time to time so that they were apparently treated as a convenient temporary investment.

The point of view of the men responsible for the management in these early days was that when the price of securities was high it was wise to put a substantial amount on mortgage and some investments in short-dated securities: but when a good return could be obtained from permanent investments in the Public Funds mortgages were neglected. In making this remark I may, perhaps, be accused of reading into past actions thoughts which were not in the minds of those of whom I am writing. But there is evidence in some of the addresses of William Morgan, and in his little book on the early history of the Old Equitable, that he at any rate appreciated these points and it was partly his appreciation of them that made him fearful lest the Society to which he had devoted so much energy should suffer from having to make investments less profitably, and it is, I think, clear that this was one of the reasons which led him to advise the Directors and members to restrict the new business so that such profits as had been due to temporary advantages, which might not recur, would not have to be spread over so large a number of cases as would have necessarily followed from the continuance of the large amount of new business which the Society had been obtaining from 1800.

If Seeley's maxim that history should pursue a practical object is one that it is wise for us to follow, we may endeavour to see what can be learnt from the information we have obtained.

The most obvious conclusion to be drawn is that it is possible to frame and follow a consistent investment policy which, over a period of years will avoid the dangers of depreciation and take advantage of appreciation. Possibly another conclusion which



ought to be drawn is that an insurance company can make satisfactory financial arrangements even if it has a very restricted field of investment. When set out in this way these conclusions seem obvious: but in any attempt to apply them in practice it is necessary to face the difficulty involved in deciding whether we are passing through a period of low prices or are merely at the beginning of a long fall in prices. If at the present time we knew that prices of securities would, on the average, fall 15 or 20 per cent in the next five years and then rise more or less steadily there would be little difficulty in applying general principles. But the problem in practice involves an estimate of the course of future prices and, while we might be prepared to commit ourselves to a broad statement that no permanent rise can take place till some little time after peace and therefore to justify the application of some settled policy over a long period, we are haunted by the fear that a small misjudgment of the course of prices, while unimportant over a long term of years, may place us in a difficult position as regards a few balance sheets in the meantime. In this connection it may be of interest to record that the practice of the Old Equitable was to take market prices when valuing securities—I think prices must have been taken at 1 November instead of 31 December, when the valuation was nominally made, as all valuations were based on assurances existing at 1 November on the assumption that they would remain in force till the end of the year. Morgan referred to this practice in his addresses of 1825 and 1829 when the Society was investing in mortgage whenever possible, and warned members not to look on the existing prices as permanent and not to think of surplus in terms of inflated values. However this may be, it is clear that the method of stating values of securities a hundred years ago was optional, while at the present time we are restricted in a way that may operate severely when funds depreciate even from a temporary cause. Such temporary effects cannot be foretold in advance, but if we can follow, as it seems to me Morgan and his Directors followed, an investment policy based on a broad outlook over a number of years, we might well run the risk of temporary disappointments.

At the present time, no doubt, everyone responsible for the investment of life assurance funds is trying to read rightly the trend of prices and to base his action on his opinion. I am doubtful if modern practice will yield many examples of greater foresight than that of the men responsible for investing life

insurance funds about a hundred years ago. It seems to me that Morgan and his Directors made and followed a consistent policy in times which bear some likeness to our own, and the story of their efforts may be useful to us in deciding on our present action and in making our forecast of the future.

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*A General Expansion Formula.*

DE MORGAN wrote at considerable length, and almost with enthusiasm, on Arbogast's Method of Derivations, which is virtually an extension of Taylor's Theorem. He says, "Few, even among mathematicians, are aware of the " power of this process . . . which well deserves to be made " as common as Taylor's Theorem. . . . By help of the method, " expansions which analysts usually avoid as much as possible, " at almost any expense of circumoperation, are carried on " with the greatest facility even further than is necessary. . . .

" The foundation of Arbogast's methods is a contrivance " for expediting the expansion of  $\phi(a+bx+cx^2+\dots)$  into a " series of the form  $A+Bx+Cx^2+\dots$ " De Morgan gives materials for proceeding as far as the term . . .  $x^{12}$ ; " not that " so much will often be necessary, but because it is desirable to " show with how little trouble questions of enormous labour in " the ordinary way . . . may be looked upon without dismay."

Readers who desire to study the mathematical basis of the method, or to go so far as  $x^{12}$  in their expansions, should refer to De Morgan's Differential and Integral Calculus, pp. 328-335; also to the article from his pen in the English Cyclopædia (title, "Taylor's Theorem"), and in the *Journal*, vol. xii, p. 206. He works out the matter very fully, and discusses the relation between Arbogast's process and the Multinomial Theorem when  $\phi(u)$  is  $u^n$ . Reference may also be made to a brief discussion of the method in Williamson's Differential Calculus. The object of the present Note is to give the formula in its general form, and also in some specific cases, as far as the term involving  $x^6$ , which will be amply sufficient for most practical purposes.\*

\* Those who wish to check or extend the formula by means of De Morgan's tables should notice that he omits  $d$ ; i.e., our  $a+bx+cx^2+dx^3+ex^4\dots$  is his  $a+bx+cx^2+ex^3+fx^4\dots$  so that after  $c$  each of his letters must be replaced by the preceding one.

I.—*General Formula*.—If  $\phi(a+bx+cx^2+\dots)=A+Bx+Cx^2+\dots$ , then denoting the successive differential coefficients of  $\phi(a)$  with respect to  $a$  by  $\phi', \phi'', \phi''', \&c.$

$$A=\phi(a)=\phi$$

$$B=b\phi'$$

$$C=c\phi'+\frac{b^2}{2}\phi''$$

$$D=d\phi'+bc\phi''+\frac{b^3}{6}\phi'''$$

$$E=e\phi'+\left(bd+\frac{c^2}{2}\right)\phi''+\frac{b^2c}{2}\phi''' + \frac{b^4}{24}\phi^{IV}$$

$$F=f\phi'+(be+cd)\phi''+\left(\frac{b^2d}{2}+\frac{bc^2}{2}\right)\phi''' + \frac{b^3c}{6}\phi^{IV} + \frac{b^5}{120}\phi^V$$

$$G=g\phi'+\left(bf+ce+\frac{d^2}{2}\right)\phi''+\left(\frac{b^2e}{2}+bcd+\frac{c^3}{6}\right)\phi''' \\ +\left(\frac{b^3d}{6}+\frac{b^2c^2}{4}\right)\phi^{IV} + \frac{b^4c}{24}\phi^V + \frac{b^6}{720}\phi^{VI}$$

It will be observed that in every case the coefficient of  $\phi'$  is of the first dimension, that of  $\phi''$  is of the second dimension, and so on.

The above coefficients may be written down at once from De Morgan's tables based on Arbogast's theory, and in the same way the series may be continued to any extent. But without reference to these tables, the series may be very simply derived (to the extent to which we have here used it) by the following method (*Cf.* De Morgan, *J.I.A.*, vol. xii, p. 212). Let  $b+cx+dx^2+\dots=z$ . Then, by Taylor's Theorem,

$$\phi(a+bx+cx^2\dots)=\phi(a+xz)=\phi(a)+xz.\phi'(a)+x^2z^2.\phi''(a)/2!+\dots$$

and if we expand  $z^2$  as far as  $x^4$ ,  $z^3$  as far as  $x^3$  and so on, by ordinary multiplication, and then collect coefficients of  $x$ ,  $x^2$ ,  $x^3\dots x^6$ , we shall arrive at the above values for A, B, C... G.

II.—*Exponential Function*.—When  $\phi(u)=e^u$ ,  $\phi, \phi', \phi'', \dots$  are all equal to  $e^a$ . Hence  $e^{a+bx+cx^2+\dots}$

$$\begin{aligned}
&= e^a \left[ 1 + bx + \left( c + \frac{b^2}{2} \right) x^2 + \left( d + bc + \frac{b^3}{6} \right) x^3 + \left( e + bd + \frac{c^2}{2} + \frac{b^2c}{2} + \frac{b^4}{24} \right) x^4 \right. \\
&\quad + \left( f + be + cd + \frac{b^2d}{2} + \frac{bc^2}{2} + \frac{b^3c}{6} + \frac{b^5}{120} \right) x^5 \\
&\quad + \left( g + bf + ce + \frac{d^2}{2} + \frac{b^2e}{2} + bcd + \frac{c^3}{6} + \frac{b^3d}{6} + \frac{b^2c^2}{4} + \frac{b^4c}{24} + \frac{b^6}{720} \right) x^6 \\
&\quad \left. + \dots \dots \dots \right]
\end{aligned}$$

III.—*Logarithmic Function*.—When  $\phi(u) = \log_e u$ ,  $\phi(a) = \log_e a$ ,  $\phi' = \frac{1}{a}$ ,  $\phi'' = -\frac{1}{a^2}$ ,  $\phi''' = \frac{2}{a^3}$ ,  $\phi^{IV} = -\frac{6}{a^4}$ ,  $\phi^V = \frac{24}{a^5}$ ,  $\phi^{VI} = -\frac{120}{a^6}$ , ... Hence,  $\log_e(a + bx + cx^2 + \dots)$

$$\begin{aligned}
&= \log_e a + \frac{b}{a}x + \left( \frac{c}{a} - \frac{b^2}{2a^2} \right) x^2 + \left( \frac{d}{a} - \frac{bc}{a^2} + \frac{b^3}{3a^3} \right) x^3 \\
&\quad + \left( \frac{e}{a} - \frac{2bd + c^2}{2a^2} + \frac{b^2c}{a^3} - \frac{b^4}{4a^4} \right) x^4 \\
&\quad + \left( \frac{f}{a} - \frac{be + cd}{a^2} + \frac{b^2d + bc^2}{a^3} - \frac{b^3c}{a^4} + \frac{b^5}{5a^5} \right) x^5 \\
&\quad + \left( \frac{g}{a} - \frac{2bf + 2ce + d^2}{2a^2} + \frac{3b^2e + 6bcd + c^3}{3a^3} - \frac{2b^3d + 3b^2c^2}{2a^4} + \frac{b^4c}{a^5} - \frac{b^6}{6a^6} \right) x^6 \\
&\quad + \dots \dots \dots
\end{aligned}$$

IV.—*Power Function*. (*Multinomial Theorem*).—When  $\phi(u) = u^n$ ,  $\phi(a) = a^n$ ,  $\phi' = na^{n-1}$ ,  $\phi'' = n(n-1)a^{n-2}$ ,  $\phi''' = n(n-1)(n-2)a^{n-3}$ , and so on. Hence, if we write  $C_r$  for  $\frac{n!}{r!(n-r)!}$ , the ordinary binomial coefficient of  $x^r$ , we have  $(a + bx + cx^2 + \dots)^n$

$$\begin{aligned}
&= a^n + nba^{n-1}x + (nca^{n-1} + C_2b^2a^{n-2})x^2 \\
&\quad + (nda^{n-1} + C_2 \cdot 2bca^{n-2} + C_3b^2a^{n-3})x^3 \\
&\quad + [nea^{n-1} + C_2(2bd + c^2)a^{n-2} + C_3 \cdot 3b^2ca^{n-3} + C_4b^4a^{n-4}]x^4 \\
&\quad + [nfa^{n-1} + C_2 \cdot 2(be + cd)a^{n-2} + C_3 \cdot 3(b^2d + bc^2)a^{n-3} \\
&\quad \quad \quad + C_4 \cdot 4b^3ca^{n-4} + C_5b^5a^{n-5}]x^5 \\
&\quad + [nga^{n-1} + C_2(2bf + 2ce + d^2)a^{n-2} + C_3(3b^2e + 6bcd + c^3)a^{n-3} \\
&\quad \quad \quad + C_4(4b^3d + 6b^2c^2)a^{n-4} + C_5 \cdot 5b^4ca^{n-5} + C_6b^6a^{n-6}]x^6 \\
&\quad + \dots \dots \dots
\end{aligned}$$

V.—*Reciprocal of Function*.—Putting  $n = -1$  in the preceding formula, we have  $(a + bx + cx^2 + \dots)^{-1}$

$$\begin{aligned}
 &= \frac{1}{a} - \frac{b}{a^2}x - \left(\frac{c}{a^2} - \frac{b^2}{a^3}\right)x^2 - \left(\frac{d}{a^2} - \frac{2bc}{a^3} + \frac{b^3}{a^4}\right)x^3 \\
 &\quad - \left(\frac{e}{a^2} - \frac{2bd + c^2}{a^3} + \frac{3b^2c}{a^4} - \frac{b^4}{a^5}\right)x^4 \\
 &\quad - \left(\frac{f}{a^2} - \frac{2be + 2cd}{a^3} + \frac{3b^2d + 3bc^2}{a^4} - \frac{4b^3c}{a^5} + \frac{b^5}{a^6}\right)x^5 \\
 &\quad - \left(\frac{g}{a^2} - \frac{2bf + 2ce + d^2}{a^3} + \frac{3b^2e + 6bcd + c^3}{a^4} - \frac{4b^3d + 6b^2c^2}{a^5} + \frac{5b^4c}{a^6} - \frac{b^6}{a^7}\right)x^6 \\
 &\quad - \dots\dots\dots
 \end{aligned}$$

VI.—*Reversion of Series*.—If  $y = ax + bx^2 + cx^3 + \dots$ , then

$$\begin{aligned}
 x &= \frac{y}{a} - b \cdot \frac{y^2}{a^3} + (2b^2 - ac) \frac{y^3}{a^5} - (5b^3 - 5abc + a^2d) \frac{y^4}{a^7} \\
 &\quad + (7b^2 \cdot \overline{2b^2 - 3ac} + 3a^2 \cdot \overline{2bd + c^2} - a^3e) \frac{y^5}{a^9} \\
 &\quad - (42b^3 \cdot \overline{b^2 - 2ac} + 28a^2 \cdot \overline{b^2d + bc^2} - 7a^3 \cdot \overline{be + cd} + a^4f) \frac{y^6}{a^{11}} \\
 &\quad + \dots\dots\dots
 \end{aligned}$$

This series is not directly derivable from the general formula in Section I, but it is known that if  $y = xf(x)$ ,

$$x = \frac{y}{f(0)} + \sum_{n=2}^{n=\infty} \left[ \frac{y^n}{n!} \cdot \frac{d^{n-1}}{dx^{n-1}} \left[ \frac{1}{f(x)} \right]^n \right]_{x=0}$$

which, in this case becomes (since  $f(x)$  here is  $a + bx + cx^2 \dots$ )

$$y/a + \sum_{n=2}^{n=\infty} \left[ \frac{y^n}{n!} \cdot \frac{d^{n-1}}{dx^{n-1}} (a + bx + cx^2 + \dots)^{-n} \right]_{x=0}$$

and it may easily be seen that in this expansion the coefficient of  $y^n$  is  $1/n$  of the coefficient of  $x^{n-1}$  in the expansion of  $(a + bx + cx^2 + \dots)^{-n}$ . These latter coefficients may readily be written down by the aid of the formula in Section IV.

If the original series for  $y$  in terms of  $x$  be convergent within any range of values of  $x$ , the reverted series for  $x$  in terms of  $y$  will be convergent within *some* range of values of  $y$ ; though it may be difficult or impossible to say what that

range is if the general law of the coefficients in the reverted series is not ascertainable. It must not be assumed that the series gives a unique value of  $x$  corresponding to any particular value of  $y$ : there may be more than one such value of  $x$ . For example, if  $y$  is  $\sin x$  or  $\cos x$  there will be an infinite number of real values of  $x$  for any given value of  $y$ : and in the simple case where  $y$  is a quadratic expression in  $x$ , say  $y = ax + bx^2$ , there will be two real values of  $x$  for any value of  $y$  if  $\frac{y}{b} > -\frac{a^2}{4b^2}$  and no real value if  $\frac{y}{b} < -\frac{a^2}{4b^2}$ . In the latter case the series will evidently be divergent. In general, the reverted series gives the value of  $x$  which tends to zero with  $y$ .

VII.—*General Formula (alternative form).*—The following form may sometimes be more convenient than that given in Section I. If

$$\phi(a + bx + c \cdot x^2/2! + dx^3/3! + \dots) = A + Bx + C \cdot x^2/2! + D \cdot x^3/3! + \dots$$

Then

$$A = \phi(a) = \phi$$

$$B = b\phi'$$

$$C = c\phi' + b^2\phi''$$

$$D = d\phi' + 3bc\phi'' + b^3\phi'''$$

$$E = e\phi' + (4bd + 3c^2)\phi'' + 6b^2c\phi''' + b^4\phi^{IV}$$

$$F = f\phi' + 5(be + 2cd)\phi'' + 5(2b^2d + 3bc^2)\phi''' + 10b^3c\phi^{IV} + b^5\phi^V$$

$$G = g\phi' + (6bf + 15ce + 10d^2)\phi'' + 15(b^2e + 4bcd + c^3)\phi''' + 5(4b^3d + 9b^2c^2)\phi^{IV} + 15b^4c\phi^V + b^6\phi^{VI}$$

See De Morgan, *Differential and Integral Calculus*, pp. 774–6.

VIII.—*Expression of Coefficients in Determinant form.*—The writer is indebted to Professor E. T. Whittaker, F.R.S., for the remark that in many important cases the coefficients can be expressed as determinants, which show very clearly the law of the series: and (to those who are familiar with the practical rules for the evaluation of determinants) it is much easier to compute the determinants numerically than to compute the coefficients from the expanded algebraical expressions. A few examples are given below.

Professor Whittaker informs the writer that formulæ (iii) and (iv), which were the first of the determinantal formulæ to be discovered, were given by Spottiswoode and



Faure in 1853-55. Formula (ii) is due to Faà di Bruno (1855), and the corresponding formula (v) for the logarithm to Glaisher (1879). Formula (i) belongs to the same class, but appears to have been given explicitly first by Segar (1892). Formula (vi) is a combination of (i) with Burmann's theorem, and has been given as such by Professor Whittaker in his lectures, but has not previously been published.

$$(i) (a + bx + cx^2 + dx^3 + \dots)^n$$

$$= a^n + na^{n-1}bx + \frac{x^2}{2! a^{2-n}} \begin{vmatrix} nb & -a \\ 2nc & (n-1)b \end{vmatrix} \\ + \frac{x^3}{3! a^{3-n}} \begin{vmatrix} nb & -a & 0 \\ 2nc & (n-1)b & -2a \\ 3nd & (2n-1)c & (n-2)b \end{vmatrix} + \dots$$

and the coefficient of  $x^r$  is

$$\frac{1}{r! a^{r-n}} \begin{vmatrix} nb & -a & 0 & \dots \\ 2nc & (n-1)b & -2a & \dots \\ 3nd & (2n-1)c & (n-2)b & \dots \\ 4ne & (3n-1)d & (2n-2)c & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

where the determinant has  $r$  rows and  $r$  columns, and the law of its formation is obvious.

$$(ii) e^{a+bx+cx^2+\dots} = e^a \left[ 1 + bx + \frac{x^2}{2!} \begin{vmatrix} b & -1 \\ 2c & b \end{vmatrix} + \frac{x^3}{3!} \begin{vmatrix} b & -1 & 0 \\ 2c & b & -2 \\ 3d & 2c & b \end{vmatrix} \right. \\ \left. + \frac{x^4}{4!} \begin{vmatrix} b & -1 & 0 & 0 \\ 2c & b & -2 & 0 \\ 3d & 2c & b & -3 \\ 4e & 3d & 2c & b \end{vmatrix} + \dots \right]$$

$$(iii) \frac{a + \beta x + \gamma x^2 + \dots}{a + bx + cx^2 + \dots} = \frac{a}{a} + \frac{x}{a^2} \begin{vmatrix} a & a \\ b & \beta \end{vmatrix} \\ + \frac{x^2}{a^3} \begin{vmatrix} a & 0 & a \\ b & a & \beta \\ c & b & \gamma \end{vmatrix} + \frac{x^3}{a^4} \begin{vmatrix} a & 0 & 0 & a \\ b & a & 0 & \beta \\ c & b & a & \gamma \\ d & c & b & \delta \end{vmatrix} + \dots$$

(iv) Putting  $a=1, \beta=\gamma=\delta \dots =0$ , we get from (iii)

$$(a+bx+cx^2+\dots)^{-1} = \frac{1}{a} - \frac{b}{a^2}x + \frac{x^2}{a^3} \begin{vmatrix} b & a \\ c & b \end{vmatrix} - \frac{x^3}{a^4} \begin{vmatrix} b & a & 0 \\ c & b & a \\ d & c & b \end{vmatrix} + \dots$$

(v)  $\log(1+ax+bx^2+\dots) =$

$$ax - \frac{x^2}{2} \begin{vmatrix} a & 1 \\ 2b & a \end{vmatrix} + \frac{x^3}{3} \begin{vmatrix} a & 1 & 0 \\ 2b & a & 1 \\ 3c & b & a \end{vmatrix} - \frac{x^4}{4} \begin{vmatrix} a & 1 & 0 & 0 \\ 2b & a & 1 & 0 \\ 3c & b & a & 1 \\ 4d & c & b & a \end{vmatrix} \\ + \dots + \frac{(-1)^{n-1}x^n}{n} \begin{vmatrix} a & 1 & 0 & 0 & 0 \dots \\ 2b & a & 1 & 0 & 0 \dots \\ 3c & b & a & 1 & 0 \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n \text{ rows and cols.} \end{vmatrix} \dots$$

(vi) If  $y=ax+bx^2+cx^3+\dots$ , the coefficient of  $y^n$  in the expansion of  $x$  in powers of  $y$  is (see Section VI)  $1/n$  of the coefficient of  $x^{n-1}$  in the expansion of  $(a+bx+cx^2+\dots)^{-n}$ . Hence, writing down these coefficients by means of (i), and re-arranging signs:

If  $y=ax+bx^2+cx^3+\dots$

$$x = \frac{y}{a} - b \frac{y^2}{a^3} + \frac{y^3}{3! a^5} \begin{vmatrix} 3b & a \\ 6c & 4b \end{vmatrix} - \frac{y^4}{4! a^7} \begin{vmatrix} 4b & a & 0 \\ 8c & 5b & 2a \\ 12d & 9c & 6b \end{vmatrix} + \dots \\ + \frac{(-1)^{n-1}y^n}{n! a^{2n-1}} \begin{vmatrix} nb & a & 0 & 0 & 0 \dots \\ 2nc & (n+1)b & 2a & 0 & 0 \dots \\ 3nd & (2n+1)c & (n+2)b & 3a & 0 \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} - \dots$$

The determinants in (vi) can be simplified before evaluation, for the first column can be divided by  $n$ , and then the bottom row by  $(n-1)$ .

It is hoped that the above results will be of occasional use to some who may not have access to the works cited.

G. J. L.

## LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law*.

Agent's knowledge of material facts not given in proposal.

A FURTHER case on the vexed question as to whether the knowledge of the agent is deemed to be the knowledge of the Company was that of *Ayrey v. British Legal, &c., Assurance Company*, 34 T.L.R. 111. This was an appeal before A. T. Lawrence and Atkin, J.J., against a decision in the Blackpool and Fleetwood County Court.

The appellant was the executrix of one Alexander McKenzie, who had effected an insurance on his life by a policy dated 14 November 1914. She claimed to recover the amount due under the policy, but the defendants refused to pay, alleging that Alexander McKenzie had made misrepresentations on his proposal form, and had withheld material information. The misrepresentations and concealment alleged were a statement that he was a fisherman, whereas, in fact, besides being a fisherman he was a member of the Royal Naval Reserve, and at the time of making the proposal he had been called up for service.

The insured expected to go mine-sweeping, and while waiting to do so he disappeared; it was supposed that he must have fallen overboard.

At the trial evidence was given to the effect that the agent of the defendant company knew that McKenzie was going mine-sweeping, and that the agent so informed the superintendent, who was district manager of the defendants; and the superintendent said that the defendants would be willing to pay on the policy, in spite of the fact that the insured was going mine-sweeping. The plaintiff further contended that as the defendant's superintendent had accepted premiums after he had full knowledge of the facts they could not now refuse to pay.

The County Court Judge decided that there was no evidence that the agent who took the proposal ever communicated the facts to the defendants before the issue of the policy, and he therefore held that the policy was void *ab initio*, and that the receipt of premiums by the superintendent after the issue of the policy did not bind the defendants. He therefore gave judgment for the defendants.

Mr. Justice Lawrence said he did not agree with the conclusion of the County Court Judge.

The assured was a fisherman, as stated on the form, but he was also called up in the Royal Naval Reserve, a fact which was not stated on the form, and the question was whether his having been called up without the statement of that fact vitiated the policy. In his opinion it did not. True, there was a provision on the form saying that omission or concealment of material facts would render the policy void. The district manager was, however, told that McKenzie was in the Royal Naval Reserve, and it was only reasonable to suppose that telling the manager was equivalent to telling the head office. Communications made to such a person would be assumed by any reasonable assured person to be communicated to the head office. The manager had no power to make a new contract, but the old contract subsisted, and any objection that might have been taken was waived by the subsequent receipt of premiums. The appeal must therefore be allowed.

Mr. Justice Atkin delivered judgment to the same effect.

Rights of  
Mortgagees under  
Courts  
(Emergency  
Powers) Act.

The case of *In re Jobson's* application and in the matter of *Chapman's* Mortgage has attracted considerable notice, owing to the rules laid down by

Mr. Justice Eve in the course of his judgment regarding the rights of mortgagees as modified by emergency legislation. These rules are of peculiar interest to those life assurance officials who have to deal with mortgages and are sometimes faced with the problems dealt with by the learned Judge. The case has not yet appeared in the Reports, but will be found in *The Times* of 17 January 1918.

The facts are as follows:

A mortgage was created in July 1906 on premises in Eastbourne, which were valued for the purpose at £21,800. The amount advanced was £12,500 with interest at 4 per-cent, and the money was used as capital by the mortgagors, who were traders occupying a considerable portion of the mortgaged premises. Notice to pay off the mortgage was served on the mortgagors in March 1917, and a summons for leave to realize the security under the Courts (Emergency Powers) Act was issued in July 1917. All the covenants and conditions of the mortgage deed had been duly performed and observed, and the principal

money due had been voluntarily reduced by annual payments to £8,350.

Judgment was given extending the time for the repayment of the mortgage debt, on condition that the mortgagor undertook to pay 5 per-cent interest and to repay out of principal £500 in October. In the course of his judgment Mr. Justice Eve said: "Experience has shown that considerable misconception exists as to the purpose and effect of the emergency legislation with which this application is concerned. The Act is not for the relief of insolvent debtors, still less for the conscription for the benefit of debtors of the property of their creditors. It is, as its name implies, an Act designed and intended to meet some of the emergencies to which the present state of war has given rise, and the particular emergency to which the sections with which I have to deal are directed is the inability of a debtor to discharge his obligation on the date when it falls to be discharged. In all cases of this class to which the Act applies it restricts the right of the creditor to resort to his legal remedies for enforcing payment to the extent that it forbids him the exercise of any such remedy until after an application has been made to the Court for leave to exercise it. Then, by Sub-section (2), it confers on the Court an absolute discretion in dealing with the application, subject to the limitation that the discretion is only to be exercised in favour of a defaulting debtor if the Court is of opinion that time should be given to him on the ground that he is unable immediately to make the payment by reason of circumstances attributable directly or indirectly, to the war. When the Court is satisfied on this point it has power to suspend the question of any remedy for such time, and subject to such conditions, as it thinks fit.

"The present application is one by mortgagees whose mortgage debt became repayable in June last, and who apply for leave to exercise any right or powers vested in them as mortgagees to realize their security, and in what I am about to say I am confining myself to applications of a similar character, *i.e.*, applications by mortgagees against defaulting mortgagors. In the ordinary way, where a mortgage debt is called in, the mortgagor meets the demand in one of three ways: (1) he realizes the security and pays off the mortgagee out of the proceeds; or (2) he provides the money out of his other resources, and takes a reconveyance; or (3) he gets the

“ assistance of another lender. who pays off the mortgage and  
“ takes a transfer as security. The last of these courses is the  
“ one most usually adopted, and, indeed, it is almost inevitable  
“ where the advance is of a large amount, which is treated as  
“ capital more or less permanently borrowed for business  
“ purposes at a fixed and moderate rate of interest.

“ In considering, therefore, whether the mortgagor qualifies  
“ himself for the exercise in his favour of the discretion conferred  
“ by Section 2 the Court ought, in the first place, to direct its  
“ attention to the question how, in the ordinary course, the  
“ particular security would be dealt with. A man who had  
“ purchased property as an investment and borrowed a portion  
“ of the purchase-money on mortgage, might—where the  
“ mortgage money was called in—elect either to realize that  
“ investment and pay off the mortgage or he might prefer to  
“ realize other investments and pay it off, or he might just as  
“ likely determine to retain his investment and find a transferee  
“ of the mortgage. On the other hand, a man who had borrowed  
“ money on the security of property in his own occupation  
“ would probably be limited to the alternatives of paying the  
“ debt out of his other resources or of obtaining a transferee,  
“ and a man who had raised money on the security of his  
“ business premises and had employed the money in his business  
“ would, almost of necessity, be compelled to meet the  
“ mortgagee’s requirements by obtaining a transferee. Nor  
“ would these more or less personal considerations alone  
“ determine the mortgagor’s action. A more potent factor  
“ would be the value of security in relation to the amount  
“ of mortgage debt.

“ It may perhaps be helpful if I indicate the considerations  
“ and conditions which, in my opinion, can fairly be regarded  
“ and applied in ordinary cases, by which I mean cases in  
“ which the relationship of mortgagor and mortgagee subsists  
“ without any complication arising out of any other contractual  
“ relations between the parties, or because the position of either  
“ party is exceptional.

“ In such cases the following furnish good working rules :

“ (a) If the security is sufficient, and if the covenants and  
“ conditions of the mortgage deed—other than the covenant to  
“ repay the principal moneys—have been performed and  
“ observed the mortgagor ought to be given a reasonable time  
“ within which to find the money to pay the debt.



“(b) The reasonable time so given may be extended if, in cases where the interest reserved is less than 5 per-cent, the mortgagor be willing to pay interest at the rate of 5 per-cent, and still further extended if, in addition, he is willing to pay over to the mortgagee on account of principal the difference between the net rents received from the mortgaged property and the interest at 5 per-cent.

“(c) In a like case, if the mortgagor is in occupation and the proper occupation rent is in excess of 5 per-cent interest on the debt he ought to attorn tenant to the mortgagee at the proper occupation rent, and pay to the mortgagee interest at 5 per-cent and the excess of such rent over the 5 per-cent interest on account of capital.

“(d) If the security is insufficient but interest has been paid up to date, and the other covenants and conditions in the mortgage deed have been performed and observed the mortgagee should not be exposed to further loss, and time should be given to the mortgagor on the terms that interest at 5 per-cent is paid, and that the mortgagee may renew the application if and when any further depreciation in the security takes place.

“(e) If the security is insufficient and there are arrears of interest, but not to a serious amount, the mortgagor should be required to clear off such arrears in addition to complying with the terms set out in (d).

“(f) “If in either of the last two cases the rent of the mortgaged property or a proper occupation rent therefor exceeds the interest the mortgagee should be allowed, if he so desires, to appoint a receiver, but in the event of his so doing, in a case falling under (e), the stipulation requiring the mortgagor to pay off the arrears of interest may have to be modified by giving him a reasonable time within which to clear them off.

“(g) In cases where there are substantial arrears of interest, or where the covenants to keep down ground rents and other outgoings, or to keep in repair or insure have been broken, the mortgagee should not be restrained from exercising his powers, in the absence of very special circumstances, unless the mortgagor is prepared to make good all such breaches forthwith and to continue under conditions similar to those enumerated under (d), (e), and (f).

“Now it is clear that in this case if there was no war the mortgagors would have had to provide for repayment by a

“ transfer. It was argued that they had not brought themselves  
 “ within the protection of the section. But I am satisfied, on  
 “ the evidence and having regard to the general conditions of  
 “ which I am bound to take judicial notice, that it is almost  
 “ impossible to obtain the necessary advance of money by any  
 “ transfer of the mortgage except on terms which would leave  
 “ the borrower in a position which is many times worse than  
 “ that which he occupies in existing circumstances. I am quite  
 “ satisfied that they have established a case within the section.  
 “ The mortgagors have brought themselves within rule (a)  
 “ mentioned above, and they ought to be given a reasonable  
 “ time within which to find the necessary money.

“ On the mortgagors undertaking to increase the interest to  
 “ 5 per-cent and to repay out of principal £500 per annum in  
 “ October, I defer the operation of the mortgagee’s remedies for  
 “ one year from to-day, if the war so long continues, with liberty  
 “ to apply in the event of the mortgagors failing to keep their  
 “ undertaking. The costs must be added to the security.”

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I am indebted to Mr. G. J. Lidstone for a copy of  
 correspondence with the Board of Inland Revenue  
 on the subject of stamp duty on mortgages for  
 unlimited amounts.

Stamp duty on  
 mortgages for  
 unlimited  
 amounts.

The previous practice of the Inland Revenue Authorities is dealt with by Mr. Barrand in “Further Notes on Some Legal Aspects of Life Assurance Practice”, *J.I.A.*, vol. xli, pp. 210, 211, and the extract given below from a letter from them dated 7 January 1918, indicates an alteration in their attitude.

In the case in question a mortgage, dated 1 July 1884, for an unlimited amount was stamped £6, that is to cover advances up to £4,800, and the security consisted of life policies in several offices. By assignment, dated 30 January 1885, the mortgagees sold all the policies to third parties for a total consideration of £1,524—which it will be observed was well within the amount covered by the stamp duty—but the policies were kept up by the purchasers until 1917 when the life assured died, and the total amount, which has recently become payable to the assignees, is considerably in excess of £4,800, the amount covered by the stamp. In the first instance the Inland Revenue Authorities ruled that the offices collectively could not pay a larger total amount than £4,800 without requiring the mortgage to be further stamped.

Further correspondence ensued and finally the Board wrote :

“ Where a mortgagee exercised his powers of sale under a mortgage and sold the policies comprised therein the Board will not hold insurance companies responsible for penalty under Section 118 of the Stamp Act of 1891 if they pay the full amount of the policy claims, even though that amount may exceed the amount which the mortgage is stamped to cover, provided that at the date of the assignment to the purchaser from the mortgagee the total of the surrender values of the policies did not exceed the amount which the mortgage is stamped to cover. In the present case it appears that the mortgagee sold the policies under the powers of sale conferred by the mortgage, and that at the date of sale the surrender values of the policies amounted to less than £4,800, the amount on which the mortgage is stamped, and the Board will accordingly offer no objection so far as they are concerned to the payment of the policy moneys.”

Policy issued to cover funeral expenses.  
Conflict of claim.

The unreported case of *Da Costa v. Prudential Assurance Company* will probably be of interest to industrial assurance officials by reason of the decision of the Court of Appeal upon the meaning of Section 36, sub-section (2) of the Assurance Companies Act 1909.

The facts are as follows :

Two policies of assurance for a total sum of £21 were taken out in 1900 and 1905 respectively upon the life of one Mary Nicholl. They were effected by her son Joseph Sylvester Nicholl, who paid premiums upon them. Mary Nicholl died on 25 December 1916, and the sum assured was paid by the Company to Joseph Sylvester Nicholl, who delivered up the policies. An action was brought by the executrix of Mary Nicholl, a Mrs. Da Costa, who obtained judgment in the Liverpool Court of Passage for £21 against the Company. Section 36, sub-section (2) of the Assurance Companies Act, 1909, provides :

“ No policy effected before the passing of this Act with a collecting society or industrial assurance company shall be deemed to be void by reason only that the person effecting the policy had not, at the time the policy was effected, an insurable interest in the life of the person assured, or that the name of the person interested, or for whose benefit or

“ on whose account the policy was effected, was not inserted  
“ in the policy, or that the insurance was not one authorized  
“ by the Acts relating to friendly societies, if the policy was  
“ effected by or on account of a person who had at the time a  
“ *bonâ fide* expectation that he would incur expenses in con-  
“ nection with the death or funeral of the assured, and if the  
“ sum assured is not unreasonable for the purpose of covering  
“ those expenses, and any such policy shall enure for the  
“ benefit of the person for whose benefit it was effected or  
“ his assigns.”

The Company having appealed from the judgment of the Liverpool Court, the appeal was heard before Swinfen Eady, L.J., Bankes, L.J., and Eve, J., and judgment given for the appellants.

In the course of his judgment Lord Justice Swinfen Eady said :

“ The learned Judge in the Court below has found that Mrs. Nicholl's son took out the policies in his mother's name in relation to any expenses in connection with her death he might be called upon to bear. I do not think there is any doubt about that. He also finds that the two sums for which the policies were effected were not unreasonable.

“ Then it appears that after the death of Mary Nicholl, which took place on the 25 December 1916, the son made good his title to both policies to the satisfaction of the Prudential Assurance Company, and they paid him the amount and took from him a receipt, and he delivered up the policies. The language of the policies is that upon payment of the premiums and on satisfactory proof of the death of the assured the company will pay the amount assured to the executors or administrators. Then that is followed by a proviso : Provided always that the production by the company of a receipt for any sum payable hereunder, signed by any person being either an executor or administrator or the husband or wife or a relation by blood or connection by marriage of the assured shall be a discharge to the company for the same, and shall be final and conclusive evidence to all intents and purposes that such sum has been duly paid and received by the person or persons lawfully and rightfully entitled to the same, and that all claims and demands whatsoever against the company in respect of this policy have been fully satisfied. What has happened is that the Prudential Assurance Company had produced receipts signed by the son, who is, of

“ course, a relation by blood, and they say that is a discharge  
 “ to the company.

“ Of course if that defence is well founded, it is a good  
 “ discharge. But it does not rest there, because upon the facts,  
 “ as the learned Judge has found them, the policy enures to the  
 “ benefit of the son, and if the plaintiff were to recover she  
 “ would only recover as trustee for the son, and her duty then  
 “ would be to pay the sum that she recovered over to the son.  
 “ But Joseph Sylvester Nicholl has already been paid direct by  
 “ the Insurance Company the whole amount, and where money  
 “ in that way has got home to the *cestui que trust* it is not  
 “ competent for an executor or trustee to sue again for the  
 “ money with the intent that he or she shall pay it over again  
 “ to the *cestui que trust*.

“ Then it was urged that the present case was not within the  
 “ section, because the name of Joseph Sylvester Nicholl does not  
 “ appear in the policy itself. But the section is intended to meet  
 “ a case of that kind. It says that ‘No policy effected before  
 “ the passing of this Act’—and both these policies fulfilled this  
 “ condition—‘shall be deemed to be void by reason only that  
 “ ‘the person effecting the policy had not, at the time the policy  
 “ ‘was effected, an insurable interest in the life of the person  
 “ ‘assured, or that the name of the person interested, or for  
 “ ‘whose benefit or on whose account the policy was effected,  
 “ ‘was not inserted in the policy.’ The section continues ‘any  
 “ ‘such policy shall enure for the benefit of the person for whose  
 “ ‘benefit it was effected’, that is to say, although the name was  
 “ not inserted in the policy, and although the policy itself is  
 “ silent on the question, nevertheless it shall enure for the  
 “ benefit of the person for whose benefit it was effected.

“ Therefore, in my opinion, the appeal should be allowed,  
 “ and judgment entered for the defendants.”

Lord Justice Bankes and Mr. Justice Eve both agreed.

### *Annuity Reserves and Rates in the United States.*

On 8 November last Mr. D. P. Fackler wrote as follows to  
 Mr. J. D. Watson, Joint Hon. Secretary of the Institute :

“ As Corresponding Member of the Institute for the United  
 “ States, it appears to be my duty to inform you and the Institute  
 “ of the following facts regarding Annuities :



“ The Convention of the Insurance Commissioners of the several States at their Meeting in the autumn of 1916 adopted a resolution to the effect that in computing the legal reserves to be held for life annuities the rate of interest assumed should be one-half of 1 per cent. higher than the legal rate prescribed for the computation of the reserves for life insurances. The resolution stated the reasons for this recommendation, which was made on the assumption that the annuity reserves would be computed upon a table fairly representing annuity experience.

“ In consequence of the above action, the Legislature of the State of New York early in this year (1917) amended the laws governing the proper legal reserves to be held for insurances and annuities, so that the legal rate to be used in the computation of annuity reserves should be 4 per cent., whilst that to be used for life insurances is  $3\frac{1}{2}$  per-cent.

“ Many other States will, doubtless, follow New York in the above action when their Legislatures meet this winter. Several insurance companies have already reduced their premiums for annuities in consequence of the above change in the New York Law, and it is likely that many others will soon do the same.

“ Your Corresponding Member has been much interested in the general subject of annuities, and presented articles thereon at the Meetings of the Actuarial Society in May 1914, and in October 1915, which may be found in the Society's *Transactions*, and may have, in some measure at least, contributed to bring about the above legislation.”

*The United States Government Life Assurance Scheme for Officers, Men and Nurses on Active Service.*

[We are indebted to Mr. W. A. Hutcheson, F.I.A., F.F.A. F.A.S., for a copy of the Act of the United States Legislature by which provision was made for the assurance of officers and men—and the nursing staff (male and female)—of the United States Army and Navy by the Treasury War Risk Insurance Bureau. We extract the following sections from Article IV (Insurance).—Eds. *J.I.A.*]:

SEC. 400. That in order to give to every commissioned officer and enlisted man and to every member of the Army Nurse Corps (female) and of the Navy Nurse Corps (female) when employed in active service under the War Department or Navy Department greater protection for themselves and their dependents than is provided in Article III,\* the United States, upon application to the bureau and without medical examination, shall grant insurance

\* This article provides (without payment of any premium) for compensation in the event of disablement, and for benefits to dependent relatives in the event of death.—Eds. *J.I.A.*



against the death or total permanent disability of any such person in any multiple of \$500, and not less than \$1,000 or more than \$10,000, upon the payment of the premiums as hereinafter provided.

SEC. 401. That such insurance must be applied for within one hundred and twenty days after enlistment or after entrance into or employment in the active service and before discharge or resignation, except that those persons who are in the active war service at the time of the publication of the terms and conditions of such contract of insurance may apply at any time within one hundred and twenty days thereafter and while in such service. Any person in the active service on or after the sixth day of April, nineteen hundred and seventeen, who, while in such service and before the expiration of one hundred and twenty days from and after such publication, becomes or has become totally and permanently disabled or dies, or has died, without having applied for insurance, shall be deemed to have applied for and to have been granted insurance, payable to such person during his life in monthly instalments of \$25 each. If he shall die either before he shall have received any of such monthly instalments or before he shall have received two hundred and forty of such monthly instalments, then \$25 per month shall be paid to his wife from the time of his death and during her widowhood, or to his child, or widowed mother if and while they survive him: Provided, however, That not more than two hundred and forty of such monthly instalments, including those received by such person during his total and permanent disability, shall be so paid; and in that event the amount of the monthly instalments shall be apportioned between them as may be provided by regulations.

SEC. 402. That the director, subject to the general direction of the Secretary of the Treasury, shall promptly determine upon and publish the full and exact terms and conditions of such contract of insurance. The insurance shall not be assignable, and shall not be subject to the claims of creditors of the insured or of the beneficiary. It shall be payable only to a spouse, child, grandchild, parent, brother or sister, and also during total and permanent disability to the injured person, or to any or all of them. The insurance shall be payable in two hundred and forty equal monthly instalments. Provisions for maturity at certain ages, for continuous instalments during the life of the insured or beneficiaries, or both, for cash, loan, paid-up and extended values, dividends from gains and savings, and such other provisions for the protection and advantage of and for alternative benefits to the insured and the beneficiaries as may be found to be reasonable and practicable, may be provided for in the contract of insurance, or from time to time by regulations. All calculations shall be based upon the American Experience Table of Mortality, and interest at three and one-half per centum per annum, except that no deduction shall be made for continuous instalments during the life of the insured in case his total and permanent disability continues more than two hundred and forty months. Subject to regulations, the insured shall at all times have the right to change the beneficiary or beneficiaries of such insurance without the consent

of such beneficiary or beneficiaries, but only within the classes herein provided. If no beneficiary within the permitted class be designated by the insured, either in his lifetime or by his last will and testament, or if the designated beneficiary does not survive the insured, the insurance shall be payable to such person or persons, within the permitted class of beneficiaries as would under the laws of the State of the residence of the insured, be entitled to his personal property in case of intestacy. If no such person survive the insured, then there shall be paid to the estate of the insured an amount equal to the reserve value, if any, of the insurance at the time of his death, calculated on the basis of the American Experience Table of Mortality and three and one-half per centum interest in full of all obligations under the contract of insurance.

SEC. 403. That the United States shall bear the expenses of administration and the excess mortality and disability cost resulting from the hazards of war. The premium rates shall be the net rates based upon the American Experience Tables of Mortality and interest at three and one-half per centum per annum.

SEC. 404. That during the period of war and thereafter until converted the insurance shall be term insurance for successive terms of one year each. Not later than five years after the date of the termination of the war as declared by proclamation of the President of the United States, the term insurance shall be converted, without medical examination, into such form or forms of insurance as may be prescribed by regulations and as the insured may request. Regulations shall provide for the right to convert into ordinary life, twenty payment life, endowment maturing at age sixty-two and into other usual forms of insurance and shall prescribe the time and method of payment of the premiums thereon, but payments of premiums in advance shall not be required for periods of more than one month each and may be deducted from the pay or deposit of the insured or be otherwise made at his election.

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## REVIEWS.

*The Mathematical Theory of Population, of its Character and Fluctuations, and of the Factors which influence them.* By G. H. KNIBBS, C.M.G., F.S.S., F.R.A.S., &c.

[Pp. xvi + 466. Melbourne: McCarron, Bird & Co.]

THIS lengthy appendix to the census of the Commonwealth of Australia discusses the use of algebraic or trigonometric functions in interpreting population statistics, and applies the methods to the Australian census. After an introductory chapter Mr. Knibbs proceeds to examine the way in which population fluctuates, and although he gives some expressions that may take a more general form, he seems to think that the normal basis of change of population is of the nature of a geometrical progression. He implies that the

geometrical form may not be realized, but harks back to it, and must, we think, feel that this is the proper starting point. There is much to be said for this assumption on *a priori* argument, but such assumptions are apt to be upset by a "disagreeable little fact." When, however, Mr. Knibbs finds that a variation in the rate of change is necessary he comes much nearer to the facts of the case, and he suggests the use of  $1 + \eta^{tm+nt}$  as an expression for dealing with the rate of change of population: the constants being obtained from three observations "at any suitable intervals."

In a subsequent chapter Mr. Knibbs apparently leaves the question of curves suitable for rates of increase and turns to fitting the facts themselves. He refers to the Pearson type curves, and after saying "although the type curves . . . fulfil their general "purpose fairly well, experience shews that their '*fitting power*' "is somewhat limited", he gives what he calls a "flexible curve" of the form  $Ax^m e^{nx^p}$ . This curve does not lend itself to fitting by moments or by a systematic method, and Mr. Knibbs says we must take several series of ordinates; each set "will give a value "for  $p$  . . . and a mean (geometric, arithmetic or other) can be "taken." In spite of Mr. Knibbs's experience of the Pearson-curves there is surely ample evidence that their fitting power has a wide range, and it is hard to see why we should desert a proved group of curves which can be fitted systematically, and has some theoretical justification, and use instead an arbitrary curve, the constants of which must be found by the method indicated above. The Pearson-curves may have to give place in the future to some still more useful group, but at the present time they stand out as the most practical attempt to find curves suitable for the description of frequency distributions, and except when Mr. Knibbs's curve takes the form of the Pearson Types III and V, we do not think he will find other people anxious to use it.

Mr. Knibbs discusses the result of certain projections of the normal curve—a method to which Professor Edgeworth has given much attention—but he seems to us to project the normal curve on to surfaces chosen arbitrarily, and if we make the form of the surface on which we project sufficiently complex we can, doubtless, obtain a wide variety of forms, most of which will, however, be useless for curve fitting in practice. This part of the work, in common with many other parts, gives the impression of notes of ideas that have occurred to the author rather than considered results of research.

The next sections discuss group values, integration for statistical purposes and smoothing in population statistics, and then after a preliminary chapter we come to the arithmetical part of the work, although several pages dealing with the summation method of graduation and various other incidental or algebraic work are interspersed. Some of the tables give interesting information, and the tables dealing with errors in age and with marriages, as well as some of the tables on fertility, &c., are specially of value.

The book is well printed and contains some excellent diagrams, but it is far too long, and the excess of matter leaves the reader with

a feeling of bewilderment. If the more important investigations and tables could be separately printed in a volume of greatly reduced size, they would be more likely to receive the study they deserve.

W. P. E.

*Du Calcul dans les Jeux de Hasard. Rédigé par D. J. KORTEWEG.  
Extrait des œuvres complètes de Christiaan Huygens: Tome XIV.*

THE direct contribution of Huygens to actuarial science does not seem to have been of great importance, although it is of some interest historically as the earliest recorded application of probabilities to life contingencies.

In the correspondence of the brothers Christiaan and Lodewijk (*J.I.A.*, vol. xxxiv, p. 387) on the subject of the calculation of expectations of life from Graunt's Mortality Table, Lodewijk gave the more practical solution of the problem—a computation, rough certainly, but based on correct principles, of what we now call the expectation of life—if Christiaan had the better of the argument.\* It is true that Christiaan, with his singular sincerity and indefatigability in clearing up difficulties, discovered in the end that they were both right (“je trouve que nous avons tous deux raison en prenant la chose en different sens”) and that Lodewijk's expectation of life and his own *Vie Probable* were two distinct things. But at the outset, influenced probably by the fact that the infant science of probabilities had up to then been applied almost exclusively to games, he seems to have regarded the problem as of the nature of a bet (“Qui gagerait qu'un enfant de 6 ans vivra jusqu'à 26”—Lodewijk having computed the expectation at age 6 at 20 years 10 months—“peut mettre 25 contre 39, “puisque de 64 enfans de 6 ans, il y en a 25 qui parviennent à l'age de 26 ans contre 39 qui meurent au dessous”), and it was only towards the end of the correspondence that he realized that it had a practical connection with the important subject of life annuities (“Le premier”, *i.e.*, Lodewijk's expectation, “est pour regler les rentes à vie, et l'autre pour les gageures”.) With reference to this correspondence Dr. Korteweg suggests that the “courbe de vie”, given by Huygens in his concluding letter, is the first example of the graphic representation of mortality; but the curve in question appears to be an interpolation curve for the purpose of finding the values of the expectation of life at intermediate ages from the computed values for ages 0, 6, 16, 26, &c., rather than a mortality curve as ordinarily understood.

The only other recorded instance of Huygens having interested himself in the actuarial applications of probabilities is in connection with Hudde's investigations (*Memoires pour servir*, &c., pp. 76, 82), in which however it would not appear from Hudde's letters that he took any more active part than that of sympathetic appreciation.

\* This correspondence was published in vol. vi of the *Oeuvres Complètes*, and was reproduced shortly afterwards, by permission of the Editors, in the *Memoires pour servir*.



But if Huygens contributed little directly to actuarial science, the value of his indirect contribution can hardly be over-estimated, for it is to him that we are indebted for the first clear statement of the elementary principles of probability. Although Pascal and Fermat had already proposed and solved some of the problems with which Huygens dealt, their methods had not been disclosed at the time when his attention was directed to the subject ("ces savants, quoiqu'ils se missent à l'épreuve l'un l'autre en se proposant beaucoup de questions difficile à résoudre, ont cependant caché leurs méthodes"), and it was left to Huygens to construct from first principles a systematic method of procedure. The result was his classical "*De Ratiociniis in Ludo Aleæ*"—to give the tract which forms the subject of this notice its better known Latin title\*—a model of clearness and lucidity, which "continued to form the best account of the subject until it was superseded by the more elaborate works of James Bernouilli, Montmort and De Moivre" (*History of the Theory of Probability*, p. 25). In Bernouilli's *Ars Conjectandi*, published more than half a century later, an annotated reprint of the "*De Ratiociniis*" occupies the first part, and in many later works the influence which it exercised on the development of the subject of probability may be traced. English translations from the original Dutch and Latin appeared in 1692 and 1714, and in recent times a French translation has been published in the *Mémoires pour servir*, and an English translation (by Mr. E. W. Scott) in the Transactions of the Actuarial Society of America (Vol. IV). But it has appropriately, if somewhat exceptionally, been reserved to Huygens' country to render due honour to the tract, and this has been most handsomely done in the definitive edition of his works which has been in course of publication by the Société Hollandaise des Sciences during the last quarter of a century. The extract which has now been separately issued under the title given at the head of this notice renders available to students of the early history of probability practically all that Huygens wrote on the subject, including, in addition to the "*De Ratiociniis*," nine appendixes containing various other researches. The value and interest of the publication are much enhanced by the Editor's historical introduction and mathematical commentary—and, it may be added, by such luxuries, unusual in these days, as large paper, wide margins, and an admirable fount of type.

The problems discussed by Huygens are for the most part special and comparatively simple cases of problems which have since become historical, such as the Problem of Points, the Problem of Dice, and the great Duration of Play Problem. They have long since been generalized, and have been solved in their general form by more powerful methods of analysis than were available in the early days of probabilities. Nevertheless the

\* A copy of the original Latin edition of Van Schooten's *Exercitationes Mathematicæ* (1657), in which the *De Ratiociniis* was first published, has recently been added to the Institute Library.

solutions given by Huygens are instructive on account of the simplicity of his analytical apparatus and the lucidity of his reasoning. Starting from the hypothesis that the chance that a player has of winning in a game has a value such that if he possessed that value he could secure the same chance in a game in which he had an equal chance of winning or losing, he establishes three elementary propositions of the category represented by the general principle that, if A's chances of winning  $a_1, a_2, \&c.$ , are  $p_1, p_2, \&c.$ , respectively, the value of his expectation is  $\Sigma pa / \Sigma p$ . This forms the basis of his argument. His one method of solving the various problems proposed by Pascal and others, or devised by himself, consists, as the Editor of the present reprint points out, "dans une application continuelle, répétée autant de fois que le "problème l'exige, de ces propositions." It is remarkable how much he achieves by such simple means, and if the method becomes laborious and its repetition monotonous as the problems become more complicated, it could, in some cases, be simply used in association with the method of induction, as, for example, in the Problem of Points, or in the Duration of Play Problem when the game is supposed to continue until one player has nothing left.

We may add, in conclusion, that at the end of the Editor's introduction will be found an interesting discussion of the "dédale des problèmes" exchanged by Huygens and Hudde on the advantage or disadvantage of playing first in certain games of chance. In a game, for example, of the pitch and toss variety in which two players toss in turn and each puts one in the pool when a head turns up and takes one out when tail turns up, and it is a question of the disadvantage of playing first (when there is nothing to take out) or generally of how the pool should be divided at any time, both Huygens and Hudde assumed that the expectations of the two players must be together equal to the amount in the pool. But Dr. Korteweg points out that the game may be prolonged indefinitely, and that in such cases there is a third expectation which he proposes to call "le part de diable." The devil's share in the case in question is the whole amount in the pool, the players having definitely lost—in the absence of any arrangement for ending the game—all that they have put in.

*Interest and Bond-values.* By M. A. MACKENZIE, M.A., F.I.A., A.A.S. *Second Edition. Revised and Enlarged.*

[Pp. 107 and Tables pp. x. University of Toronto. Price \$2.]

(London: C. & E. Layton. 8s. 6d. net.)

WE welcome a second edition of Prof. Mackenzie's useful little text-book on the theory of interest and its practical application to the valuation of bonds and other securities.

Some alterations have been made in the earlier chapters, a certain amount of new matter has been inserted (we notice, for



example, an interesting discussion on p. 72 of the question whether it is better from the borrower's point of view to issue a loan at a premium or at a discount when repayment has to be provided for by a sinking-fund accumulated at a rate lower than the yield on the loan), and the exercises at the end have been supplemented by a few examination papers, which may assist students to test their knowledge of the subject.

A feature of the book which will commend it to those who find it easier to understand principles and methods when illustrated arithmetically than when expressed in symbols is the prominence given throughout to practical numerical examples. In valuing redeemable securities Prof. Mackenzie shows how Bond-value Tables may be used in some cases in which the conditions are not exactly those assumed in the construction of the tables, but we gather that he considers it simpler in such cases—and in fact safer in all cases unless the Bond-value Tables have been checked and the assumptions on which they have been constructed are clearly stated—to use ordinary interest and annuity-tables. The question of *income-tax*, which is now in the United Kingdom (and may in course of time become in America) such a very material factor in the valuation of, and the calculation of the yield on, redeemable securities, is not discussed.

In the probable event of future editions of the book being required, the table on p. 32—showing on a simple interest basis equivalent yearly, half-yearly and quarterly payments at various rates of interest—might perhaps be reconsidered. Although the simple interest equivalents may be near enough for practical purposes, a table of this kind, in which some of the tabulated amounts are correct (to the number of places of decimals retained), and others are only approximate, and its employment in many of the numerical examples, seems to us to make it harder for the student to understand clearly the important distinction between effective and nominal rates. On p. 18 it is stated that “if the interest-rate be 4 per-cent compounded half-yearly, the payments of \$100 at the end of each quarter must be regarded as payments of \$201 at the end of each half-year; while payments of \$100 at the end of each year must be regarded as payments of \$49·505 at the end of each half-year.” If the equivalent payment is required to three places it would seem no less important in the former case than in the latter that it should be calculated accurately. The general statement that “a payment of 1 at the end of each year is equivalent, at a rate of  $i$  per year, to a payment of  $x$  at the end of each half-year where  $x + x\left(1 + \frac{i}{2}\right) = 1$ ” is equally open to question. We can quite understand that in a practical book it may be desirable to avoid the complications of  $i/j(p)$ , but since—as Prof. Mackenzie states on p. 44—“the investment yield compounded half-yearly has become almost an institution”, the practical course would appear to be to deal only with a half-yearly effective rate, and to work out everything correctly on that basis.

Another point that may be mentioned is the retention of cents, *i.e.*, two places of decimals, whether they are justified or not, in the results of most of the numerical examples. Thus, on p. 43, the value to yield  $4\frac{1}{2}$  per-cent, compounded half-yearly, of a \$10,000 bond due in 25 years, and bearing quarterly coupons at 4 per-cent, is worked out at \$9287.55, when in the calculation the equivalent half-quarterly coupon is taken as \$201.12 and multiplied by  $a_{\overline{50}|}$  at  $2\frac{1}{4}$  per-cent. And on p. 66 the value to pay  $4\frac{1}{2}$  per-cent compounded half-yearly of a \$50,000 25-year 5 per-cent annuity bond with quarterly payments is given as \$52,726.63, this being the product of  $a_{\overline{50}|}$  (at  $2\frac{1}{4}$  per-cent) and an approximation (to two places on a simple interest basis) to the equivalent half-yearly payment of the annuity. Clearly the cents cannot be reliable in either case. In some cases, where the factors to be multiplied are exact, Prof. Mackenzie obtains correct results by using interest functions to more places than are given in the tables at the end of the book or are to be found in most practical tables. It would in our view be more instructive to base the examples on the tabulated values, and to let the student see what degree of accuracy can be obtained with the means available. The resulting values would in most cases be practical—cents being quite meaningless in connection with tens of thousands of dollars. In the few cases in which, for comparison of methods or for making a loan-schedule work out neatly, greater accuracy is necessary, the values of the functions to the requisite number of places could be specially stated.

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## CORRESPONDENCE.

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### ANNUITY-VALUES ON MAKEHAM'S HYPOTHESIS.

*To the Editors of the Journal of the Institute of Actuaries.*

SIRS,—The review (*J.I.A.*, vol. 1, p. 320) of my paper “Su una relazione fra l'annualità vitalizia di gruppo e l'annualità semplice, nell'ipotesi di Makeham,” suggests the following observations:

(1). The reader might suppose that I had repeated—although without being aware of the fact—McClintock's analysis in “On the computation of Annuities on Mr. Makeham's Hypothesis” (*J.I.A.*, vol. xviii, p. 242). But that is not the case either in form or in substance.

By means of the B-function—which McClintock did not consider—I endeavour to establish whether, and if so on what conditions, continuous annuity-values for  $m$  joint lives can be obtained by the formula

$$\bar{a}_{x_1 x_2 \dots x_m} = \frac{1}{p_m \log c} [F(1 - p_m, q_m) - c^m q_m^m \Gamma(1 - p_m)] \quad (1)$$

where  $F$  is a hypergeometrical function and  $\Gamma$  the  $\Gamma$ -function.

Neither van der Belt, to whom the corresponding formula for  $m=1$  is attributed (Enc. des Sciences Mathematiques, T.I., vol. 4, p. 531), nor McClintock, whose priority ought to be recognized—nor, so far as I know, any other writer—ever suggested the generalization and conditions established by me.

(2). Formula (1) is valid so long as the function  $F$  is convergent, and so also is the formula

$$a_{xy} = K a_x \dots (2)$$

which can be readily deduced from (1). The coefficient  $K$  depends on the functions  $F$  and  $\Gamma$ , and the calculation of its numerical value can be carried to any degree of approximation.

In the case of the *Text-Book*  $3\frac{1}{2}$  per-cent Table, formulas (1) and (2) are applicable so long as the number of lives  $m$  does not exceed 9. If  $m=2$  the formulas are applicable so long as  $w$  does not exceed 67, where  $2c^w = c^x + c^y$ ; and if  $m=3$  they are applicable so long as  $w$  does not exceed 63, where  $3c^w = c^x + c^y + c^z$ . The resulting coefficient  $K$  is always positive.

It follows that the reviewer's statement that the formula "seems to be inapplicable (the numerator becoming negative) to such a practical case as the evaluation of  $a_{60.60.60}$ ", cannot refer to formulas (1) and (2), and I do not know how it can refer to formula IV', because the numerator of this formula is not negative under the stated conditions, and I had anticipated the reviewer by stating that the formula is available for values of  $q_m$  "abbastanza piccoli" (rather small) and consequently not for greater values of  $m$  and older ages.

(3). When commutation-tables for a particular mortality-table are not available, and it is not convenient for any reason to undertake the rather laborious work of tabulation, it seems undeniable that formula (1) with the indicated limitations, and the passage from  $\bar{a}_x$  to  $a_x$ , will suffice for all requirements. Further, if one has a single-life commutation-table or the single-life annuity-values for all ages, and it is inconvenient to construct commutation-tables for two or more lives of equal ages, formula (2)—which admits, by the simple process of calculating the coefficient  $K$ , of the passage to joint-life annuities—is not to be despised.

The coefficient  $K$  can of course be represented by several approximate expressions. For instance, I gave a first approximation applicable to the RF  $3\frac{1}{2}$  per-cent Table. The corresponding approximation for the *Text-book*  $3\frac{1}{2}$  per-cent Table—to pass from  $a_x$  to  $a_{xx}$ —would be

$$K = \frac{\cdot 12 + \cdot 49f - \cdot 31(1+2f)f^{\cdot 51}}{\cdot 14 + \cdot 25f - \cdot 22(1+f)f^{\cdot 45}}$$

where

$$f = \cdot 00105 + \cdot 000096x + \cdot 0000044x^2$$

and it can hardly be said that this involves "a somewhat laborious calculation."

(4) Finally, the writer of the review appears to give the preference to the formula  $a_{xy} = a'_w$ . But for the application of this

formula it is necessary to have annuity-values for all ages and for several rates of interest, or to seek the assistance of other more or less approximate formulas. Therefore, generally, one must be content with an approximation which has not been proved to be satisfactory in all cases in its results. Formula (2) on the other hand is always applicable when the single-life annuity-value is given at the same rate of interest; and formula (1) is applicable in every case without any preliminary tabulation.

I am, Sirs, &c.,

F. INSOLERA.

*R. Ist. Sup. di Commercio, Turin,*  
23 January 1918.

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[We are glad to publish Prof. Insolera's letter, but we do not think that there was anything in our review to suggest that his analysis was the same as McClintock's. The reviewer's statement that Prof. Insolera had used McClintock's method (*i.e.*, the method of evaluating the integral for  $\bar{a}$  in an infinite series) for the purpose of obtaining the ratio of  $a_{xyz} \dots$  to  $a_x$  appears to be in accordance with the facts. McClintock did not restrict his investigation to the case of a single life; he indicated that the method could be applied, by a simple modification, to  $m$  lives, and his formula, modified accordingly, is identical in substance with Prof. Insolera's generalised formula (1). With regard to the conditions of applicability of the formula, the limitations imposed by Prof. Insolera do not appear to be necessary. Although the hypergeometrical series, in its general form, is divergent if  $x$  is  $> 1$ , the special type of hypergeometrical series entering into McClintock's and Prof. Insolera's formula is convergent for all values of  $\Sigma c^x$ . It follows that the formula is valid for lives of any ages, and it may be extended to any number of lives by further integrations by parts. The objection to the formula is not that it is subject to any limitations in theory, but that it involves an impracticable amount of calculation—owing to the slow convergency of the series—except for young lives. We are indebted to Mr. G. J. Lidstone for the information that when  $\Sigma c^x \log_e 1/g$  is large a good result can be obtained by means of Schlömilch's series for the incomplete  $\Gamma$ -function (*see* Bromwich's "*Introduction to the Theory of Infinite Series*"). There would seem to remain, however, a considerable interval between the age at which McClintock's formula ceases to be

of practical utility and that at which the Schlömilch series becomes applicable.

The statement quoted by Prof. Insolera in (2) refers to his formula IV'. This formula is given for  $a_{x_1 x_2 \dots x_m}$  without any explicit limitation of its applicability, and the words at the beginning of the investigation "quando si abbia da fare con gruppi di pochi elementi, così che  $q_m$  sia abbastanza piccolo" (when one has to do with combinations of a few lives so that  $q_m$  is sufficiently small) would not, we think, lead the ordinary reader to suppose that the formula does not apply to the calculation of a joint-life annuity on three lives of 60.

With regard to the approximation to K given in (3) it should be borne in mind that the expression is derived from formula IV' and is of limited application. It appears to give  $a_{65.65} = 6.375$ , the true value being 5.486. The approximate formula  $a_{xy} = a_w + \log_e s(1a)_w$ , gives (without using tables at more than one rate of interest) the correct result 5.486. —EBS. *J.I.A.*].

## MORTALITY AMONG NEUTRALS IN WAR-TIME.

*To the Editors of the Journal of the Institute of Actuaries.*

DEAR SIRS,—Those Members of the Institute who read Professor Hersch's paper "La Mortalité chez les Neutres en Temps de Guerre", reviewed in *J.I.A.*, vol. 1, p. 72, will remember that in this paper the author endeavoured to answer the question: "Which classes of a population are most seriously affected by the indirect effect of a War?"

The method adopted by the author was to consider the increase of mortality due to a War as the *absolute difference* between the mortality experienced in a time of War and the normal mortality of a time of peace, and to compare the results thus obtained for the different age groups. The method was, in fact, equivalent to a comparison of  $q_{ng} - q_n$  age-group by age-group, where  $q_{ng}$  represents the mortality from all causes, including the indirect effect of a War, and  $q_n$  the normal mortality.

The same subject was dealt with by Mr. J. W. Nixon in his paper "War and National Vital Statistics with Special Reference to the Franco-Prussian War"—*Journal of the Royal Statistical Society*, vol. lxxix, part 4. In this paper the author contended that the proper method of comparison was to compare, not the absolute, but the percentage increase in mortality, *i.e.*, not  $q_{ng} - q_n$ , but

$$\frac{q_{ng} - q_n}{q_n}$$



By adopting this method Mr. Nixon arrives at the following conclusions:

"It will be seen that the highest age-groups generally show the smallest increases in mortality, the highest increases being shown at periods of young and middle life. This, I think, is contrary to expectation. It is also contrary to certain definite conclusions based on the same figures recently made in a Swiss publication. This pamphlet states that the effects of War on mortality are the greatest at the two extremes of life, and are lowest at the ages 10 to 14. . . . . The method by which these conclusions are reached is fallacious."

Mr. Nixon then discusses Prof. Hersch's method in a foot note:

"As showing the incidence of mortality on the two sexes the author seems quite sound, but in dealing with the incidence of mortality at different ages of life his analysis is statistically quite unsound. . . . . The author takes the mean number of deaths in the years 1870-71, and finds the excess in these two years over the year 1869 per 1,000 of the population living at each age-group. He thus obtains a curve which reproduces in general outline the usual U-shaped mortality curve for ages, namely, a very high death-rate at the two extremes of life, and a low death-rate for the intervening ages. As persons over 65 had a high death-rate in 1869, and also a high death-rate in the years 1870-71, the difference between these two is likely to be much higher than the difference between the death-rates of a young age-group, *e.g.*, 15-20. By using this method of absolute increase in death-rates the Author, in effect, says that a rise in the death-rate from (say) 10 to 15 per 1,000—*i.e.*, a rise of 5 per 1,000—is much more serious than a rise in the death-rate from (say) 3 to 6 per 1,000—*i.e.*, one of 3 per 1,000. The former, however, is a rise of 50 per cent, the latter of 100 per-cent, and these figures are the ones which should be compared."

These remarks of Mr. Nixon's appear to have aroused considerable interest in Italy, and have given rise to two papers, the first of which, by Corrado Gini, appeared in the September-December 1916 number of the "*Rivista Italiana di Sociologia*", while the other by F. P. Cantelli, entitled "*Sull'Aumento di Mortalità dovuto alla guerra. Riflessioni critiche di metodologia statistica*", originally appeared in the November 1917 number of the "*Giornale degli Economisti e Rivista di Statistica*", and has now been published in pamphlet form. Both of these writers take exception to the criterion proposed by Mr. Nixon, and put forward alternative solutions of the problem.

The course adopted by Signor Gini is, in effect, to compare the increase in mortality with the probable error, so that if in two sections A and B of the populations, the rates of mortality are increased from  $m_1$  and  $m_3$  respectively, to  $m_2$  and  $m_4$ , then the increase in A must be regarded as  $>$  or  $<$  the increase in B according as

$$\frac{m_2 - m_1}{\sqrt{m_1(1 - m_1)}} \text{ is } > \text{ or } < \frac{m_4 - m_3}{\sqrt{m_3(1 - m_3)}}$$



The objections to the criterion proposed by Gini are, I think, obvious. They are put with great force by Prof. Cantelli on pages 13-15 of his pamphlet, to which I would refer anyone who is interested in the matter.

Prof. Cantelli points out that the criterion proposed by Nixon cannot give a satisfactory answer to the question: "Which classes of a population are the most seriously affected by the indirect effects of a War"? For "Consider a case—fictitious or not does not matter—which exposes the weakness of this criterion. Two populations, A and B, have suffered severely from the effects of a War. In the population A the mortality has increased from 10 per thousand to 100 per thousand, while in the population B it has increased from 10 per hundred to 100 per 100. Which of the two populations has been more severely hit by the War? That is, which population has suffered the heavier mortality? Clearly, it seems to me, population B, which has been completely wiped out by the effects of the War. But the expression used by Nixon gives the same result for the two populations, namely:

$$\frac{100 - 10}{10}$$

"But even excluding the above example, it does not seem to me that the expression

$$\frac{q_{ng} - q_n}{q_n} \quad \text{or} \quad \frac{m_{ng} - m_n}{m_n}$$

"can answer questions of the kind asked above, since it, as we see from its very nature, ignores the question of the exposed to risk."

Prof. Cantelli goes on to show that this question can only be answered by a comparison, age-group by age-group, of the values of  $q_g$ , where  $q_g$  is the probability of death on the assumption that the exposed to risk are only subject to mortality caused by the indirect effects of the war. He finds the value of  $q_g$  as follows:

"Let us suppose that out of  $l$  persons exposed to risk  $m$  die in one year from all causes, including the indirect effects of war. Let us further suppose that we know that if no deaths were due to the effects of war and that only the ordinary rate of mortality was in operation  $n$  persons would die in the year. We can then say that during the whole year of risk,  $l - n$  persons are not liable to die from normal causes. Hence it follows that the  $m - n$  persons, who die out of the  $l - n$  under consideration, die from causes due solely to the war, *i.e.*,

$$q_g = \frac{m - n}{l - n}$$

"But

$$\frac{m}{l} = q_{ng} : \frac{n}{l} = q_n$$

$$\therefore q_g = \frac{q_{ng} - q_n}{1 - q_n} \quad "$$

In a foot-note the author points out that this formula can be derived directly from Karup's theorem

$$p_{ng} = p_n \times p_g.$$

In order to compare the results given by his formula with those given by the formulæ of Hersch and Nixon, Gini calculated values of  $q_n$  from the mean annual number of deaths in the decennium 1876-85 and the census of 1880, and gives a series of tables, for the age-groups used by Prof. Hersch, for Switzerland, Holland and Belgium, distinguishing in each case between males and females.

I reproduce here Gini's tables for Swiss males, and have added, for purposes of comparison, a table of  $q_g$  calculated according to the formula given by Prof. Cantelli.

*The indirect effect of the Franco-Prussian War, 1870-71, on the Mortality of the Swiss Male Population.*

Age Group	The absolute excess of Mortality, namely, 10,000 ( $q_{ng} - q_n$ ) Prof. Hersch's method	Normal Mortality, namely, 10,000 $q_n$ as calculated by Signor Gini (see above)	Percentage increase in Mortality $\frac{q_{ng} - q_n}{q_n}$ Mr. Nixon's method	Relative increase in Mortality $\frac{m_2 - m_1}{\sqrt{m_1(1 - m_1)}}$ Gini's method	$q_g$ $= \frac{q_{ng} - q_n}{1 - q_n}$ Prof. Cantelli's method
	(1)	(2)	(3)	(4)	(5)
0-1	424	2,583	16	·097	·05716
1-5	74	205	36	·052	·00755
5-10	36	61	59	·046	·00362
10-15	11	35	31	·018	·00110
15-20	13	49	27	·018	·00130
20-25	65	73	89	·077	·00653
25-30	69	87	79	·074	·00696
30-35	38	99	38	·038	·00384
35-45	40	125	32	·036	·00405
45-55	51	198	26	·037	·00520
55-65	70	369	19	·037	·00726
65-75	132	774	17	·049	·01430
75-	252	1,722	15	·067	·03044

It will be noticed that this latter table supports the conclusion drawn by Prof. Hersch, and militates against those drawn by Mr. Nixon.

Yours faithfully,

D. S. SAVORY.

3, Queen's Gardens,  
Ealing, W. 5.

## INDUSTRIAL MORTALITY IN 1915-1917.

*To the Editors of the Journal of the Institute of Actuaries.*

DEAR SIRS,—Recently I had occasion to investigate the mortality experience of a very large number of policies on male lives embodying industrial assurance contracts of various descriptions.

The general results obtained were sufficiently remarkable, I think, to be of interest to readers of the *Journal*. I have therefore, in the accompanying diagram, exhibited them graphically in comparison with two standard population curves, namely, Dr. Farr's English Life Table No. 3 and the English Life Table No. 8 published in the supplement to the 75th Annual Report of the Registrar-General. The function chosen for comparison is the rate of mortality  $q_x$  as shown by the following experience of males :

Rate of mortality experienced among industrial policies during 1913					
"	"	"	"	"	" 1915
"	"	"	"	"	" 1916
"	"	"	"	"	" 1917
"	as shown by the English Life Table No. 8.				"
"	"	"	"	"	No. 3.

The exposed to risk and deaths were obtained from the valuation class books which were very convenient for the purpose, the policies being tabulated on the assumption that on the average exact ages were attained at the date of valuation.

This method gives a possible range of nearly two years in the tabulated age, *i.e.*, the assumed exact age  $x$  represents ages between  $x-1$  and  $x+1$ . Apart from abnormal disturbances in the law of mortality this method gives results reliable enough for most purposes. When, however, a section of the experience is subjected to a very violent change in the progression of the rates of mortality over a series of ages a certain amount of error is introduced. In the present instance the method of classification undoubtedly has the effect of over-estimating the rates of mortality at ages from 16 to 20, and somewhat under-estimating them thereafter.

The 1913 curve shows a close correspondence with that of the general population particularly at the military ages.

The curves for the three complete war years 1915, 1916, and 1917 each exhibit certain special features, and it is believed are the only published statistics showing the incidence of the War on the mortality of a section of the community sufficiently large to be representative of the whole male population.

The experiences of separate years were large enough to give results that did not require graduation to enable a reliable comparison to be made between them, and no such adjustment has been attempted. The close correspondence of the 1913 curve with the English No. 8 curve is sufficient evidence in support of this statement.

The remarkable feature of the 1915 curve is the continuance of high rates at ages over 40. The 1916 curve, while showing the

same main features as that for 1915, shows lower rates than the 1913 curve at ages over 40. The causes that gave rise to these features suggest some interesting speculations as to ages of men in the "Old Army" and the effect of war conditions on the vitality of the general population. I have not the necessary statistics available however to investigate these points. The 1917 curve exhibits two features of interest, namely, the moving of the maximum point to an age one year younger, accompanied by a reduction in the rate of mortality below age 20, that is, at the training ages.

The curves of course do not show the relative rates of mortality among combatants in the three years under review, but I think it may be claimed that they are a fair index of the toll that the War has taken of the manhood of the nation during that period.

I am, Dear Sirs,

Yours faithfully,

*Prudential Assurance Co., Limited,*  
*Holborn Bars, E.C. 1.*

J. BURN.

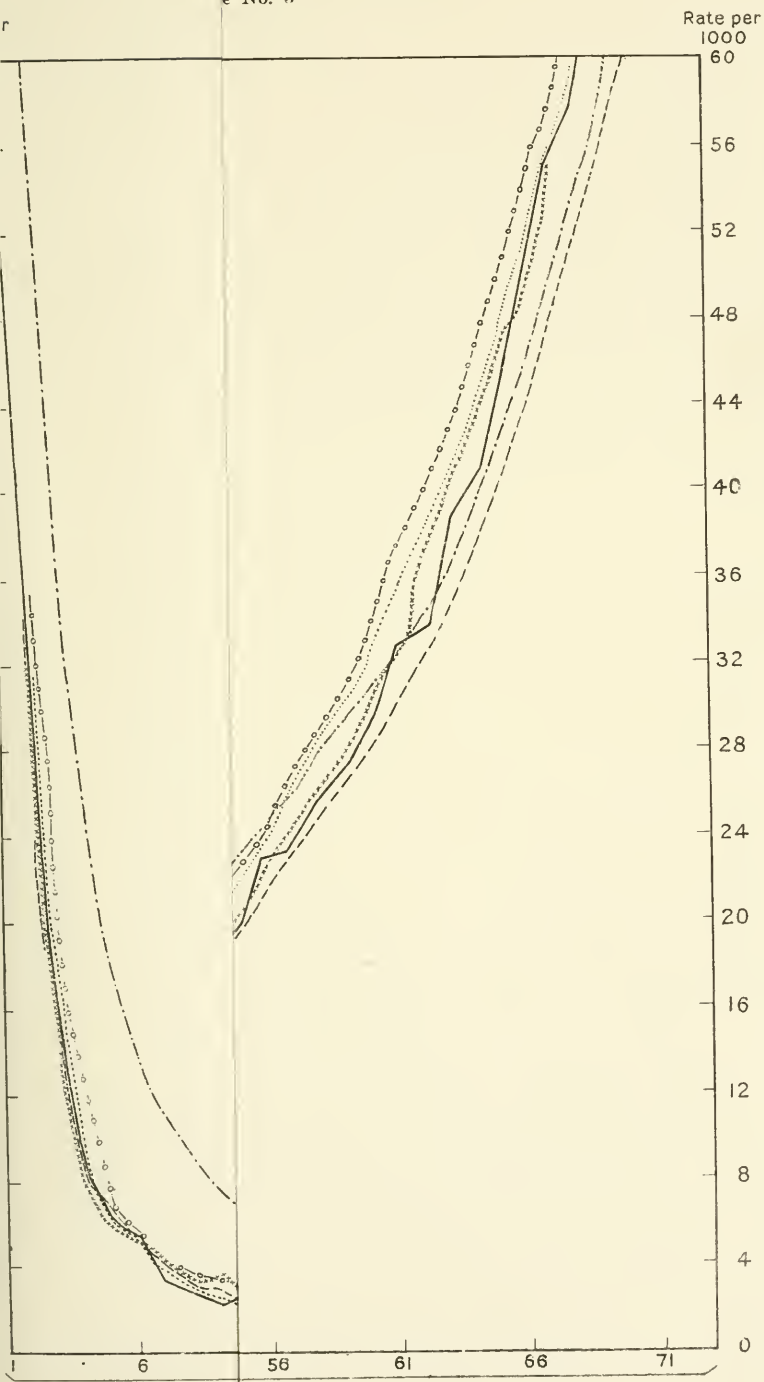
20 March 1918.

\*\_\* Mr. Burn has kindly supplied the actual rates of mortality experienced by the Prudential at ages 16 to 60 in the years 1913 and 1915-1917. These rates are given in the following Table, with the addition (in the first column) of the English Life No. 8 rates :

*Values of  $q_x$ . Ages 16-60. Male Lives.*

Age	English Life No. 8	PRUDENTIAL EXPERIENCE			
		1913	1915	1916	1917
16	·00259	·00290	·00467	·00404	·00374
17	·00279	·00317	·00790	·00840	·00483
18	·00302	·00379	·01386	·01856	·01283
19	·00326	·00372	·01845	·03201	·03834
20	·00348	·00397	·02008	·03731	·04742
21	·00366	·00386	·01838	·03772	·04559
22	·00378	·00386	·01695	·03460	·04394
23	·00386	·00377	·01665	·03124	·04063
24	·00392	·00399	·01534	·02805	·03769
25	·00400	·00441	·01376	·02572	·03507
26	·00411	·00442	·01340	·02379	·03248
27	·00425	·00451	·01311	·02192	·03014
28	·00440	·00475	·01199	·02098	·02886
29	·00458	·00532	·01246	·02001	·02794
30	·00478	·00499	·01168	·01915	·02601

e No. 8      - - - - -  
 e No. 3      - - - - -



*Tables of Mortality amongst Male Industrial Policyholders in the Prudential Assurance Company, Ltd*

Rate of Mortality experienced during the year 1913

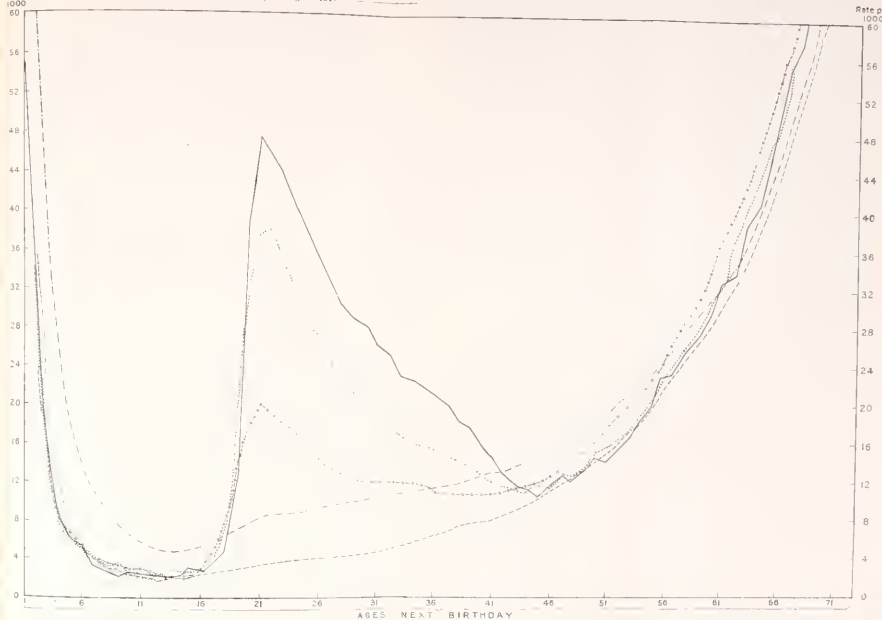
"	"	"	"	"	1915
"	"	"	"	"	1916
"	"	"	"	"	1917

Rate of Mortality, English Life Table No 8

Rate of Mortality, English Life Table No 3

Rate per 1000

Rate per 1000





*Values of  $q_x$ . Ages 16-60. Male Lives—continued.*

Age	English Life No. 8	PRUDENTIAL EXPERIENCE			
		1913	1915	1916	1917
31	·00502	·00562	·01182	·01773	·02501
32	·00528	·00520	·01177	·01675	·02304
33	·00558	·00605	·01225	·01576	·02265
34	·00590	·00594	·01145	·01565	·02194
35	·00624	·00642	·01118	·01488	·02097
36	·00659	·00667	·01087	·01440	·01977
37	·00695	·00714	·01069	·01369	·01816
38	·00731	·00742	·01065	·01286	·01744
39	·00769	·00811	·01130	·01216	·01588
40	·00811	·00798	·01119	·01152	·01452
41	·00858	·00902	·01094	·01140	·01286
42	·00909	·00908	·01137	·01099	·01159
43	·00964	·01005	·01123	·01065	·01111
44	·01024	·01063	·01169	·01173	·01043
45	·01089	·01121	·01258	·01130	·01154
46	·01158	·01184	·01288	·01253	·01236
47	·01231	·01308	·01438	·01233	·01205
48	·01308	·01391	·01469	·01316	·01290
49	·01391	·01520	·01660	·01479	·01427
50	·01482	·01609	·01781	·01551	·01413
51	·01586	·01717	·01831	·01643	·01533
52	·01701	·01877	·01873	·01710	·01637
53	·01827	·01861	·02107	·01898	·01854
54	·01963	·02269	·02255	·02008	·01987
55	·02111	·02423	·02372	·02234	·02283
56	·02272	·02637	·02642	·02397	·02318
57	·02444	·02852	·02869	·02629	·02558
58	·02629	·02991	·03191	·02786	·02731
59	·02827	·03307	·03559	·03075	·02996
60	·03042	·03462	·03802	·03268	·03291

## THE INSTITUTE OF ACTUARIES.

THE Council of the Institute have recently had under consideration the position of candidates for the Examinations in view of the situation created by the War. In order to afford candidates every possible help, under the special conditions which have arisen, the Council have decided:

- (1) To reduce the number of subjects of examination, and in this way to restrict the scope of the necessary reading.

(2) To request the Board of Examiners to afford candidates some guidance in the reading required ; and

(3) To hold Examinations more frequently.

They have, therefore, suspended the existing Regulations and Syllabus of Examinations, and have adopted, until further notice, the appended revised Regulations and Syllabus. Recommendations by the Board of Examiners as to the reading required for the Examinations are in course of preparation.

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*REGULATIONS and SYLLABUS of Examinations for admission to the Classes of Student, Associate, and Fellow.*

*(These Regulations will continue in force for not less than four years from the date of the first Examinations held after the termination of the War. All previous Regulations are hereby suspended.)*

1. The Examinations held by the Institute are as set out in the annexed Syllabus.

2. An applicant for admission to the Class of Student must (a) have furnished such evidence of general education as the Council may from time to time prescribe, and (b) have passed Part I of the Examinations under these or previous Regulations.

As evidence of general education the Council will (subject as hereinafter provided) require a Certificate showing that the applicant has passed the Matriculation or Senior School Examination of the University of London, or a similar Examination of any University in the British Empire, or the Oxford or Cambridge Senior Local Examination, or some other Examination approved by the Council from time to time or accepted on individual application. Provided that if in any case the Council are satisfied that an applicant has been precluded by special circumstances from passing one of the Examinations hereinbefore mentioned they may, if otherwise satisfied as to his general education, dispense with a Certificate of his having passed such Examination.

3. An applicant for admission to the Class of Associate must (a) be a Student and (b) have passed Parts I and II of the Examinations under these or previous Regulations.

4. An applicant for admission to the Class of Fellow must (a) be a Student or Associate and (b) have passed Parts I, II, III, and (subject as hereinafter provided) Part IV of the Examinations under these or previous Regulations. Provided that an applicant who has passed Parts I and II under previous Regulations shall be required to pass only Part III and Part IV, Section A, under these Regulations, and that an applicant who has passed Parts I, II, and III

under previous Regulations shall be required to pass only Part IV, Section A, under these Regulations.

5. An applicant who, under the Bye-laws or previous Regulations, has been exempted from passing a specified Part of the Examinations, shall be considered, for the purposes of these Regulations, to have passed that Part.

6. An applicant for admission to any Class shall, after having passed the prescribed Examinations, be admitted to that Class on signing the proper Form of Obligation or of Transfer, as the case may be, and on paying the subscription of the Class for the current year.

7. A Candidate shall not (except as hereinafter provided) present himself for Examination in a Part or a Section thereof until he has passed the previous Part. Provided that a Candidate, who is a Graduate in Mathematical Honours of a University in the British Empire, or any other Candidate on the recommendation of a Tutor of the Institute or of two Fellows having personal knowledge of his qualifications, may take Part I (or the remaining Section of Part I if he shall have passed in one Section) and Part II at the same Examination, but no such Candidate shall be considered to have passed Part II until he shall have passed both Part I and Part II.

8. The Fee for a Part of the Examination is £1. 1s., and for one Section of a Part 10s. 6d.

9. A Candidate will not be allowed to present himself for Examination until he has paid all entrance fees, subscriptions, and examination fees that may be due and complied with the requirements of the Bye-laws and these Regulations.

10. Examinations will be held in London and at such other places as the Council may appoint, in June and December of each year, or at such other times as the Council may prescribe. The first Examinations under these Regulations will be held as soon as possible after the termination of the War.

At least four months' notice will be given by public advertisement of the places at which Examinations will be held and of the dates of such Examinations.

A Candidate for Examination at any place in the United Kingdom must give notice in writing to the Assistant Secretary so that such notice shall reach the Assistant Secretary at least fourteen days before the date of Examination, and a Candidate for Examination at any place outside the United Kingdom must give such notice so that it shall reach the Assistant Secretary at least two months before the date of Examination. At the time of giving notice a Candidate must specify the Part or Section of a Part for which he intends to present himself, and must pay the prescribed fee.

11. The names of successful Candidates in each Part or Section of a Part will be arranged in alphabetical order, without distinction of Class.

*March 1918.*

## SYLLABUS OF EXAMINATIONS.

## PART I.

*Section A.*

Arithmetic and Algebra ; the theory and use of Logarithms ; the Elements of the Theory of Probabilities.

The Elements of the Calculus of Finite Differences, including Interpolation and Summation ; Elementary Differential and Integral Calculus, excluding questions necessitating the use of Trigonometry.

*Section B.*

Compound Interest and Annuities-Certain, including the construction and use of relative Tables.

*A Candidate must pass in both Sections. The two Sections may be taken at the same Examination, or either Section may be taken at one Examination and the remaining Section at a subsequent Examination.*

*Two Papers will be set in Section A and One in Section B. Three hours will be allowed for each Paper.*

## PART II.

Life Contingencies, including Life Annuities and Assurances, and the construction and use of the Life-Table and monetary and other Tables based thereon ; excluding questions on the compilation of Tables from Statistics, or on Graduation.

*Either two or three Papers will be set at the discretion of the Examiners. Three hours will be allowed for each Paper.*

## PART III.

*Section A.*

The Methods of compiling Mortality, Sickness, Accident, and other similar Statistics, of deducing Tables therefrom, and of adjusting or graduating such Tables.

The Distinctive Features of existing Tables now in general use.

*Section B.*

The Valuation of the Liabilities of Life Assurance and Employers' Liability Insurance Companies.

The Principles and Methods of the Distribution of Surplus.

*A Candidate must pass in both Sections. The two Sections may be taken at the same Examination, or either Section may be taken at one Examination and the remaining Section at a subsequent Examination.*

*Two papers will be set in each Section. Three hours will be allowed for each Paper.*

## PART IV.

*Section A.*

The general provisions of the Acts relating to Life Assurance Companies, Employers' Liability Insurance Companies and Friendly Societies.

The practical valuation of Life Interests and Reversions, and of Policies for Surrender or Purchase.

*Section B.*

The law relating to Life Assurance Contracts.

Life Assurance Accounts ; preparation of Schedules, Statements and Reports ; drafting of Policies and endorsements ; and other practical matters arising in Life Office Administration.

The determination of Office Rates of Premium for Assurances and Annuities, but excluding rates of Contribution for Sickness, Pension and Widows' and Orphans' Funds.

Extra premiums for under-average lives, hazardous occupations and residence in unhealthy climates ; and the materials available for their determination.

*Section C.*

The formation and valuation of and calculation of rates of contribution for Friendly Societies, Pension Funds, and Widows' and Orphans' Funds.

The National Health Insurance Acts, with special reference to their financial provisions.

The Elements of Statistics ; Official Statistics, with special reference to their employment in the solution of Actuarial Problems.

*A Candidate must pass in Section A and (subject to the provision in Rule 4) in either Section B or Section C. Section A and Section B (or Section C) may be taken at the same Examination, or either Section may be taken at one Examination and the remaining Section at a subsequent Examination.*

*Two Papers will be set in each Section. Three hours will be allowed for each Paper.*

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REGULATIONS FOR PROBATIONERS.

Probationers, while not being Members of the Institute, are allowed the following privileges, namely :

They are entitled to join the classes for Students, in accordance with the rules prescribed for such classes, and to attend the Ordinary General Meetings of the Institute, but not to vote or take part in the discussions thereat.

They may borrow books from the Library for the purposes of their studies, but this privilege is subject to the discretion of the Librarians, and to the rules which the Council may from time to time prescribe.



Application to become a Probationer must be made on a prescribed form, and the applicant must satisfy the Council as to his general education by furnishing evidence thereof similar to that required from an applicant for admission to the Class of Student. If the application is approved the applicant shall become a Probationer on payment of an entrance fee of 10s. 6d., but the Council may at any time withdraw their approval, and thereupon he shall cease to be a Probationer. Should a Probationer subsequently be admitted a Member of the Institute, the fee of 10s. 6d. paid by him on becoming a Probationer, will be taken as paid on account of the entrance fee as Student.

The annual subscription for Probationers is 10s. 6d., payable on admission and on 1st October in each year. If the subscription for any year be not paid before the 31st December, then the defaulter shall no longer be a Probationer.

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## Obituary.

HARRY ORRELL, Student of the Institute, Private, 20th Battalion, Manchester Regiment.

*Killed in Action 23 April 1917.*

BRIAN NEEDELL, Student of the Institute, Sergeant, 5th Battalion, City of London Regiment.

*Killed in Action 2 May 1917.*

WALTER ERNEST SMITH, Probationer of the Institute, Captain, King's Own Yorkshire Light Infantry.

*Died of Wounds 5 July 1917.*

SIDNEY GEORGE WEATHERDON, Probationer of the Institute, 2nd Lieut., 11th Battalion, Essex Regiment.

*Killed in Action 19 September 1917.*

THOMAS NOWELL ASKWITH, Student of the Institute, 2nd Lieut., Royal Field Artillery.

*Killed in Action 26 October 1917.*

GERALD DANBY DOUCET, Fellow of the Institute, 2nd Lieut., 7th Battalion, Northumberland Fusiliers.

*Killed in Action 26 October 1917.*

HUGH JERROLD HAMMOND, Student of the Institute, Captain, 12th Battalion, Royal Warwicks (attached 2nd Battalion, Gloucestershire Regiment).

*Died of Wounds 23 March 1918.*

WILLIAM ASKHAM, Probationer of the Institute, Lieut. (Acting Captain), Army Cyclist Corps.

*Killed in Action 11 April 1918.*

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# JOURNAL

OF THE

## INSTITUTE OF ACTUARIES.

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*Newton's Interpolation Formulas.* By DUNCAN C. FRASER,  
M.A., F.I.A.

FOLLOWING upon a suggestion which was made in a previous number of the *Journal* (vol. I, p. 20), an endeavour has been made in the following pages to bring together the whole of Newton's work on the subject of interpolation by means of formulas of finite differences. His contributions to this subject are comprised in three items (1) the "*Methodus Differentialis*"; (2) a letter written in 1675, giving detailed instructions for the calculation of certain tables; and (3) the celebrated Lemma No. 5 in Book III of the "*Principia*."

There are also various references to the subject in the "*Commercium Epistolicum*", a collection of letters relating to the controversy between Newton and Leibnitz as to the origin of the differential calculus.

*Methodus Differentialis*.—This is a short treatise, complete in itself, on central formulas of interpolation and their applications. It was first published in the year 1711 by William Jones in a volume in which he collected a number of Newton's shorter works under the title "*Analysis Per Quantitatum Series, Fluxiones ac Differentias: cum Enumeratio Linearum Tertii Ordinis*."

A photographic reproduction of the original Latin text,\*

\* The scale of the original has been reduced by one-fifth to admit of inclusion in the *Journal* page.

taken from a copy of the first edition in the Institute Library, and a translation are given. It is believed that no previous translation of this little work has been published.

Although the "Methodus Differentialis" was not printed until the year 1711, it was composed many years earlier. In the Latin preface (from which I translate), William Jones says :

"The book is brought to a graceful close by the addition of a little tract entitled 'Methodus Differentialis', which I have transcribed by permission of the distinguished author from his own autograph. . . . This 'Methodus Differentialis' depends upon the problem of drawing a Parabolic Curve through a given number of points, reference to which had been made by the distinguished author in his letter to Oldenburg, sent in 1676, and a solution of which he gave in Lemma 5, Book III of his 'Principia' by means of a construction which is not at all the same as that which we now present."

The letter to which William Jones refers is a letter dated 24 October 1676, which is included in the "Commercium Epistolicum." In the course of the letter Newton describes a method by which the function  $\sqrt{a^2 - ax + \frac{x^2}{2}}$  might be

expanded in a series of powers of  $x$ ; and then goes on to say (I translate freely from the original Latin), "But I attach little importance to this method because when simple series are not obtainable with sufficient ease, I have another method not yet published by which the problem is easily dealt with. It is based upon a convenient, ready and general solution of this problem, *To describe a geometrical curve which shall pass through any given points.*"

He then refers to cases in which such a problem can be solved by geometrical constructions without calculation; and adds: "but the above problem is of another kind; and although it may seem to be intractable at first sight, it is nevertheless quite the contrary; perhaps indeed it is one of the prettiest problems that I can ever hope to solve." This fixes the date of composition of the "Methodus" as prior to October 1676; and there is some reason to think that its date may be several years earlier.

A short account of the "Methodus Differentialis" has previously been given in the *Journal* (vol. xv, pp. 145 and 177),

by Professor Ludwig Oppermann, of Copenhagen. He there gives the date of publication in error as 1715. He gives his opinion that this little treatise was written many years before the Lemma, basing his view apparently on internal evidence only. The preface, which is conclusive as to the priority of the "Methodus", is not printed in Horsley's complete edition of Newton's works and may not have been seen by Professor Oppermann.

#### NOTES.

In *Proposition I* it is shown that if the ordinate corresponding to the abscissa  $A+x$  is  $a+bx+cx^2+dx^3+ex^4+\dots$  then expressions for all the divided differences can be exactly obtained. This is proved by actual division for the case when the highest power of  $x$  involved in the expression for the ordinate is the fourth.

*Proposition II.*—In the same case as in Proposition I, the values of five ordinates being known, full directions are given for the solution of the five simultaneous equations, from which can be obtained the values of the coefficients in terms of an ordinate and of divided differences of the ordinates.

In these two Propositions there is some confusion in the original text as to the first term in the expression for the ordinate. In the enunciation of Proposition I it is omitted, and I have supplied it. In the Table it is given as  $A$ , the abscissa being  $A+x$ . In the demonstrations it is not mentioned until the end of Proposition II, and an error occurs there, it being stated that the final operation in the solution of the simultaneous equation gives the first term of the abscissa  $A$ . It is quite clear that what is obtained is the first term of the ordinate, and in the translation this has been called  $a$  in correspondence with the remaining coefficients  $b, c, d$ , &c.

*Proposition III. Case I.*—The ordinates being equidistant and the number being odd a central difference formula is given in terms of the central ordinate and the central differences which are in line with it. The coefficients of the formula are

$$1, x, \frac{1}{2}x^2, \frac{x(x^2-1)}{6}, \frac{x(x^2-1)}{24}, \&c.,$$

and it will be recognized that this is the formula which is

commonly called Stirling's. The differences used in both cases of this proposition are not divided differences but simple differences without division. It will be found on examination that Newton takes his differences and measures the values of  $x$  in a sense opposite to that which is now customary. To bring the details of the work into conformity with our present practice the signs of the odd powers of  $x$  and the signs of the odd differences would have to be altered; but as it happens that the odd powers of  $x$  always occur in combination with odd differences, this makes no difference in the formula.

*Case II.*—The ordinates being equidistant and their number being even, a formula is given in terms of the mean of the two central ordinates and of the central differences opposite to that mean. The coefficients of the formula are

$$1, x, \frac{4x^2-1}{8}, \frac{x(4x^2-1)}{24}, \frac{(4x^2-1)(4x^2-9)}{384}, \&c.$$

This formula is now commonly known by the name of Bessel's formula. There is a misprint in the original,  $e_2 + e_3$  being printed for  $\frac{e_2 + e_3}{2}$ .

*Proposition IV.*—A misprint occurs in the original, where the expression for the difference  $b6$  ought to be  $\frac{A6B6 + A7B7}{A6A7}$  and not  $\frac{A6B6 - A7B7}{A6A7}$ . Newton uses the expressions  $A6$ ,  $B6$ , &c., to represent the arithmetical values of the lengths of the ordinates without reference to sign, and an example of the same practice will be found in the Lemma.

*Case I.*—An odd number of ordinates being given at points on the abscissa  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , &c., which are separated by unequal intervals, a central difference formula is given in terms of the ordinate at the central point  $\delta$  and of the central divided differences which are in line with it, the coefficients being

$$1, x - \delta, (x - \delta) \times \frac{1}{2} \left\{ \frac{(x - \gamma)}{+ (x - \epsilon)} \right\}, (x - \delta)(x - \gamma)(x - \epsilon),$$

$$(x - \delta)(x - \gamma)(x - \epsilon) \times \frac{1}{2} \left\{ \frac{(x - \beta)}{+ (x - \zeta)} \right\}, \&c.$$

*Case II.*—The number of ordinates being even and the two central ordinates being at the points  $\delta$  and  $\epsilon$ , a formula is given in terms of the mean of the two central ordinates and of the divided differences in line with that mean. The coefficients are

$$1, \frac{1}{2} \left\{ \frac{(x-\delta)}{+(x-\epsilon)} \right\}, (x-\delta)(x-\epsilon), (x-\delta)(x-\epsilon) \times \frac{1}{2} \left\{ \frac{(x-\gamma)}{+(x-\zeta)} \right\}, \\ (x-\delta)(x-\epsilon)(x-\gamma)(x-\zeta), \text{ \&c.}$$

The analogy of these two formulas for divided differences with Stirling's formula and Bessel's formula will be easily seen.

In *Proposition V* Newton points out the application of the above four formulas when it is required to find any intermediate term of a series, of which certain terms are given.

In *Proposition VI* he points out that approximate expressions for the area of a curve, of which certain ordinates are known, can be derived from the preceding formulas.

In the *Scholium* Newton gives well-known formulas for the bisection of an interval, and for finding the area, when four ordinates are known. He then goes on to describe a process by which the problem of finding the approximate area when  $2n+1$  ordinates are known can be reduced to the case of finding the area in terms of  $n+1$  ordinates. It will be found on examination that Newton's process amounts to exactly the same thing as applying the formula for  $n+1$  ordinates separately to the two halves of the curve of which  $2n+1$  ordinates are given.

The meaning of his next paragraph is not entirely clear, but Newton's idea may have been to simplify the process of finding the approximate area by taking the sums of the ordinates in two's or three's, &c., using these sums as new ordinates and passing through their extremities a new curve, the area of which, taken between suitable limits, would approximate to the area required.

*Letter on the Construction of Tables.*—The two letters here printed, the first of which is simply a letter from William Jones to Professor Cotes, enclosing a letter from Newton, dated 8 May 1675, to Mr. John Smith, are taken from a

volume published by J. Edleston, M.A., in 1850, and entitled "Correspondence of Sir Isaac Newton and Professor Cotes, including letters of other eminent men, now first published from the originals in the Library of Trinity College, Cambridge", &c. The directions for the construction of tables given by Newton are of a very practical character, and will be easily followed by anyone who wishes to examine his method in detail.

The formulas employed can readily be obtained by applying the binomial theorem and by using Stirling's interpolation formula.

The two principal formulas, namely :

$$s = \omega + \frac{1}{2}st + \frac{1}{6}m$$

$$\zeta = \frac{\omega}{10} + \frac{2st}{100} + \frac{m}{6000}$$

expressed in modern notation are as follows :

$$F(x+1) - F(x) = F'(x) + \frac{1}{2}F''(x) + \frac{1}{6}F'''(x)$$

$$F\left(x + \frac{1}{10}\right) - F(x) = \frac{1}{10}F'(x) + \frac{1}{2} \cdot \frac{1}{100}F''(x) + \frac{1}{6} \cdot \frac{1}{1000}F'''(x)$$

The last term in this second formula, added by Mr. Edleston for the sake of completeness, appears to be superfluous.

It will be noticed that they are in fact formulas of the differential calculus and can be written down at once from Taylor's theorem. But Newton's method of obtaining them was more probably that suggested above. Brook Taylor, the discoverer of Taylor's theorem, was not born until 1685, and the theorem, which he obtained as a simple corollary to Newton's descending difference formula by making the differences indefinitely small, was first published in 1715.

*Lemma No. V, Book III of the "Principia."*—This has been previously translated in an English version of the "Principia" published by Motte in 1729. The version here given is new.

The Lemma gives the well-known propositions of



interpolation by means of descending differences, for equal and for unequal intervals, which have always been regarded as laying the foundation of the calculus of finite differences. It appears from Newton's own statements that the whole of the "*Principia*" was written between December 1684 and May 1686, with the exception of 14 specified propositions among which the Lemma is not included. The date of the composition of the Lemma was therefore shortly before May 1686. It is difficult to suppose that he was not previously aware of the propositions stated in the Lemma; and it is remarkable that he had not included them in the "*Methodus*" which was composed many years before. The explanation may be that at the time he wrote the "*Methodus*" his mind was much engrossed with schemes for the calculation of extensive tables, for which the formulas of central differences were of greater practical use than formulas proceeding by descending differences. At the time he composed the Lemma, the particular point he had in view, as will be found by reference to the immediately succeeding proposition in the "*Principia*", was its application to an isolated case of interpolation.

In Newton's letter to Mr. John Smith, the notation has been reproduced without change, and it may be necessary to warn the reader that in such symbols as  $2F$  and  $F^2$  the 2 is a suffix merely. In the versions given of the "*Methodus*" and of the Lemma, suffixes have been printed in the way now usual.

I have to express my acknowledgments to Mr. Walter Stott and Mr. R. O'Donovan, both of the Royal Insurance Company, Ltd., for their valuable assistance in the translation of the "*Methodus*." For the final form of that translation and for any defects which may be found in it, and for the translation of the Lemma, &c., I must take the entire responsibility.

In a future number of the *Journal* I hope to discuss the references to the subject of these notes which are to be found in the letters of Newton included in the "*Commercium Epistolicum*."

Reference may appropriately be made here to the valuable and interesting historical notes included in a

contribution to vol. xviii of the Transactions of the Actuarial Society of America, by Mr. S. A. Joffe, under the title "Interpolation Formulæ and Central-Difference Notation", in which he traces the history of the subject from the time of Newton, and draws particular attention to the connection between Newton's general formula for unequal differences and the general interpolation formulæ of Euler and Lagrange.



# METHODUS. DIFFERENTIALIS.

## PROP. I.



*I figuræ curvilineæ Abscissa componatur ex quantitate quavis data A, & quantitate indeterminata x, & Ordinata constet ex datis quocunque quantitatibus b, c, d, e, &c. in totidem terminos hujus progressionis Geometricæ  $x, x^2, x^3, x^4, &c.$  respective ductis. & ad Abscissæ puncta totidem data erigantur Ordinatum applicatæ: dico quod Ordinarum differentia prima dividi possint per earum intervalla, & differentiarum sic divisarum*

A a

*farum differentię dividi possint per Ordinarum binarum intervalla, & harum differentiarum sic divisarum differentię dividi possint per Ordinarum ternarum intervalla, & sic deinceps in infinitum.*

Etenim si pro Abscissę parte indeterminata  $x$  ponantur quantitates quęvis datę  $p, q, r, s, t$ , &c. successive, & ad Abscissarum sic datarum terminos erigantur Ordinatę  $a, \beta, \gamma, \delta, \epsilon$ , &c. Hę Abscissę & Ordinatę & Ordinarum differentię divisę per Abscissarum differentias (quę utique sunt Ordinarum intervalla) & quotorum differentię divisę per Ordinarum alternarum differentias, & sic deinceps, exhibentur per Tabulam sequentem.

Abscissę	Ordinatę
$A+p$	$A+bp+cp^2+dp^3+ep^4=a$
$A+q$	$A+bq+cq^2+dq^3+eq^4=\beta$
$A+r$	$A+br+cr^2+dr^3+er^4=\gamma$
$A+s$	$A+bs+cs^2+ds^3+es^4=\delta$
$A+t$	$A+bt+ct^2+dt^3+et^4=\epsilon$
Divisor. Diff. Ord.	Quoti per divisionem prodeuntes.
$p-q) a-\beta$	$b+c \times \overline{p+q}+d \times \overline{pp+pq+q^2}+e \times \overline{p^3+p^2q+pq^2+q^3}=\zeta$
$q-r) \beta-\gamma$	$b+c \times \overline{q+r}+d \times \overline{qq+qr+r^2}+e \times \overline{q^3+q^2r+qr^2+r^3}=v$
$r-s) \gamma-\delta$	$b+c \times \overline{r+s}+d \times \overline{rr+rs+ss}+e \times \overline{r^3+r^2s+rs^2+s^3}=b$
$s-t) \delta-\epsilon$	$b+c \times \overline{s+t}+d \times \overline{ss+st+tt}+e \times \overline{s^3+s^2t+st^2+t^3}=v$
$p-r) \zeta-\eta$	$c+d \times \overline{p+r}+e \times \overline{pp+pq+qq+pr+qr+rr}=\lambda$
$q-s) \eta-\theta$	$c+d \times \overline{q+s}+e \times \overline{qq+qr+rr+qs+rs+ss}=\mu$
$r-t) \theta-\kappa$	$c+d \times \overline{r+s+t}+e \times \overline{rr+rs+ss+rt+st+tt}=v$
$p-s) \lambda-\mu$	$d+e \times \overline{p+q+r+s}=\xi$
$q-t) \mu-v$	$d+e \times \overline{q+r+s+t}=\pi$
$p-t) \xi-\pi$	$e=\sigma$

PROP.

## METHODUS DIFFERENTIALIS.

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## P R O P. II.

*Isdem positis, & quod numerus terminorum  $b, c, d$ ,  $e$ , &c. sit finitus, dico quod Quotorum ultimus æqualis erit ultimo terminorum  $b, c, d, e$ , &c. et quod per Quotos reliquos dabuntur termini reliqui  $b, c, d, e$ , &c. et his datis dabitur Linea Curva generis Parabolici quæ per Ordinatarum omnium terminos transibit.*

Etenim in Tabula superiore Quotus ultimus  $r$  æqualis erat termino ultimo  $e$ . Et hic terminus ductus in summam datam  $p + q + r + s$ , & ablatu de Quoto  $\xi$  relinquit terminum penultimum  $d$ . Et quantitates jam datæ  $d \times p + q + r + e \times pp + pq + qq + pr + qr + rr$ , si auferantur de Quoto  $\lambda$ , relinquunt terminorum antepenultimum  $c$ . Et quantitates jam datæ  $c \times p + q + d \times pp + pq + qq + e \times p^3 + ppq + pqq + q^3$ , si auferantur de Quoto  $\zeta$ , relinquunt terminum  $b$ . Et simili computo si plures essent termini, colligerentur omnes per Quotorum Ordines totidem. Deinde quantitates datæ  $bp + cpp + dp^3 + ep^4$ , si subducantur de Ordinata prima  $a$ , relinquunt Abscissæ terminum primum  $A$ . Et quantitas  $A + bx + cx^2 + dx^3 + ex^4 + \&c.$  est Ordinata Curvæ generis Parabolici quæ per Ordinatarum omnium datarum terminos transibit, existente Abscissâ  $A + x$ .

Ex his Propositionibus quæ sequuntur facile colligi possunt.

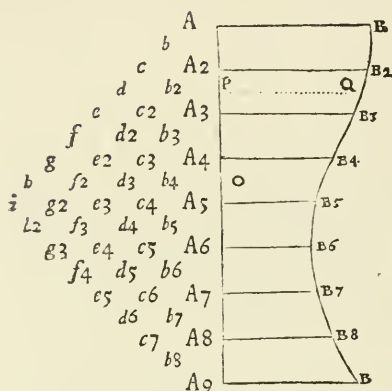
## P R O P. III.

*Si Recta aliqua  $AA_9$  in æquales quotcunque partes  $AA_2, A_2A_3, A_3A_4, A_4A_5$ , &c. dividatur, & ad puncta divisionum erigantur parallelæ  $AB, A_2B_2, A_3B_3$ , &c. Invenire curvam Geometricam generis Parabolici quæ per omnium erectarum terminos  $B, B_2, B_3$ , &c. transibit.*

Erectarum  $AB, A_2B_2, A_3B_3$ , &c. quære differentias Primas,  $b, b_2, b_3$ , &c. Secundas  $c, c_2, c_3$ , &c. Tertias  $d, d_2, d_3$ , &c. et sic deinceps usque dum veneris ad ultimam differentiam, quæ hic sit  $i$ .

Tunc.

## METHODUS DIFFERENTIALIS.



Tunc incipiendo ab ultima differentia excerpe medias differentias in alternis Columnis vel Ordinibus differentiarum, & Arithmetica media inter duas medias reliquarum, Ordine pergendo usque ad Seriem primorum terminorum AB, A<sub>2</sub>B<sub>2</sub>, A<sub>3</sub>B<sub>3</sub>, &c. sint hæc *k, l, m, n, o, p, q, r, s*, &c. quorum ultimus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas differentias; antepenultimus medium trium antepenultimarum differentiarum, & sic deinceps usque ad primum

quod erit vel medius terminorum A, A<sub>2</sub>, A<sub>3</sub>, &c. vel Arithmeticus medius inter duos medios. Prius accidit ubi numerus terminorum A, A<sub>2</sub>, A<sub>3</sub>, &c. est impar; posterius ubi par.

## C A S. I.

In Casu priori, sit A<sub>5</sub>B<sub>5</sub> iste medius terminus, hoc est, A<sub>5</sub>B<sub>5</sub> = *k*,  $\frac{b_4 + b_5}{2} = l$ ,  $c_4 = m$ ,  $\frac{d_3 + d_4}{2} = n$ ,  $e_3 = o$ ,  $\frac{f_2 + f_3}{2} = p$ ,  $g_2 = q$ ,  $\frac{h + h_2}{2} = r$ ,  $i = s$ . Et erecta Ordinatum applicata PQ, dic A<sub>5</sub>P = *x*; & duc terminos hujus Progressionis

$1 \times \frac{x}{1} \times \frac{x}{2} \times \frac{x^2 - 1}{3 \cdot 2} \times \frac{x}{4} \times \frac{x^2 - 4}{5 \cdot 2} \times \frac{x}{6} \times \frac{x^2 - 9}{7 \cdot 2} \times \frac{x}{8} \times \frac{x^2 - 16}{9 \cdot 2} \times \frac{x}{10} \times \frac{x^2 - 25}{11 \cdot 2} \times \frac{x}{12} \times \frac{x^2 - 36}{13 \cdot 2} \&c.$   
in se continuo; & orientur termini

$1. x. \frac{x^2}{2} \cdot \frac{x^3 - x}{6} \cdot \frac{x^4 - x^2}{24} \cdot \frac{x^5 - 5x^3 + 4x}{120} \cdot \frac{x^6 - 5x^4 + 4x^2}{720} \cdot \frac{x^7 - 14x^5 + 49x^3 - 36x}{5040} \cdot \&c.$

per quos si termini seriei *k, l, m, n, o, p*, &c. respective multiplicentur, aggregatum factorum  $k + xl + \frac{x^2}{2}m + \frac{x^3}{6}n + \frac{x^4 - x^2}{24}o + \frac{x^5 - 5x^3 + 4x}{120}p + \&c.$  erit longitudo Ordinatum applicatæ PQ.

## C A S. II.

In Casu posteriori, sint A<sub>4</sub>B<sub>4</sub>, A<sub>5</sub>B<sub>5</sub> duo medii termini, hoc est, sit  $\frac{A_4B_4 + A_5B_5}{2} = k$ ,  $b_4 = l$ ,  $\frac{c_3 + c_4}{2} = m$ ,  $d_3 = n$ ,  $e_2 + e_3 = o$ ,  $f_2 = p$ ,  $\frac{g + g_2}{2} = q$ , &  $h = r$ .



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&c.  $b = r$ . Et erecta Ordinatum applicata PQ, biseca  $A_4A_5$  in O, & dicto  $OP = x$ , duc Terminos hujus Progressionis

$1 \times \frac{x}{1} \times \frac{xx - \frac{1}{4}}{2x} \times \frac{x}{3} \times \frac{xx - \frac{9}{4}}{4x} \times \frac{x}{5} \times \frac{xx - \frac{25}{4}}{6x} \times \frac{x}{7} \times \frac{xx - \frac{49}{4}}{8x}$ , &c. in se continuo; et orientur termini 1.  $x$ .  $\frac{4xx-1}{8}$ .  $\frac{4x^3-x}{24}$ .  $\frac{16x^4-40x^2+9}{384}$ . &c. per quos si termini series  $k, l, m, n, o, p, q$ , &c. respective multiplicentur, aggregatum factorum  $k + xl + \frac{4x^2-1}{8}m + \frac{4x^3-x}{24}n + \frac{16x^4-40x^2+9}{384}o + \&c.$  erit Longitudo Ordinatum applicata PQ.

Sed hic notandum est quod intervalla  $AA_2, A_2A_3, A_3A_4$ , &c. hic supponantur esse unitates, & quod differentia colligi debent auferendo inferiores quantitates de superioribus,  $A_2B_2$  de  $AB$ ,  $A_3B_3$  de  $A_2B_2$ ,  $b_2$  de  $b$ , &c. et faciendo ut sint  $AB - A_2B_2 = b$ ,  $A_2B_2 - A_3B_3 = b_2$ ,  $b - b_2 = c$ , &c. adeoque quando differentia illa hoc modo prodeunt negativae signa earum mutanda sunt.

## P R O P. IV.

*Si recta aliqua in partes quocunque inaequales  $AA_2, A_2A_3, A_3A_4, A_4A_5$ , &c. dividatur, & ad puncta divisionum erigantur parallelae  $AB, A_2B_2, A_3B_3$ , &c. Invenire Curvam Geometricam generis Parabolici quae per omnium erectarum terminos  $B, B_2, B_3$ , &c. transibit.*

Sunto puncta data  $B, B_2, B_3, B_4, B_5, B_6, B_7$ , &c. et ad Abscissam quamvis  $AA_7$  demitte Ordinatam perpendiculariter  $BA, B_2A_2$ , &c.

$$\text{Et fac } \frac{AB - A_2B_2}{AA_2} = b, \frac{A_2B_2 - A_3B_3}{A_2A_3} = b_2,$$

$$\frac{A_3B_3 - A_4B_4}{A_3A_4} = b_3, \frac{A_4B_4 - A_5B_5}{A_4A_5} = b_4,$$

$$\frac{A_5B_5 - A_6B_6}{A_5A_6} = b_5, \frac{A_6B_6 - A_7B_7}{A_6A_7} = b_6,$$

$$\frac{A_7B_7 - A_8B_8}{A_7A_8} = b_7.$$

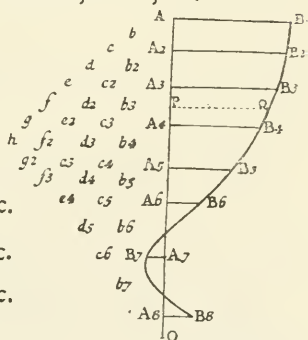
$$\text{Deinde } \frac{b - b_2}{A_2A_3} = c, \frac{b_2 - b_3}{A_2A_4} = c_2, \frac{b_3 - b_4}{A_3A_5} = c_3, \&c.$$

$$\text{Tunc } \frac{c - c_2}{A_4A_5} = d, \frac{c_2 - c_3}{A_2A_5} = d_2, \frac{c_3 - c_4}{A_3A_6} = d_3, \&c.$$

$$\text{Et } \frac{d - d_2}{A_2A_5} = e, \frac{d_2 - d_3}{A_2A_6} = e_2, \frac{d_3 - d_4}{A_3A_7} = e_3, \&c.$$

Sic pergendum est ad ultimam differentiam.

B b



Differen-

## METHODUS DIFFERENTIALIS

Differentiis sit collectis & divisis per intervalla Ordinatum applicatarum ; in alternis earum Columnis five Seriebus vel Ordinibus excerpe medias, incipiendo ab ultima, & in reliquis Columnis excerpe media Arithmetica inter duas medias, pergendo usque ad seriem primorum terminorum, AB, A<sub>2</sub>B<sub>2</sub>, &c. Sinto hæc  $k, l, m, n, o, p, q, r$ , &c. quorum ultimus terminus significet ultimam differentiam ; penultimus medium Arithmeticum inter duas penultimas ; antepenultimus mediam trium antepenultimarum, &c. Et primus  $k$  erit media Ordinatum applicata, si numerus datorum punctorum est impar ; vel medium Arithmeticum inter duas medias, si numerus eorum est par.

## C A S. I.

In Casu priori, sit A<sub>4</sub>B<sub>4</sub> ista media Ordinatum applicata, hoc est, sit A<sub>4</sub>B<sub>4</sub> =  $k$ ,  $\frac{b_3 + b_4}{2} = l$ ,  $c_3 = m$ ,  $\frac{d_2 + d_3}{2} = r$ ,  $e_2 = o$ ,  $\frac{f + f_2}{2} = p$ ,  $g = q$ . Et erecta Ordinatum applicata PQ, & in Basi AA<sub>5</sub> sumpto quovis puncto O, dic OP =  $x$ , & duc in se gradatim terminos hujus Progressionis

$$1 \times x - OA_4 \times x - \frac{OA_4 + OA_5}{2} \times \frac{x - OA_3 \times x - OA_5}{x - \frac{1}{2}OA_3 + OA_5} \times x - \frac{OA_2 + OA_6}{2} \times \&c.$$

et oriam Progressionem afferva ; vel quod perinde est duc terminos hujus Progressionis

$$1 \times x - OA_4 \times x - OA_3 \times x - OA_5 \times x - OA_2 \times x - OA_6 \times x - OA \times x - OA_7 \times \&c.$$

in se gradatim, & terminos exinde ortos duc respective in terminos hujus Progressionis

$$1. x - \frac{1}{2}OA_3 + OA_5. x - \frac{1}{2}OA_2 + OA_5. x - \frac{1}{2}OA_4 + OA_7, \&c. \text{ et orientur termini}$$

intermedii tota Progressione existente

$$1. x - OA_4. x^2 - \frac{1}{2}OA_3 + 2OA_4 + OA_5 \times x + \frac{OA_3 + OA_5}{2} \times OA_4, \&c.$$

Vel dic OA =  $\alpha$ , OA<sub>2</sub> =  $\beta$ , OA<sub>3</sub> =  $\gamma$ , OA<sub>4</sub> =  $\delta$ , OA<sub>5</sub> =  $\epsilon$ , OA<sub>6</sub> =  $\zeta$ , OA<sub>7</sub> =  $\eta$  :  $\frac{OA_3 + OA_5}{2} = \theta$ ,  $\frac{OA_2 + OA_6}{2} = \chi$ ,  $\frac{OA_4 + OA_7}{2} = \lambda$ . Et ex Progressione

$1 \times x - \delta \times x - \gamma \times x - \epsilon \times x - \beta \times x - \zeta \times x - \alpha \times x - \eta$  &c. collige terminos quibus multiplicatis per  $1. x - \theta$ ,  $x - \chi$ ,  $x - \lambda$ , &c collige alios terminos intermedios, tota serie prodeunte

$1, x - \delta, x^2 - \delta + \theta x + \delta \theta, x^3 - \delta + 2\theta x^2 + \gamma \epsilon + 2\delta \theta x - \gamma \delta \epsilon$ , &c. per cujus terminos multiplica series  $k, l, m, n, o$ , &c. Et aggregatum productorum  $k + x - \delta \times l + x^2 - \delta + \theta x + \delta \theta \times m + \&c.$  erit longitudo Ordinatum applicatæ PQ.

C A S.

## METHODUS DIFFERENTIALIS.

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## C A S. II.

In Casu posteriori, sint  $A_4B_4$ ,  $A_5B_5$  duæ mediæ Ordinatum applicatæ, hoc est,  $\frac{A_4B_4 + A_5B_5}{2} = k$ ,  $b_4 = l$ ,  $\frac{c_3 + c_4}{2} = m$ ,  $d_3 = n$ ,  $\frac{e_2 + e_3}{2} = o$ ,  $f_2 = p$ , &c. Et alternorum  $k$ ,  $m$ ,  $o$ ,  $q$ , &c. Coefficientes orientur ex multiplicatione terminorum hujus Progressionis in se

$1 \times x - OA_4 \times x - OA_5 \times x - OA_3 \times x - OA_6 \times x - OA_2 \times x - OA_7 \times x - OA_1 \times x - OA_8$  &c. Et reliquorum Coefficientes ex multiplicatione horum per terminos hujus Progressionis

$$x - \frac{+OA_4 + OA_5}{2}, x - \frac{+OA_3 + OA_6}{2}, x - \frac{+OA_2 + OA_7}{2}, x - \frac{+OA_1 + OA_8}{2}, \&c.$$

Hoc est, erit  $k + x - \frac{+OA_4 + OA_5}{2} \times l + x^2 - OA_4 \times OA_5 \times m$ , &c. Ordinatum applicata PQ,

$$\text{vel } PQ = k + \frac{x \times l}{-\frac{1}{2}OA_4 - \frac{1}{2}OA_5} + \frac{x \times x}{-OA_4} + \frac{x \times m}{-OA_5} + \frac{x \times x}{-OA_4} + \frac{x \times x}{-OA_5} + \frac{x \times n}{-\frac{1}{2}OA_3 - \frac{1}{2}OA_6} \&c.$$

$$\text{Sive dic } x - \frac{+OA_4 + OA_5}{2} = \pi, \quad x - OA_4 \times x - OA_5 = \rho,$$

$$\rho \times x - \frac{+OA_3 + OA_6}{2} = \sigma, \quad \rho \times x - OA_3 \times x - OA_6 = \tau,$$

$$\tau \times x - \frac{+OA_2 + OA_7}{2} = \upsilon, \quad \tau \times x - OA_2 \times x - OA_7 = \phi,$$

$$\phi \times x - \frac{+OA_1 + OA_8}{2} = \chi, \quad \phi \times x - OA_1 \times x - OA_8 = \psi,$$

$$\text{Et erit } k + \pi l + \rho m + \sigma n + \tau o + \upsilon p + \phi q + \chi r + \psi s = PQ.$$

## P R O P. V.

*Datis aliquot terminis seriei cujuscunque ad data intervalla dispositis, invenire terminum quemvis intermedium quamproxime.*

Ad rectam positione datam erigantur termini dati in dato angulo, interpositis datis intervallis, & per eorum puncta extrema, per Propositiones præcedentes, ducatur linea Curva generis Parabolici. Hæc enim continget terminos omnes intermedios per feriem totam.

PROP.

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## METHODUS DIFFERENTIALIS.

## P R O P. VI.

*Figuram quamcunque Curvilineam quadrare quamproxime, cujus Ordinatæ aliquot inveniri possunt.*

Per terminos Ordinarum ducatur linea Curva generis Parabolici ope Propositionum præcedentium. Hæc enim figuram terminabit quæ semper quadrari potest, et cujus Area æquabitur Area figuræ propositæ quamproxime.

## S C H O L I U M.

Utiles sunt hæ Propositiones ad Tabulas construendas per interpolationem Serierum, ut & ad solutiones Problematum quæ a quadraturis Curvarum dependent, præsertim si Ordinarum intervalla & parva sint & æqualia inter se, & Regulæ computentur, & in usum reserventur pro dato quocunque numero Ordinarum. Ut si quatuor sint Ordinatæ ad æqualia intervalla sitæ, sit A summa primæ & quartæ, B summa secundæ & tertiæ, & R intervallum inter primam & quartam, & Ordinata nova in medio omnium erit  $\frac{9B-A}{15}$ , & Area tota inter primam & quartam erit  $\frac{A+3B}{8} R$ .

Et nota quod ubi Ordinatæ stant ad æquales ab invicem distantias, sumendo summas Ordinarum quæ ab Ordinata media hinc inde æqualiter distant, & duplum Ordinatæ mediæ, componitur Curva nova cujus Area per pauciores Ordinatas determinatur, & æqualis est Area Curvæ prioris quam invenire oportuit. Quinetiam si pro Ordinatis novis sumantur summa Ordinatæ primæ & secundæ, et summa tertiæ & quartæ, et summa quintæ & sextæ, & sic deinceps; vel si sumantur summa trium primarum Ordinarum, & summa trium proximarum, & summa trium quæ sunt deinceps; vel si sumantur summæ quaternarum Ordinarum, vel summæ quinarum: Area Curvæ novæ æqualis erit Area Curvæ primo propositæ. Et sic habitis Curvæ quadrandæ Ordinatis quocunque quadratura ejus ad quadraturam Curvæ alterius per pauciores Ordinatas reducetur.

Per

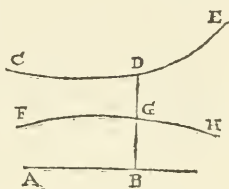
## METHODUS DIFFERENTIALIS.

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Per data vero puncta quotcunque non solum Curvæ lineæ generis *Parabolici*, sed etiam Curvæ aliæ innumeræ diverforum generum duci possunt.

Sunto CDE, FGH Curvæ duæ Abscissam habentes communem AB, et Ordinatas in eadem recta jacentes BD, BG, & relatio inter has Ordinatas definiatur per æquationem quamcunque. Dentur puncta quotcunque per quæ Curva CDE transire debet, & per æquationem illam dabuntur puncta totidem nova per quæ Curva FGH transibit. Per Propositiones superiores describatur Curva

FGH generis *Parabolici* quæ per puncta illa omnia nova transeat, & per æquationem eandem dabitur Curva CDE quæ per puncta omnia primo data transibit,



F I N I S.



## I.

## METHODUS DIFFERENTIALIS.

(Translation.)

## PROP. I.

If the abscissa of a curve consist of a given quantity  $A$  and an indeterminate quantity  $x$ , and if the ordinate consist of any number of quantities  $b, c, d, e$ , &c., multiplied respectively into a corresponding number of terms of the G. P.  $x, x^2, x^3, x^4$ , &c., and if ordinates be erected at as many points of the abscissa; then the first differences of the ordinates are divisible by their intervals; and the differences of the differences so divided are divisible by the intervals between alternate ordinates; and the differences of these differences so divided are divisible by the intervals between every third ordinate, and so on indefinitely.

Thus if given quantities  $p, q, r, s, t$ , &c., be substituted in succession for the indeterminate portion of the abscissa, and ordinates  $a, \beta, \gamma, \delta, \epsilon$ , &c., be erected at the extremities of the abscissæ so determined; the abscissæ and the ordinates, and the differences of ordinates divided by the differences of abscissæ (which are in fact the intervals of the ordinates), and the differences of the quotients divided by the differences of alternate ordinates, and so on, are shewn in the following table. (See p. 86.)

## PROP. II.

Making the same suppositions and assuming the number of terms  $b, c, d, e$ , &c., to be finite, the last quotient will be equal to the last of the terms  $b, c, d, e$ , &c., and the remaining terms will be found by means of the remaining quotients; and, when these terms are known, a parabolic curve is determined which passes through the extremities of all the ordinates.

Thus, in the preceding table, the last quotient  $\sigma$  is equal to the last term  $e$ ; and the product of this term by the known sum  $p+q+r+s$  when subtracted from the quotient  $\xi$  leaves as remainder  $d$ , the last term but one. The quantities  $d(p+q+r) + e(p^2+pq+q^2+pr+qr+r^2)$  which are then known, being deducted from the quotient  $\lambda$ , give the term  $c$ . The quantities  $c(p+q) + d(p^2+pq+q^2) + e(p^3+p^2q+pq^2+q^3)$ , which are then known, being deducted from the quotient  $\zeta$ , leave the term  $b$ . By a similar calculation, other terms, if any, would be obtainable by means of a corresponding series of quotients. Finally, the ascertained quantities  $bp+cp^2+dp^3+ep^4$ , when deducted from the first ordinate  $a$ , leave the first term  $a$  of the expression for the ordinate. And the quantity  $a+bx+cx^2+dx^3+ex^4$ , &c., is the ordinate of a parabolic curve which passes through the extremities of all the given ordinates, the abscissa being  $A+x$ .

From these propositions those which follow are easily inferred.



## PROP. III.

If a straight line  $A_1A_9$  be divided into any number of equal parts  $A_1A_2, A_2A_3, A_3A_4, A_4A_5, \&c.$ , and if parallel straight lines  $A_1B_1, A_2B_2, A_3B_3, \&c.$ , be erected at the points of division, it is required to find a parabolic curve which shall pass through the extremities of all these lines.

(See figure on p. 88).

Find the first differences  $b_1, b_2, b_3, \&c.$ , of the ordinates  $A_1B_1, A_2B_2, A_3B_3, \&c.$ ; the second differences  $c_1, c_2, c_3, \&c.$ ; the third differences  $d_1, d_2, d_3, \&c.$ ; and so on, up to the last difference and let that difference be called  $i$ .

Then beginning at the last difference, take the central differences in the alternate columns or orders of differences, and the arithmetic means between the two central differences in the remaining columns, proceeding in order up to the series of primary terms  $A_1B_1, A_2B_2, A_3B_3, A_4B_4, \&c.$  Call the terms extracted  $k, l, m, n, o, p, q, r, s, \&c.$ ; the last of these symbols representing the last difference; the last but one, the mean between the two differences in the last column but one, the last but two, the central difference of the three differences in the last column but two, and so on to the first of the symbols, which will represent either the central ordinate of the series  $A_1B_1, A_2B_2, A_3B_3, \&c.$ , or the mean between the two central ordinates, the former happening when the number of the ordinates is odd, and the latter when the number is even.

## CASE I.

In the former case let  $A_5B_5$  be the central ordinate, that is put

$$A_5B_5 = k, \quad \frac{b_4 + b_5}{2} = l, \quad c_4 = m, \quad \frac{d_3 + d_4}{2} = n, \\ e_3 = o, \quad \frac{f_2 + f_3}{2} = p, \quad g_2 = q, \quad \frac{b_1 + b_2}{2} = r, \quad i = s.$$

Erect the ordinate PQ and let  $AP = x$ . Now multiply continuously into one another the terms of the progression,

$$1, x, \frac{x}{2}, \frac{x^2 - 1}{3x}, \frac{x}{4}, \frac{x^2 - 4}{5x}, \frac{x}{6}, \frac{x^2 - 9}{7x}, \frac{x}{8}, \frac{x^2 - 16}{9x}, \frac{x}{10}, \frac{x^2 - 25}{11x}, \frac{x}{12}, \frac{x^2 - 36}{13x},$$

&c., the resulting terms being

$$1, x, \frac{x^2}{2}, \frac{x^3 - x}{6}, \frac{x^4 - x^2}{24}, \frac{x^5 - 5x^3 + 4x}{120}, \frac{x^6 - 5x^4 + 4x^2}{720}, \\ \frac{x^7 - 14x^5 + 49x^3 - 36x}{5040}, \&c.$$

Then if these terms be respectively multiplied into the terms of the series  $k, l, m, n, o, p$ , &c., the sum of the products, namely

$$k + x \cdot l + \frac{x^2}{2} \cdot m + \frac{x^3 - x}{6} \cdot n + \frac{x^4 - x^2}{24} \cdot o + \frac{x^5 - 5x^3 + 4x}{120} \cdot p + \&c.$$

will be the length of the ordinate PQ.

#### CASE II.

In the latter case let  $A_4B_4$  and  $A_5B_5$  be the two central ordinates, that is, put

$$\frac{A_4B_4 + A_5B_5}{2} = k, \quad b_4 = l, \quad \frac{c_3 + c_4}{2} = m, \quad d_3 = n,$$

$$\frac{e_2 + e_3}{2} = o, \quad f_2 = p, \quad \frac{g_1 + g_2}{2} = q, \quad \& \quad h = r.$$

Erecting the ordinate PQ take the middle point O of  $A_4A_5$ , and call  $OP = x$ .

Now multiply continuously into one another the terms of the progression

$$1, x, \frac{x^2 - 1}{2x}, \frac{x}{3}, \frac{x^2 - 9}{4x}, \frac{x}{5}, \frac{x^2 - 25}{6x}, \frac{x}{7}, \frac{x^2 - 49}{8x}, \&c.$$

the resulting terms being

$$1, x, \frac{4x^2 - 1}{8}, \frac{4x^3 - x}{24}, \frac{16x^4 - 40x^2 + 9}{384}, \&c.$$

Then if these terms be respectively multiplied into the terms of the series  $k, l, m, n, o, p, q$ , &c., the sum of the products, namely,

$$k + x \cdot l + \frac{4x^2 - 1}{8} \cdot m + \frac{4x^3 - x}{24} \cdot n + \frac{16x^4 - 40x^2 + 9}{384} \cdot o + \&c.$$

will be the length of the ordinate PQ.

It is to be noted that each of the intervals  $A_1A_2, A_2A_3, A_3A_4$ , &c., is here assumed to be unity; also that the differences are to be obtained by deducting the lower quantities from the upper,  $A_2B_2$  from  $A_1B_1$ ,  $A_3B_3$  from  $A_2B_2$ ,  $b_2$  from  $b_1$ , &c., so that  $A_1B_1 - A_2B_2 = b_1$ ,  $A_2B_2 - A_3B_3 = b_2$ ,  $b_1 - b_2 = c_1$ , &c., and further that when any of the differences taken in this way turn out to be negative, effect must be given to the negative signs.

## PROP. IV.

If a straight line be divided into any number of unequal parts  $A_1A_2$ ,  $A_2A_3$ ,  $A_3A_4$ ,  $A_4A_5$ , &c., and if parallel straight lines  $A_1B_1$ ,  $A_2B_2$ ,  $A_3B_3$ , &c., be erected at the points of division; it is required to find a parabolic curve which shall pass through the extremities of all the lines so erected.

(See figure on p. 89.)

Let the given points be  $B_1$ ,  $B_2$ ,  $B_3$ ,  $B_4$ ,  $B_5$ ,  $B_6$ ,  $B_7$ , &c., and let fall ordinates  $B_1A_1$ ,  $B_2A_2$ , &c., perpendicularly on the abscissa  $A_1A_7$ . Put

$$\frac{A_1B_1 - A_2B_2}{A_1A_2} = b_1, \quad \frac{A_2B_2 - A_3B_3}{A_2A_3} = b_2, \quad \frac{A_3B_3 - A_4B_4}{A_3A_4} = b_3,$$

$$\frac{A_4B_4 - A_5B_5}{A_4A_5} = b_4, \quad \frac{A_5B_5 - A_6B_6}{A_5A_6} = b_5, \quad \frac{A_6B_6 + A_7B_7}{A_6A_7} = b_6, \quad \frac{-A_7B_7 - A_8B_8}{A_7A_8} = b_7.$$

Thence derive

$$\frac{b_1 - b_2}{A_1A_3} = c_1, \quad \frac{b_2 - b_3}{A_2A_4} = c_2, \quad \frac{b_3 - b_4}{A_3A_5} = c_3, \quad \&c. ;$$

and then

$$\frac{c_1 - c_2}{A_1A_4} = d_1, \quad \frac{c_2 - c_3}{A_2A_5} = d_2, \quad \frac{c_3 - c_4}{A_3A_6} = d_3, \quad \&c. ;$$

and

$$\frac{d_1 - d_2}{A_1A_5} = e_1, \quad \frac{d_2 - d_3}{A_2A_6} = e_2, \quad \frac{d_3 - d_4}{A_3A_7} = e_3, \quad \&c. ;$$

the process being continued in the same way until the last difference is reached.

After the differences have been collected and divided by the intervals between the ordinates, the next step is to pick out the central terms in the alternate columns (or series, or lines), reckoning from the last difference, and the arithmetic means between the two central terms in the remaining columns, right up to the series of primary terms  $A_1B_1$ ,  $A_2B_2$ , &c. Let the terms extracted be  $k$ ,  $l$ ,  $m$ ,  $n$ ,  $o$ ,  $p$ ,  $q$ ,  $r$ , &c., of which the last symbol denotes the last difference; the last but one, the mean between the two differences in the last column but one; the last but two, the central difference of the three differences in the last column but two, &c. Then the first symbol  $k$  will represent the central ordinate if the number of ordinates is odd, or the arithmetic mean between the two central ordinates if their number is even.

## CASE I.

In the former case let  $A_4B_4$  be the central ordinate, that is, put  $A_4B_4 = k$ ,  $\frac{b_3 + b_4}{2} = l$ ,  $c_3 = m$ ,  $\frac{d_2 + d_3}{2} = n$ ,  $e_2 = o$ ,  $\frac{f_1 + f_2}{2} = p$ ,  $g = q$ . Having erected the ordinate PQ, take a fixed point O in the base  $A_1A_5$  and call  $OP = x$ . Then multiply into one another in succession the terms of the progression,

$$1, \quad x - OA_4, \quad x - \frac{OA_3 + OA_5}{2}, \quad \frac{(x - OA_3)(x - OA_5)}{x - \frac{1}{2}(OA_3 + OA_5)},$$

$$x - \frac{OA_2 + OA_6}{2}, \quad \&c.,$$

and take the resulting progression.

Or, what comes to the same thing, multiply into one another in succession the terms of the progression,

$$1, \quad x - OA_4, \quad (x - OA_3)(x - OA_5), \quad (x - OA_2)(x - OA_6),$$

$$(x - OA_1)(x - OA_7), \quad \&c.$$

Then multiply the resulting terms respectively into the terms of the progression,

$$1, \quad x - \frac{OA_3 + OA_5}{2}, \quad x - \frac{OA_2 + OA_6}{2}, \quad x - \frac{OA_1 + OA_7}{2}, \quad \&c.,$$

and intermediate terms will be obtained, the complete progression being

$$1, \quad x - OA_4, \quad x^2 - \frac{OA_3 + 2OA_4 + OA_5}{2}x + \frac{OA_3 + OA_5}{2} \times OA_4, \quad \&c.$$

Otherwise, let  $OA_1 = a$ ,  $OA_2 = \beta$ ,  $OA_3 = \gamma$ ,  $OA_4 = \delta$ ,  $OA_5 = \epsilon$ ,

$$OA_6 = \zeta, \quad OA_7 = \eta, \quad \frac{OA_3 + OA_5}{2} = \theta, \quad \frac{OA_2 + OA_6}{2} = \chi, \quad \frac{OA_1 + OA_7}{2} = \lambda.$$

Obtain terms by continuous multiplication from the progression

$$1, \quad x - \delta, \quad x - \gamma, \quad x - \epsilon, \quad x - \beta, \quad x - \zeta, \quad x - a, \quad x - \eta, \quad \&c.,$$

and obtain intermediate terms by multiplying the results respectively by

$$1, \quad x - \theta, \quad x - \chi, \quad x - \lambda, \quad \&c.,$$

the whole series being

$$1, \quad x - \delta, \quad x^2 - (\delta + \theta)x + \delta\theta, \quad x^3 - (\delta + 2\theta)x^2 + (\gamma\epsilon + 2\delta\theta)x - \gamma\delta\epsilon, \quad \&c.,$$

the terms of which are to be multiplied respectively by  $k, l, m, n, o, p$ , &c. Then the sum of the products, namely

$$k + (x - \delta).l + \{x^2 - (\delta + \theta)x + \delta\theta\}.m + \&c.$$

will be the length of the ordinate PQ.

### CASE II.

In the latter case let  $A_4B_4, A_5B_5$ , be the two middle ordinates, that is, put

$$\frac{A_4B_4 + A_5B_5}{2} = k, \quad b_4 = l, \quad \frac{c_3 + c_4}{2} = m, \quad d_3 = n, \quad \frac{e_2 + e_3}{2} = o, \quad f_2 = p, \quad \&c.$$

The co-efficients of the alternate terms are obtained by the continuous multiplication of the terms of the progression,

$$1, (x - OA_4)(x - OA_5), (x - OA_3)(x - OA_6), (x - OA_2)(x - OA_7), \\ (x - OA_1)(x - OA_8), \&c..$$

and those of the remaining terms are obtained by multiplying the above co-efficients by the terms of the progression,

$$x - \frac{OA_4 + OA_5}{2}, \quad x - \frac{OA_3 + OA_6}{2}, \quad x - \frac{OA_2 + OA_7}{2}, \quad x - \frac{OA_1 + OA_8}{2}.$$

Then

$$k + \left(x - \frac{OA_4 + OA_5}{2}\right).l + \{x^2 - (OA_4 + OA_5).x + OA_4 \times OA_5\}.m + \&c.,$$

will be the ordinate PQ ;

$$\text{or,} \quad PQ = k + \left(x - \frac{1}{2}OA_4 - \frac{1}{2}OA_5\right).l + (x - OA_4)(x - OA_5).m \\ + (x - OA_4)(x - OA_5)\left(x - \frac{1}{2}OA_5 - \frac{1}{2}OA_6\right).n, \&c.$$

$$\text{Thus putting} \quad x - \frac{OA_4 + OA_5}{2} = \pi, \quad (x - OA_4)(x - OA_5) = \rho,$$

$$\rho \cdot \left(x - \frac{OA_3 + OA_6}{2}\right) = \sigma, \quad \rho \cdot (x - OA_3)(x - OA_6) = \tau,$$

$$\tau \cdot \left(x - \frac{OA_2 + OA_7}{2}\right) = v, \quad \tau \cdot (x - OA_2)(x - OA_7) = \phi,$$

$$\phi \cdot \left(x - \frac{OA_1 + OA_8}{2}\right) = \chi, \quad \phi \cdot (x - OA_1)(x - OA_8) = \psi.$$

the equation to the curve will be

$$k + \pi.l + \rho.m + \sigma.n + \tau.o + v.p + \phi.q + \chi.r + \psi.s = PQ.$$

## PROP. V.

*Certain terms out of a sequence of values being given, arranged at known intervals, it is required to find any intermediate term as closely as possible.*

On a fixed straight line erect at a constant angle the given terms arranged at the given intervals; and let a parabolic curve be drawn through their extremities by means of the preceding propositions. This curve will pass through the extremities of all the intermediate terms.

## PROP. VI.

*To find the approximate area of any curve a number of whose ordinates can be ascertained.*

Let a parabolic curve be drawn through the extremities of the ordinates by means of the preceding propositions. This will form the boundary of a figure whose area can always be ascertained, and its area will be approximately equal to the area required.

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 SCHOLIUM.

These propositions are useful for the construction of tables by the interpolation of series, as also for the solution of problems which depend on finding the areas of curves, especially if the intervals between the ordinates are small and equal to one another; and rules applicable to any given number of ordinates can be derived and recorded for reference. For example: If there are four ordinates at equal intervals, let A be the sum of the first and fourth, B the sum of the second and third, and R the interval between the first and fourth; then the central ordinate will be  $\frac{9B - A}{16}$ , and the area between the first and fourth ordinates will be  $\frac{A + 3B}{8} R$ .

Note also that when the ordinates stand at equal distances from one another, the sums of ordinates which are equally distant from the central ordinate, along with twice the central ordinate, supply data for a new curve the area of which is determined by means of a smaller number of ordinates and is equal to the area of the original curve. Moreover, if the sum of the first and second ordinates, the sum of the third and fourth, the sum of the fifth and sixth, and so on in succession are taken as new ordinates; or if the sum of the first three ordinates, the sum of the next three, and the sum of the succeeding three are taken; or if the sums of the ordinates are taken four at a time, or five at a time, the area of the new curve will be equal to that of the original curve. And in this way, when a number of ordinates are given of a curve whose area is required, the calculation of the area is reduced to that of the area of another curve by means of a smaller number of ordinates.



Through a given number of points not only parabolic curves but an infinity of other curves of different kinds can be drawn.

(See figure on page 93.)

Let CDE, FGH, be two curves having a common abscissa AB, and ordinates BD, BG, lying in the same straight line ; and let the relation between the ordinates be defined by any equation whatever. Let any number of points be given through which the curve CDE is required to pass, and by that equation an equal number of points will be given through which the curve FGH will pass. By means of the foregoing propositions let a parabolic curve FGH be described passing through those points, and by the same equation a curve CDE will be given which will pass through all the points first given.

## II.

### LETTER ON CONSTRUCTION OF TABLES.

#### LETTER CIX.

*W. Jones to Prof. Cotes.*

LONDON, Jan. 1st, 17 <sup>$\frac{11}{12}$</sup> .

Dr. Sr.

¶ I have sent you here inclos'd, the Copy of a Letter, that I found among Mr. Collins's papers, from Sr. Is. Newton to one Mr. Smith ; the contents thereof seem to have, in some measure, relation to what you are about, as being the application of the Doctrine of Differences to the making of Tables ; and for that reason I thought it might be of use to you, so far as to see what has bin done already : I shew'd this to Sr. Isaac, he remembers y<sup>t</sup>. he apply'd it to all sorts of Tables, but has nothing by him more than what is printed : I have more papers of Mr. Mercator's and others, upon this subject, tho, I think, none so material, as this. I shou'd be very glad to see what you have done of this kind all publish'd. And I must confess, that, unless you design a considerable large Volume, 'twere much better to put them into the Transactions ; for that wou'd sufficiently preserve them from being lost, which is y<sup>e</sup>. common fate of small single Traets ; and at y<sup>e</sup>. same time save the trouble and expense of printing them, since the subject is too curious to expect any profit by it : and besides, now, as the R. Society having done themselves the honour of choosing you a Member, something from you cannot but be acceptable to them : Sr. Isaac himself expects those things of yours that I formerly mentioned to him as your promise.

I am, Sr. your much oblig'd

friend, & humble Serv<sup>t</sup>.

W. JONES.

LETTER CIX. (*bis*).

Newton to J. Smith.

[Enclosed in Letter CIX.]

[COPY.]

TRIN. COLL. CAMBRIDGE, May 8th, 1675.

Sr.

I have consider'd y<sup>e</sup>. buisness of computing Tables of Square, Cube, & Sq. Sq<sup>r</sup>. Roots; and y<sup>e</sup>. best way of p<sup>r</sup>forming it, y<sup>t</sup>. I can think of is y<sup>t</sup>. which follows:

If y<sup>u</sup>. w<sup>o</sup>d compute a Table to 8 decimal places, let y<sup>e</sup>. roots of every hundredth number be extracted to ten decimal places, and then compute every tenth numb<sup>r</sup>. and afterwards every number by the following methods:

TAB. I.				TAB. II.			
$n-50$	$*o$	$a$		$n-6$	4E		
	$m$	$o$			5e		
$n-40$	$op$	$\beta$		$n-5$	5E. F5		
	$m$	$\pi$			$\zeta^4$		
$n-30$	$pq$	$\gamma$		$n-4$	F4		
	$m$	$\chi$			$\zeta^3$		
$n-20$	$qr$	$\delta$		$n-3$	F3		
	$m$	$\rho$			$\zeta^2$		
$n-10$	$rs$	$\epsilon$		$n-2$	F2		
	$m$	$\sigma$	$\frac{m}{10}$		$\zeta^1$		
$n$	$st$	$\zeta$		$n-1$	F1		
	$m$	$\tau$			$\zeta$		
$n+10$	$tr$	$\eta$		$n$	F	$\frac{st}{100}$	
	$m$	$v$			1 $\zeta$		
$n+20$	$rx$	$\theta$		$n+1$	1F		
	$m$	$\phi$			2 $\zeta$		
$n+30$	$xy$	$\iota$		$n+2$	2F		
	$m$	$\psi$			3 $\zeta$		
$n+40$	$yz$	$\kappa$		$n+3$	3F		
	$m$	$\omega$			4 $\zeta$		
$n+50$	$z^*$	$\lambda$		$n+4$	4F		
					5 $\zeta$		
				$n+5$	5F. G5		
					$\eta^4$		
				$n+6$	G4		
					$\eta^3$		
				$n+7$	G3		
					$\eta^2$		
				$n+8$	G2		
					$\eta^1$		
				$n+9$	G1		
					$\eta$		
				$n+10$	G	$\frac{tr}{100}$	
					1 $\eta$		
				$n+11$	1G		
					2 $\eta$		
				$n+12$	2G		

*In the First Table.*

Let  $n$  signify every 100<sup>th</sup> numb<sup>r</sup>. &  $F$  its root, whether Square, Cube, or Sq. Square; and  $n - 50$ ,  $n - 40$ ,  $n - 30$ , &c., every tenth numb<sup>r</sup>; and  $A$ ,  $B$ ,  $C$ ,  $D$ , &c., their roots; and  $o$ ,  $p$ ,  $q$ ,  $r$ , &c., the differences of these roots: and  $op$ ,  $pq$ ,  $qr$ , &c., their second differences, (that is  $op$ , the diff. of  $o$  &  $p$ ,  $pq$  the diff. of  $p$  &  $q$ , &c.) and  $m$  their third difference, that is y<sup>e</sup>. common difference of  $^*o$  &  $op$ ,  $op$  &  $pq$ ,  $pq$  &  $qr$ , &c.

Further, let  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , &c., signify y<sup>e</sup>. the differences of these Roots from those next less, namely,  $\alpha$  the difference of y<sup>e</sup>. root of  $n - 50$  & y<sup>e</sup>. like root of  $n - 51$ ,  $\beta$ , the diff. of y<sup>e</sup>. roots  $n - 40$  &  $n - 41$ ,  $\zeta$  the diff. of y<sup>e</sup>. roots of  $n$  &  $n - 1$ ,  $\eta$  the diff. of y<sup>e</sup>. roots of  $n + 10$  &  $n + 9$ , &c. And let  $o$ ,  $\pi$ ,  $\chi$ ,  $\rho$ , &c., signify the diff. of  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , &c. And  $\frac{m}{10}$  the common diff. of  $o$ ,  $\pi$ ,  $\chi$ ,  $\rho$ , &c.

*In the Second Table.*

Let  $n - 6$ ,  $n - 5$ ,  $n - 4$ ,  $n - 3$  &c. signify y<sup>e</sup>. single numbers,

4E, 5E, or F5, F4, F3 &c. their roots,

5e,  $\zeta 4$ ,  $\zeta 3$ ,  $\zeta 2$  &c. the diff. of those roots;

$\frac{st}{100}$  the common diff. of those differences for y<sup>e</sup>. ten numbers

between  $n - 5$  &  $n + 5$ .

And so for y<sup>e</sup>. ten numbers between  $n + 5$  &  $n + 15$ ; let G5, G4, G3, &c. signify y<sup>e</sup>. roots;  $\eta 4$ ,  $\eta 3$ ,  $\eta 2$  &c. their first differences, and  $\frac{tr}{100}$  their second differences; and the like for every denarie between  $n - 50$  &  $n + 50$ .

This explication of the Tables being p'mis'd, you may compute them thus;

$$\text{Out of } n, \left\{ \begin{array}{l} \text{Square} \\ \text{Cube} \\ \text{Sq. Sq.} \end{array} \right\} \text{Root, make } \left\{ \begin{array}{l} \frac{10F}{2n} = \omega, \quad \frac{10\omega}{2n} = st, \quad \frac{30st}{2n} = m \\ \frac{10F}{3n} = \omega, \quad \frac{20\omega}{3n} = st, \quad \frac{50st}{3n} = m \\ \frac{10F}{4n} = \omega, \quad \frac{30\omega}{4n} = st, \quad \frac{70st}{4n} = m \end{array} \right.$$

$$\omega + \frac{1}{2}st + \frac{1}{6}m = s, \quad \frac{\omega}{10} + \frac{2}{100}st + \frac{1}{6000}m = \zeta, \quad \text{and } \frac{st}{10} + \frac{55m}{1000} = \sigma$$

And these quantities  $F$ ,  $st$ ,  $m$ ,  $s$ ,  $\zeta$ , &  $\sigma$ , being thus found, y<sup>e</sup>. rest are given by Additn. & Subduct.

† *Note by Editor (J. EDLESTON).*—I have added the  $\frac{m}{6000}$ . I have also corrected some other errors of transcription.

*Note by D. C. F.*—The quantity  $\frac{m}{6000}$  is so small that Newton properly omitted it.

For  $st + m = rs$ ,  $rs + m = qr$ , &c.  $st + m = tv$ ,  $tv + m = vx$ , &c.

Again  $s + rs = r$ ,  $r + rq = q$ , &c.  $s - st = t$ ,  $t - tv = v$ , &c.

And  $F - s = E$ ,  $E - r = D$ , &c.  $F + t = G$ ,  $G + v = H$ , &c.

Further

$$\sigma + \frac{m}{10} = \rho, \rho + \frac{m}{10} = \chi, \text{ \&c. } \sigma - \frac{m}{10} = \tau, \tau - \frac{m}{10} = v, \text{ \&c. }$$

Lastly  $\zeta + \sigma = \epsilon$ ,  $\epsilon + \rho = \delta$ , &c.  $\zeta - \tau = \eta$ ,  $\eta - v = \theta$ , &c.

These quantities being thus computed in the first Table, to every 10<sup>th</sup> number, the roots may be computed in y<sup>e</sup>. 2<sup>d</sup> Table to every numb<sup>r</sup>. by Addition and Subduction only ;

$$\text{For } \zeta + \frac{st}{100} = \zeta 1, \zeta 1 + \frac{st}{100} = \zeta 2, \text{ \&c. }$$

$$\zeta - \frac{st}{100} = 1\zeta, 1\zeta - \frac{st}{100} = 2\zeta, \text{ \&c. }$$

$$\text{Again } F - \zeta = F1, F1 - \zeta 1 = F2, \text{ \&c. }$$

$$F + 1\zeta = 1F, 1F + 2\zeta = 2F, \text{ \&c. }$$

Thus you must proceed to five Figures on either hand, and then do the like in the next ten Figures, saying

$$\eta + \frac{tv}{100} = \eta 1, \eta 1 + \frac{tv}{100} = \eta 2, \text{ \&c. }$$

And the like for every Denarie between  $n - 50$  &  $n + 50$ .

In these Computations, Note, 1<sup>st</sup>.—That they must be done everywhere to 10 or 11 decimal places, if you will have a Table of Roots exact to 8 of these places.

2<sup>dly</sup>.—If 5F & G5, the roots of  $n + 5$  found two ways agree to 8 decimal places, it argues the whole works from which they were derived to be true. And so of y<sup>e</sup>. roots of  $n + 15$ ,  $n + 25$ ,  $n - 5$ , &c. And also of y<sup>e</sup>. Terms A, \* $\phi$ , &  $a$ ; L,  $z^*$ , &  $\lambda$  where two works meet. Let this therefore be y<sup>e</sup>. Proof of y<sup>e</sup>. work.

This Sr. is w<sup>t</sup>. has occurred to me about your design, which I hope will do your business, the whole work being p<sup>r</sup>form'd by Addit. & Subduct: excepting y<sup>t</sup> in y<sup>e</sup>. computation of every 100<sup>th</sup> number, there is required y<sup>e</sup>. Extraction of one root, & three divisions to find F,  $\omega$ ,  $st$ , &  $m$ .

Sr. I am,

Your humble Serv<sup>t</sup>.

IS. NEWTON.

[Note appended by Editor, J. EDLESTON.]

The person to whom this letter is written may be conjectured to be "John Smith, Philo-Accomptant", author of *Stereometrie*, Lond. 1673. (He must not

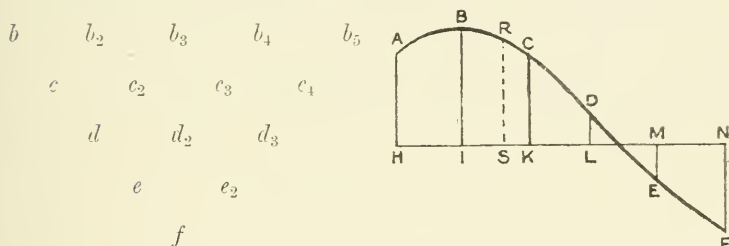
be confounded with Cotes's uncle). In the Macclesfield Correspondence, II, 370-374, there are two other letters on the extraction of roots from Newton to this same person (not to Collins, as there printed) dated July 24 and Aug. 27, 1675, in the former of which he refers to the method given in the foregoing letter. Mr. J. Smith seems to have had a design of constructing Tables of Square, Cube and Biquadr. Roots, and consulted Newton as to the best mode of computing them. The Tables, if ever made, do not appear to have been published. The earliest Tables of Roots are Briggs's MS. Tables of the Square Roots of Numbers up to 1000 mentioned in Mayne's Merchant's Companion (London, 1674), p. 80.

## III.

## PRINCIPIA. BOOK III. LEMMA V.

(Translation.)

To find a parabolic curve which shall pass through any given points :



Let the points be A, B, C, D, E, F, &c., and from them let fall perpendiculars AH, BI, CK, DL, EM, FN, on any given straight line HN.

*Case I.*—If the intervals HI, IK, KL, &c., between the points H, I, K, L, M, N, are equal, find the first differences  $b, b_2, b_3, b_4, b_5$ , &c., of the perpendiculars; the second differences  $c, c_2, c_3, c_4$ , &c.; the third differences  $d, d_2, d_3$ , &c., so that  $AH - BI = b$ ,  $BI - CK = b_2$ ,  $CK - DL = b_3$ ,  $DL + EM = b_4$ ,  $-EM + FM = b_5$ , &c.;  $b - b_2 = c$ , &c.; and so on, to the last difference, which in this case is  $f$ . Then having erected any perpendicular RS, which shall be an ordinate to the required curve, in order to find its length put each of the intervals HI, IK, KL, LM, equal to unity and let  $AH = a$ ,

$$-HS = p, \quad \frac{1}{2}p \times (-IS) = q, \quad \frac{1}{3}q \times SK = r, \quad \frac{1}{4}r \times SL = s, \quad \frac{1}{5}s \times SM = t,$$

proceeding in the same way to the last but one of the perpendiculars ME, and prefixing the negative sign to the terms HS, IS, &c., which lie on the same side of the point S as A, and the positive signs to the terms SK, SL, &c., which lie on the other side of the point S. Then, observing the correct signs, RS will be  $= a + bp + cq + dr + es + ft$ , &c.

*Case II.*—But if the intervals HI, IK, &c., between the points H, I, K, L are unequal, find  $b, b_2, b_3, b_4, b_5$ , being the first differences of the perpendiculars AH, BI, CK, &c., divided by the

intervals between the perpendiculars;  $c, c_2, c_3, c_4$ , &c., the second differences divided by the intervals between the alternate perpendiculars;  $d, d_2, d_3$ , &c., the third differences divided by the intervals between every third perpendicular;  $e, e_2$ , &c., the fourth differences divided by the intervals between every fourth perpendicular, and so on; so that  $b = \frac{AH - BI}{HI}$ ,  $b_2 = \frac{BI - CK}{IK}$ ,  $b_3 = \frac{CK - DL}{KL}$ , &c.;  $c = \frac{b - b_2}{HK}$ ,  $c_2 = \frac{b_2 - b}{IL}$ ,  $c_3 = \frac{b_3 - b_4}{KM}$ , &c.;  $d = \frac{c - c_2}{HL}$ ,  $d_2 = \frac{c_2 - c_3}{IM}$ , &c. The differences having been ascertained, put  $AH = a$ ,  $-HS = p$ ,  $p \times (-IS) = q$ ,  $q \times SK = r$ ,  $r \times SL = s$ ,  $s \times SM = t$ ; proceeding, it will be understood, to the last perpendicular but one, ME; then the ordinate RS will be  $= a + bp + cq + dr + es + ft$ , &c.

*Corollary.*—Hence the areas of all curves can be ascertained approximately. For if several points are found of the curve whose area is required, and a parabolic curve be supposed drawn through these points, its area will be approximately the same as that of the given curve. But the area of the parabolic curve can always be found geometrically by methods that are very well known.

*Notes on some recent developments of Pension Problems in America.* By W. J. H. WHITTALL, F.I.A., F.A.S.

INTRODUCTORY.

SOME interesting reports have come to hand recently from America showing that pension questions, and more particularly the proper methods of financial provision, are receiving a great deal of attention throughout the United States and Canada. A thorough exploration of these sources of information by actuarial enquirers is clearly desirable in the public interest, especially in view of the enormous accretions that are being made to the pension liabilities of this country as a consequence of the war, both directly on the military side and indirectly through the great increase in the numbers of the bureaucracy and the rising scales of all salaries and wages.

For instance, the Prime Minister on a recent Saturday afternoon promised the Metropolitan Police, who were on strike, an increase of pensionable pay of 13s. a week. The additional liability for pensions alone thus placed on the taxpayer must represent a present capital value of some



millions of pounds; but no information has been given on this point, nor is it known even whether a calculation of the cost was made before the liability was incurred. A still more disturbing incident is the issue of an official statement by the President of the Board of Education (Cd. 9141), that he proposes, obviously with the consent of the Treasury, to introduce a Bill which "will bring within one comprehensive system of State pensions, on a non-contributory basis, the certificated teachers, the uncertificated teachers, and the teachers of special subjects in elementary schools, and the teachers in all other schools aided by the Board of Education, including those training colleges which are not departments of universities." The scale of benefits will closely resemble that of the Civil Service Superannuation Act of 1909, and, speaking generally, all existing staffs, male and female, are to be included in respect of past service. As I understand the statement, the contributory scheme for all secondary schools recommended by a departmental committee (Cd. 7365), to be effected by means of insurance policies, will now be superseded. As the total salary list in the aided secondary schools and training colleges is over £2,500,000, I should not be surprised if in their case the total present value of Mr. Fisher's proposed benefits amounted to as much as £8,000,000 or £10,000,000 for the existing staffs. As regards the elementary schools, I have not salary statistics at hand, but Sir George Hardy and Mr. D. C. Fraser found over 100,000 teachers in service in 1912, and an estimated present salary list of £15,000,000 would be not unreasonable. The present value of the pensions in their case might thus amount to £50,000,000 or more. But this would be for existing staffs only. If the service be regarded as a going concern, and the cost be reckoned of replacing outgoing members, the present value would be largely increased. In the case of some of the State services in America, it has been found that this consideration has more than doubled the actuarial estimates of the present capital cost. I only mention these rough guesses to show what large sums are in question, and the urgent need of proper information.

It will probably be found that a great deal is taking place in America which has a distinct bearing on the numerous schemes now being introduced or contemplated in this country. By way of a beginning, I propose to give a short account of

the Carnegie Foundation for the Advancement of Teaching ; and to supplement this later by drawing attention to a few of the recent developments in the field of municipal and state pensions.

THE CARNEGIE FOUNDATION FOR THE ADVANCEMENT  
OF TEACHING.

In 1905 Mr. Andrew Carnegie set aside £2,000,000 in 5 per-cent Steel Bonds (the dollar being reckoned at 4s.) to benefit the teachers engaged in higher education in the United States and Canada, and the "Carnegie Foundation for the Advancement of Teaching" was established. Its Twelfth Report, written by Dr. H. S. Pritchett, the President, and issued from 576, Fifth Avenue, New York City, contains a good deal of information apart from their own experience. Except where otherwise stated, my facts are taken from this Report. For the following historical summary of the proceedings of the Foundation I am partly indebted to a memorandum prepared by Mr. James Robb, the Secretary of the Carnegie Universities Trust in Edinburgh.

The revenue of £100,000, in the words of the Certificate of Incorporation, was mainly "to provide retiring pensions, " without regard to race, sex, creed, or colour, for the teachers " of universities, colleges, and technical schools in the United " States, the Dominion of Canada, and Newfoundland, who, " by reason of long and meritorious service in these " institutions, shall be deemed by the Board of Directors to " be entitled to the assistance and aid of this Corporation, or " who, by reason of old age or disability, may be prevented " from continuing in the active work of their profession ; to " provide for the care and maintenance of the widows and " families of the said teachers. . . ." Mr. Carnegie, in his letter to the trustees, of date 16 April 1905, gave his reasons for the Foundation. He had reached the conclusion that the least rewarded of all the professions was that of the teacher in the higher educational institutions of America, that able men hesitated to adopt teaching as a career, and many old professors, whose places should be occupied by younger men, could not be retired, and that expert calculation showed the revenue would be ample for the purpose he had in view.

There were to be no contractual relations and no legal promises as to future pensions. At the same time, in order

to preserve the self-respect of the recipients, it was considered necessary to avoid any eleemosynary taint, and the pensions were to be regarded as fairly earned by the past service. After mature deliberation and consultation with actuaries, administrators, teachers, and publicists, the trustees adopted a system of retiring allowances which was non-contributory and stipendiary in character, namely, one-half of the average salary for the last five years plus £80; they admitted a selected list of institutions to the permanent privileges of the pension endowment, believing that it was better to establish a fair retiring allowance system in a limited number of colleges than a very poor system in a large number; they decided to award these pensions at the minimum age of 65, and to give a smaller pension after 25 years of service as a professor, on the ground of disability; and they extended the privileges of the Foundation to instructors and to the widows of teachers.

The experience acquired in the working of this system for a few years banished some delusions, such as the expectation on the part of many teachers that they would be able to retire in the fifties and spend the rest of their lives in literary art or scientific research, the anticipation of college presidents that inefficient men could be disposed of by a pension, as well as the expectation of the trustees that scientific research could be stimulated by a judicious use of retiring allowances. It was also soon found out that the revenue, which expert calculation estimated would be ample for the purpose, was to be quite insufficient to meet the demands made upon it by eligible applicants. It had been stated in the First Annual Report that the average pension paid by the Trust was then £290, and was expected to be less in future. At the close of 1912, six years after the Trust was initiated, the general average had increased to fully £335, and was likely to increase further. It was also estimated that the pension system could be maintained at a cost of somewhere between 7 and 10 per-cent of the salary roll, and that an income of £100,000 would maintain such a system for a group of between 3,000 and 4,000 teachers. At 30 September 1912 the amount distributed annually had already grown to £114,085 for 429 retiring allowances, and 90 widows' pensions in all, and the cost then amounted in general to about 4 per-cent of the salary roll of the accepted institutions. Finally, on 1 April

1917 there were 6,593 teachers, including professors and instructors, in the associated colleges and universities (now 73 in number), and at 30 June, when the general endowment of the Foundation stood at almost £2,900,000 the total number of allowances in force was 480, and the cost £147,872. As regards the future, the estimate now is that for nearly the whole of the coming fifty years the load upon the Foundation, arising from the continuation of the present rule, even if limited to existing staffs, would exceed the present income of the Foundation. It was assumed that the income for the purposes of pension payments would be approximately £150,000 per annum, and it was calculated that in forty-five years there would be a deficit of some millions. Thus the work, begun with the best intentions, and with the utmost desire to serve the teaching body, resulted in conclusions which were quite different from the assumptions upon which it started.

Clearly the whole problem of pensions had to be faced by the Trust, and a new basis found if possible. Very careful consideration was given to the problem in all its bearings from time to time. On 15 April 1916 Dr. H. S. Pritchett, the President, by direction of the trustees, issued for general distribution his pamphlet entitled, "A Comprehensive Plan of Insurance and Annuities for College Teachers." This plan claimed to be based upon the results of a study of the personal statements of some 4,000 college teachers who had furnished full information as to their experience in the effort to protect their families against the risks incident to their calling. The conclusion of the enquiry was that a contributory system in which both teacher and college joined, and which was so constructed that it would not restrict migration from one college to another, was the only system of retiring annuities which was at once socially wise, economically sound, and permanently secure.

Criticism of the pamphlet and the advice of all interested—teachers, actuaries, and publicists—were invited. Voluminous correspondence followed, and it was then resolved that the plan be referred to a Joint Commission consisting of six trustees of the Foundation, two representatives of the American Association of University Professors, and one representative each from the Association of American Universities, the National Association of State Universities,

and the Association of American Colleges, assisted by an actuary. This body of eleven individuals was thus representative of the leading educational authorities of America. On 27 April 1917 it agreed upon a comprehensive plan of insurance and annuities which was recommended to the trustees, and to a description of which I will return presently.

The difficulties in which the Carnegie Foundation had found itself involved are very much those which would have been foreseen by competent actuarial opinion. Lessons derived from actual experience, however, are often more readily received than any results of *a priori* reasoning, however competent. It may, therefore, be worth while to quote at length, and in Dr. Pritchett's own words, the conclusions arrived at by the Trustees of the Foundation, as the result of their 12 years' experience of "hit or miss" methods of granting pensions.

The past year has been a notable one in the history of the Carnegie Foundation. It has marked the culmination of some twelve years of study and investigation.

When the Carnegie Foundation was established, the founder and those associated with him had no thought of any other form of pension system than that of a teacher's pension paid entirely from an endowment provided for that purpose, and without the participation of the teacher. The concern of the trustees and of the founder was so to introduce such a pension to the colleges and teachers of the United States, Canada, and Newfoundland, that it might not seem a gratuity—that it might come as a right, not as a favour . . . . .

The experience of twelve years and the examination of the pension problem in all countries has led the trustees to no different conclusion as to the need for making any pension a matter of right, not a matter of favour. The exhaustive study made in these intervening years has shown clearly that no system of free pensions can be devised which will not in the end affect the teacher's pay. The experience of the world has also now accepted the economic truth that the members of any group in the body politic receive their best service at the hands of society when the machinery is provided by which they may attain therein economic independence, rather than have the risk of dependence lifted from them by free gift. It is also clear that the opportunity to protect themselves against dependence should be open to the great body of teachers, not to a selected minority.

Under the pressure of these conclusions the officers and trustees of the Carnegie Foundation were faced with two duties: first, to carry out fairly and to the best of their ability the obligations assumed in the associated institutions; and secondly, to establish



as quickly as possible a system of benefits open to the great body of teachers in the three English-speaking countries of North America, upon terms which should be permanent, economically sound, and within the reach of the teacher and of his college . . . . The second question—namely, what are the sound bases for the protection of the lives of teachers against dependence?—has been difficult to deal with, but the conclusions to which all authorities have come have, on the whole, been more generally accepted. The result of the twelve years' work of the Foundation has been epoch-making in this respect. The studies brought together from all sources have demonstrated that the problem of insurance and the problem of the annuity cannot be financially separated. This study has made clear the fact that a contributory system of annuities is the only one which society can permanently support, and under which the teacher shall be sure of his protection. Finally, these studies have made clear that the establishment of a co-operation for the common protection of college teachers in the United States, Canada, and Newfoundland, would mean in a generation a security for the profession of the teacher, and a solidarity of feeling which nothing else can bring about.

While the work of these years has been difficult, while questions have necessarily been raised which brought criticism and alarm, the results of this work have nevertheless been the greatest contribution to the profession of the teacher which the Carnegie Foundation could possibly accomplish. Nothing will count so much for the future as to bring about a form of co-operation under which each man who enters the teacher's profession may be able to enter into a contract whereby he and his family shall be protected against dependence, and at a cost within his ability to pay. It is to be regretted that this knowledge was not available when the Foundation began. The one thing which such an agency can do is to proceed sincerely and courageously, once the information is in its hands. This is what the Carnegie Foundation has undertaken to do. It is encouraging to find from the widespread correspondence which comes to the Foundation, that the great body of teachers have realized the effort which has been made, have sympathized with the difficulties to be overcome, and have accepted in the best spirit the results to which the studies have led.

The past year of the Foundation—marking as it does the acceptance of the fundamental principles worked out in the ten years past, committing the Foundation definitely to the principles of the contributory pension, to a plan of insurance and annuities which mutually support each other, and to an organization open to the great body of college teachers of the whole Continent—is the combination of twelve years of work, begun with the best intentions and with the highest desires to serve the teaching body, but resting in the end upon conclusions quite different from the assumptions upon which the work started. In this day of world trial, the Carnegie Foundation has sought to deal sincerely both with its obligations and its mistakes.



It will now be interesting to revert to the detailed conclusions of the Joint Commission to which reference has been made. In the first instance, it had tables submitted by actuaries showing that the probable future "load" of the pensions exceeded the income in prospect. The following extract is made from these statements as illustrating a point discussed a good deal in this country in reference to State pensions, namely, the impossibility of accepting the relation of the non-effective votes to the effective votes—that is, of current pensions to current salaries—as any guide to future commitments where the numbers in service are increasing rapidly. It was found that the Foundation had cognizance of 2,898 members of existing male staffs between the ages of 35 and 64. If their age distribution had been that of a normal population (McClintock's Table) it would have been as set out in column 2. Their actual age distribution was found to be as set out in column 3.

Age period (1)	Estimated Distribution of a Stationary Population (2)	Actual Distribution (3)
35-39	570	904
40-44	540	715
45-49	500	550
50-54	473	359
55-59	430	211
60-64	376	159
Total	2,898	2,898

It is, of course, obvious that the numbers approaching age 65 at the present time are no guide to the number of future pensioners.

After consideration of this question, the Joint Commission frankly agreed that the obligations of the Foundation need not be regarded as extending in any case beyond existing teachers; and while hoping that future and further subsidies from extraneous sources might enable it to bear the heavy but limited future load as regards existing teachers, it recognized that the ever-increasing burden which obviously would accrue with future appointments could not be borne without contributions. The Joint Commission thereupon, with typical American thoroughness and effort to get down to

logical foundations, voted a series of fundamental principles for a sound pension system which may be given in full.

## I.

1. The function of a pension system is to secure to the individual who participates in it protection against the risk of dependence due to old age or to disability.

2. The obligation to secure this protection for himself and for his family rests first upon the individual. This is one of the primary obligations of the existing social order. Society has done its best for the individual when it provides the machinery by which he may obtain this protection at a cost within his reasonable ability to pay.

3. Men either on salary or on wages, are, in the economic sense, employees. The employer, whether a government, a corporation, or an individual, has a direct financial interest in the establishment of some pension system which shall enable old or disabled employees to retire under satisfactory conditions. In addition, society demands to-day that the employer assume some part in the moral and social betterment of his employees. The obligation of the employer to co-operate in sustaining a pension system is primarily a financial one, and in the second place, a moral one.

4. A pension system designed for any group of industrial or vocational workers should rest upon the co-operation of employee and employer.

5. Teachers' pensions should be stipendiary in character, amounting to a fair proportion of the active pay.

## II.

1. In actuarial terms a pension is a deferred annuity upon the life of one or more individuals, payable upon the fulfilment of certain conditions.

2. In order that an individual participating in a pension system may be assured of his annuity when due, one condition is indispensable: There must be set aside, year by year, the reserve necessary, with its accumulated interest, to provide the annuity at the age agreed upon. On no other conditions can the participator obtain a satisfactory contract. The man of thirty who participates in a pension plan under which he expects an annuity thirty-five or forty years in the future will take some risk of disappointment in accepting any arrangement less secured than a contractual one.

3. A pension system conducted upon the actuarial basis of setting aside, year by year, the necessary reserve is the only pension system whose cost can be accurately estimated in advance.

4. A method by which a pension is paid for in advance in annual or monthly instalments is the most practical plan which can be devised for purchasing a deferred annuity, provided that the contributions begin early in the employee's career, and provided also that the contributions paid in year by year receive the benefit of the current interest for safe investments.

5. As a matter of practical administration, a pension system should apply to a group whose members live under comparable financial and economic conditions. To attain its full purpose, participation in the pension system to the extent of an agreed minimum should form a condition of entering the service or employment the members of which are co-operating in the pension system.

As to the machinery by which these principles could be put in operation, the Commission recommended that a new Teachers' Insurance Association should be formed under the laws of the State of New York. The joint contributions of employer and employed would be paid to this Association in exchange for:

- (1) *Term Insurances*, for family provision, to last until age 65, and of "flexible" amounts to suit varying circumstances, and
- (2) *Deferred Annuities*, to begin at age 65, with return of premiums and interest in the event of previous death or retirement.

It was proposed that the annuities should be receivable by monthly payments and according to one of four alternative methods arranged to suit varying needs. These are (i) an ordinary annuity for the life of the grantee; (ii) the same, with continuation to his executor until the total annuity payments shall equal the contributions, or rather the purchase price as accumulated up to age 65; (iii) an annuity for the life of the grantee, with continuation of half the amount to his widow, if any; and (iv) the same, with special provision as in method (ii).

It was to be a provision of the Charter of Incorporation that the business should be conducted without profit to the stockholders, and that the Association should issue only non-participating policies; but it is understood that this would not prevent the ultimate division of surplus profits, if any, among insuring members. The Carnegie Foundation proposes to find the share capital and an initial reserve fund, to pay all expenses incurred beyond the free interest income, and to guarantee certain rates of interest on surrendered annuity policies. Among some concluding practical suggestions made by the Joint Commission were (i) that contributions should be made jointly by the institution and the teacher, with liberty to vary the proportion, (ii) that while the deferred annuities should be compulsory, the insurances should be optional, and

(iii) that on migration to an institution within the scheme or likely to be so the benefit of the policies should enure to the member.

These suggestions follow very closely the practice of the Federated Universities System in this country, though in one respect they fall short of it. Under the latter system the whole benefit of past contributions enures to the member upon withdrawal arising from any cause, whereas the Carnegie Scheme apparently contemplates that some option may be reserved to the employing institutions in respect of its own past contributions in the event of withdrawal from teaching altogether. I will postpone what I have to say on the general principle here involved, which appears to me to lie at the root of all pension problems and to have received hitherto quite insufficient attention.

In view of the difficulty of dealing with disablement pensions in an insurance scheme, as has been found in the working of the Federated Universities System and similar schemes in this country, the following conclusions of the Joint Commission on this matter may be quoted.

The basis of these pensions or the definition of disability may vary all the way from "total and permanent" disability, as understood by insurance companies that have a disability clause in their policies, to a much more liberal definition under which a man is regarded as disabled when he is physically or mentally disqualified to continue the profession of teaching. The Commission thinks that in devising a permanent pension system the first thing to be aimed at is sufficiency of funds; and the definition of disability should be made with due regard to the funds available for the purpose of paying disability pensions. It is the sense of the Commission that as liberal definition as seems to be consistent with the funds available for this purpose is desirable.

The risk of disablement, the Joint Commission add, will not be dealt with by the proposed new Insurance Association, but will be provided for by the Carnegie Foundation. I have not found in the report any definite statement that this proposal has been accepted by the Trustees, but it seems clear that the idea is adumbrated with their consent and is under consideration.

Viewing the financial position of the Carnegie Foundation as it exists to-day, apparently a complete actuarial valuation has not been made, but it is estimated that its outgoings on

the basis of present commitments for the next 45 years will be £13,800,000. The similar estimate of its prospective income is £6,800,000. There is thus, on a 45 years' budget, a prospective deficit of £7,000,000 as the result of granting or promising pensions during the last 12 years. It is thus not surprising that the Trustees of the Foundation should have acted on the conclusions of the Joint Commission and, as mentioned in the extract already quoted, decided (i) to promote, for the use of all members of the profession to be appointed in the future and of all institutions not yet associated with the Foundation, a new pension scheme based on contributions in advance by employers and employed; and (ii) to concentrate their own financial efforts mainly on the problem of their existing commitments. In this task they are exceptionally fortunate in being able to fall back upon another great foundation, with ample funds, the Carnegie Corporation of New York. The latter has agreed to furnish the Carnegie Foundation for Teaching with (i) for the capital and initial fund of the new Insurance Association, £200,000; (ii) for existing teachers still outside, £200,000; and (iii) for creation of Reserve towards its own accruing liabilities a present sum of £2,200,000, and an annual contribution of £120,000. These munificent contributions may not, of course, supply the whole deficiency, and they are subject to the condition that the Trustees will revise their rules and, by contributions from the younger men, or in some other way, ensure a future actuarial balance. In calculating its position over the next 45 years, the Foundation appears to have reckoned as an asset only the value of its future assured income. The *corpus* of its funds, now nearly £3,000,000, is apparently to be preserved intact if possible, so that it will still occupy a strong financial position. There thus appears to be a fair prospect that it will be able, as its own heavy but limited liabilities are gradually reduced, to assume some responsibility for providing disablement pensions in accordance with the principles laid down by the Joint Commission, namely, that the contingency of disablement is best provided for apart from superannuation, and that the definition of disablement may properly vary to some extent with the funds available.

*(To be continued.)*

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*On a method of Approximate Valuation.* By ALFRED HENRY,  
F.I.A.

IN times like the present, when the saving of clerical labour is a matter of some importance, it may be of interest to consider a method of approximate valuation which gives results very close to the true values and at the same time shortens the mechanical work very materially.

Briefly, the method aims at substituting for the numerous multiplications involved in the process of valuation, simple summations of the data which can be readily and rapidly performed on an adding machine.

If the valuation factor (*i.e.*, single premium, annuity, &c.) be denoted by  $F(x)$  and  $u_x$  be the amount to be valued at age  $x$ , then  $\sum u_x F(x)$  gives the total liabilities or assets as the case may be.

If now  $F(x)$  can be expressed as  $a + bx + cx^2$ , then the above total becomes  $a\sum u_x + b\sum xu_x + c\sum x^2 u_x$ . Clearly in this form, the valuation can be expressed in terms of the continuous summation of the  $u_x$  column from the bottom upwards, for :

$$\sum_{r=0}^{r=n} u_r = \sum_{r=0}^{r=n} u_r (r+1)$$

and 
$$\sum_{r=0}^{r=n} 3u_r = \sum_{r=0}^{r=n} u_r \frac{(r+1)(r+2)}{2}$$

Therefore 
$$\sum_{r=0}^{r=n} r^2 u_r = 2\sum_{r=0}^{r=n} 3u_r - 3\sum_{r=0}^{r=n} u_r + \sum_{r=0}^{r=n} u_r$$

and 
$$\sum_{r=0}^{r=n} r u_r = \sum_{r=0}^{r=n} u_r - \sum_{r=0}^{r=n} u_r$$

Accordingly the valuation formula can be expressed as

$$2c\sum u_r + (b-3c)\sum u_r + (a-b+c)\sum u_r$$

It follows that, given the values of the constants  $a$ ,  $b$  and  $c$  for any particular table, the valuation can be made from successive summations of the  $u_r$  column.

In the examples that follow, the values of the constants have been found for old-established offices by weighting the equations with the numbers at each fifth age as obtained from



the appropriate model office. Actually the values can, of course, be found from any three ages, as three equations only are necessary to find three unknowns. It was, however, thought better to use weighted equations for all ages, afterwards combining these equations in three groups in order to give the three equations necessary for the finding of the required values.

This method gives the best results, because the final result is accurate *either* if the distribution is the same as that of the model office *or* if the fit is perfect. The resulting deviations are the product of the misfit and the deviation from the normal, and this product must tend to be small. Thus if  $u'$  be the actual numbers and  $u$  be the normal numbers, and  $F, F'$  be the true and the approximate values of the function, so that  $u' = u + \delta u$  and  $F' = F + \delta F$

we want  $(u + \delta u)F$

and we get  $(u + \delta u)(F + \delta F)$

The error is  $(u + \delta u)\delta F = \delta u \cdot \delta F$  since  $u\delta F$  is made zero in the fitting.

In order to test the accuracy of this assumption, a curve was fitted to the H<sup>M</sup> 3 per-cent annuity table, weighting the values by the numbers living according to Mr. King's model office of 50 years of age. The values so obtained were employed to make a valuation of the model office of 10 years of age, with the following result :

True Valuation £656,445, Approximate Valuation £655,356. The error, on this severe test, is £1,089, or one-sixth of one per-cent.

The following table is of interest, as showing the closeness of the approximation at individual ages :

*Values of  $a_x$ .*

Age $x$	True Value	Approximate Value	Age $x$	True Value	Approximate Value
22	21·656	22·370	57	11·353	11·401
27	20·582	20·899	62	9·498	9·707
32	19·373	19·395	67	7·712	7·981
37	18·037	17·860	72	5·975	6·223
42	16·566	16·293	77	4·512	4·433
47	14·923	14·694	82	3·290	2·612
52	13·188	13·064	87	2·393	·758

Except at the extreme ages (where the errors are in opposite directions) the difference between the two values is nowhere more than .27. As has already been pointed out, these errors tend to balance and, in fact, the method has been found to give very good results, even in cases where the form of the valuation function did not lend itself readily to be fitted by a parabolic curve.

For endowment assurances valued by the Z method, the mean valuation age corresponding to each unexpired term has been taken from Dr. Buchanan's model office. Although the adoption of such ages introduces a further source of error, it is interesting to note that, in practice, a balance of error is secured.

In the following examples the data have been taken at random from the Board of Trade returns, with the figures cut down so as to conceal their identity. Comparison is made in each case with the valuation results as published. The data are shown in the appendix; these are summed continuously twice from the bottom upwards, and a complete cast of the final column gives the value of the third summation. In each case the test is a severe one, as re-assurances are not deducted.

*Example 1.*—Office A. Whole life. With profits. Valuation of sums assured and bonuses.  $O^M$  3 per-cent.

The origin is taken at age 15 and therefore any policies existing at younger ages must be treated as age 15 and summations made up to and including that age.

$$a = .242378 \quad b = .00880808 \quad c = .0000127587$$

$$\Sigma = 18584 \quad \Sigma^2 = 678541 \quad \Sigma^3 = 14280130$$

$$2c\Sigma^3 = 364.3$$

$$(b - 3c)\Sigma^2 = 5950.7$$

$$(a - b + c)\Sigma = 4340.9$$

Value of sum assured and				
bonuses	...	...	...	10655.9
				<u>True value = 10646</u>

*Example 2.*—Office A. Whole life. With profits. Valuation of tabular premiums.  $O^M$  3 per-cent.

The limited payment premiums must, of course, be valued separately.

*Premiums payable throughout life.*

$$a = 25.511892 \quad b = -0.302413 \quad c = -0.000438052$$

$$\Sigma = 462.8 \quad \Sigma^2 = 17797.7 \quad \Sigma^3 = 388074.4$$

$$2c\Sigma^3 = -340.0$$

$$(b-3c)\Sigma^2 = -5358.9$$

$$(a-b+c)\Sigma = 11946.7$$

$$\text{Value of premiums} \quad \dots \quad \dots \quad \underline{\underline{6247.8}}$$

*Limited payment premiums.*

The origin was taken at unexpired term 0; the value of  $a$  is therefore zero. As the summations are taken, not to the origin, but to the next higher term, the formula is slightly different.\*

$$a = 0 \quad b = 0.8918174 \quad c = -0.01007234$$

$$\Sigma = 66.6 \quad \Sigma^2 = 725.6 \quad \Sigma^3 = 5551.0$$

$$2c\Sigma^3 = -111.8$$

$$(b-c)\Sigma^2 = 654.4$$

$$a\Sigma = 0.0$$

$$\text{Value of limited payment premiums} \quad 542.6$$

$$\text{Value of "whole life" premiums} \quad 6247.8$$

(as above)

$$\text{Total value of premiums} \dots \dots \underline{\underline{6790.4}} \quad \text{True value} = 6802$$

*Example 3.*—Office B. Endowment assurances with profits. Valuation of sums assured and bonuses.  $O^M$  3 per-cent.

The policies being grouped according to year of maturity, the origin is taken as the valuation year. The first group of policies is, therefore, at unit distance from the origin, and the formula to be used is the same as that used in valuing the limited payment premiums.

$$a = 0.99227 \quad b = -0.0243313 \quad c = 0.0002376118$$

$$\Sigma = 23141 \quad \Sigma^2 = 446969 \quad \Sigma^3 = 5327002$$

$$2c\Sigma^3 = 2531.5$$

$$(b-c)\Sigma^2 = -10981.5$$

$$a\Sigma = 22962.1$$

$$\text{Value of sum assured and}$$

$$\text{bonuses} \dots \dots \dots \underline{\underline{14512.1}} \quad \text{True value} = 14499$$

\* Alternatively if 0 be inserted as the value at the origin and the summations be taken up to and including the zero value, the original formula can be used without modification.

*Example 4.*—Office B. Endowment assurances with profits. Valuation of tabular premiums.  $O^M$  3 per-cent.

$$a = -\cdot 2295125 \quad b = \cdot 8351065 \quad c = -\cdot 00815726$$

$$\Sigma = 926\cdot 7 \quad \Sigma^2 = 16451\cdot 4 \quad \Sigma^3 = 184579\cdot 3$$

$$2c\Sigma^3 = - 3011\cdot 3$$

$$(b-c)\Sigma^2 = 13872\cdot 9$$

$$a\Sigma = - 212\cdot 7$$

Value of tabular premiums	<u>10648·9</u>	True value=10677
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*Example 5.*—Office C. Valuation of annuities on female lives.  $O^{[a]}$  3 per-cent.

The distribution of annuity business is such that little weight is given to ages under 40; the fit of the approximate curve is therefore not so good for these ages, and it is best to treat all annuitants under age 40 as being of that age.

The origin is taken at age 40 and summations are therefore taken up to and including that age.

$$a = 19\cdot 544538 \quad b = -\cdot 3703151 \quad c = \cdot 00065768$$

$$\Sigma = 13108 \quad \Sigma^2 = 370693 \quad \Sigma^3 = 6303147$$

$$2c\Sigma^3 = 8290\cdot 9$$

$$(b-3c)\Sigma^2 = -138004\cdot 6$$

$$(a-b+c)\Sigma = 261052\cdot 5$$

Value of annuities ...	...	<u>131338·8</u>	True value=130946
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N.B.—The difference is almost entirely due to the one large contract at age 78.

*Example 6.*—Friendly Society D. Value of benefits.

$$a = 32\cdot 14096 \quad b = \cdot 259632 \quad c = -\cdot 01020274$$

$$\Sigma = 1000 \quad \Sigma^2 = 20100 \quad \Sigma^3 = 297367$$

$$2c\Sigma^3 = - 6068$$

$$(b-3c)\Sigma^2 = 5834$$

$$(a-b+c)\Sigma = 31871$$

Value of benefits ...	...	<u>31637</u>	True value=31636
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*Example 7.*—Friendly Society E. Value of benefits.

$$a = 32.14096 \quad b = .259632 \quad c = -.01020274$$

$$\Sigma = 100 \quad \Sigma^2 = 2.704 \quad \Sigma^3 = 44597$$

$$2c\Sigma^3 = - 910$$

$$(b - 3c)\Sigma^2 = 785$$

$$(a - b + c)\Sigma = 3187$$

Value of benefits	...	...	3062	True value = 3062
			<u><u>3062</u></u>	

The above examples could be multiplied indefinitely, but they are sufficient to show that the method yields results which are sufficiently accurate for practical purposes. In particular, for the valuation of such items as sums assured, bonuses, gross premiums or annuities payable, where the data do not vary with the basis of valuation, the results are much superior to those yielded by the use of a model office. If the constants are available for several mortality tables at different rates of interest, then a single set of summations of the data will enable valuations on several different bases to be made in a few minutes. For a net premium valuation, some approximation to the net premiums on the new basis would be necessary.

It should be added that the method is not entirely empirical, as it would be easy to calculate from the approximate annuity values, rates of mortality which would satisfy exactly the assumptions as to the shape of the annuity curve.

## Office A.

Age	Sums Assured and Bonuses	Tabular Premiums	Age	Sums Assured and Bonuses	Tabular Premiums
15	1	...	56	608	16.4
16	5	.1	57	493	13.5
17	9	.1	58	399	9.6
18	4	.1	59	397	11.0
19	3	.1	60	320	9.5
20	7	.1	61	437	13.0
21	4	.1	62	345	10.9
22	23	.3	63	359	12.7
23	63	1.1	64	222	5.9
24	65	1.1	65	281	8.1
25	85	1.4	66	339	11.6
26	34	.6	67	228	7.2
27	56	1.1	68	732	29.1
28	196	3.6	69	175	5.7
29	253	4.8	70	155	4.1
30	250	4.7	71	183	8.8
31	194	3.6	72	201	4.5
32	295	5.9	73	111	1.9
33	177	3.5	74	83	2.1
34	389	8.4	75	68	2.1
35	390	7.5	76	55	1.6
36	306	6.4	77	79	1.3
37	356	7.2	78	52	1.7
38	518	12.1	79	40	.7
39	452	9.2	80	44	.5
40	445	11.1	81	31	.5
41	584	11.5	82	30	.5
42	378	8.2	83	46	1.1
43	382	7.7	84	8	.3
44	539	12.3	85	20	.1
45	495	9.4	86	6	.1
46	465	10.5	87	18	.3
47	427	8.6	88	6	...
48	535	13.2	89	7	.2
49	476	10.1	90	2	.1
50	510	14.8	91	10	.1
51	436	11.2	92	4	.1
52	751	17.2	93	1	...
53	478	11.7	94	...	...
54	478	13.7	95	...	...
55	466	11.3	96	9	.3

## Premiums payable for limited periods.

Unexpired Term	Tabular Premiums	Unexpired Term	Tabular Premiums	Unexpired Term	Tabular Premiums
1	2.4	12	2.4	23	.5
2	1.8	13	3.6	24	.6
3	2.0	14	1.5	25	.6
4	2.2	15	4.5	26	.2
5	5.6	16	1.2	27	.1
6	4.4	17	5.4	28	.4
7	4.2	18	1.9	29	.3
8	6.9	19	3.2	30	...
9	2.4	20	.5	31	.2
10	1.8	21	.3	...	...
11	5.1	22	.4	...	...



## Office B.

$n^*$	Sums Assured and Bonuses	Tabular Premiums	$n^*$	Sums Assured and Bonuses	Tabular Premiums
1	131	7.0	24	879	30.2
2	138	7.8	25	997	34.1
3	266	13.8	26	743	23.8
4	296	16.1	27	705	22.1
5	260	15.5	28	661	20.9
6	434	23.5	29	689	21.1
7	351	17.4	30	470	14.0
8	463	22.6	31	399	11.6
9	535	24.6	32	356	9.9
10	627	30.8	33	293	8.1
11	727	32.9	34	227	6.1
12	892	40.4	35	265	7.0
13	742	33.5	36	138	3.5
14	806	37.2	37	115	2.9
15	941	43.4	38	102	2.5
16	1,044	47.9	39	70	1.7
17	1,027	44.4	40	75	1.8
18	1,052	44.9	41	19	.4
19	1,189	49.9	42	29	.7
20	1,209	48.9	43	18	.4
21	1,019	39.3	44	4	.1
22	897	32.7	45	...	...
23	833	29.1	46	8	.2

\*  $n$  = Year of maturity — Year of valuation. Valuation date is 31 December.

## Office C.

Age	Amount of Annuity	Age	Amount of Annuity	Age	Amount of Annuity
21	17	44	540	67	410
22	...	45	27	68	275
23	...	46	71	69	127
24	10	47	50	70	203
25	...	48	116	71	126
26	...	49	235	72	615
27	...	50	226	73	402
28	...	51	150	74	526
29	...	52	36	75	78
30	...	53	61	76	251
31	...	54	281	77	237
32	...	55	249	78	2,817
33	...	56	401	79	29
34	11	57	290	80	313
35	64	58	357	81	151
36	...	59	247	82	...
37	25	60	46	83	145
38	...	61	637	84	312
39	...	62	145	85	89
40	73	63	410	86	115
41	...	64	413	87	...
42	...	65	439	88	24
43	...	66	186	89	50

*Societies D and E.*

Age last Birthday	NUMBER OF MEMBERS		Age last Birthday	NUMBER OF MEMBERS	
	Society D	Society E		Society D	Society E
16	35	...	43	30	3
17	28	2	44	23	6
18	33	1	45	20	4
19	37	1	46	12	...
20	20	...	47	14	8
21	14	3	48	19	3
22	22	...	49	13	1
23	38	...	50	19	4
24	33	1	51	13	2
25	36	2	52	7	...
26	28	...	53	10	5
27	23	2	54	12	3
28	30	...	55	12	...
29	19	...	56	8	...
30	28	4	57	10	1
31	32	3	58	7	3
32	17	...	59	14	2
33	15	1	60	10	...
34	24	4	61	7	...
35	27	2	62	3	2
36	22	6	63	...	1
37	31	4	64	8	...
38	24	3	65	5	1
39	20	3	66	6	...
40	28	1	67	...	1
41	26	5	68	3	...
42	21	1	69	4	1

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*Decreasing Debts on Endowment Assurances.* By CHARLES H. ASHLEY, A.I.A., *Actuary of the London and Manchester Assurance Company.*

IN his useful note on this subject (*J.I.A.*, vol. xlix, p. 266), Mr. Todhunter starts off with the assumption that "in any case, the extra premium that would be charged has been determined according to the usual office practice," and so proceeds to find the debt. But I suggest, with Mr. Coutts (p. 278), that it is no more unscientific, having regard to the paucity of data on which to calculate extra risks, to assess a sub-standard life "by a debt in the first instance and measure the extra risk in this form."

The decreasing debt system has limitations in practice which I consider should be defined at the outset. Tables

could then be constructed with due regard thereto, showing what mortality curves can be provided for and what extra premiums will correspond.

Accordingly, in the hope of developing the study of the subject of extra rating (a subject which seems to have grown in importance during the war, I submit the appended tables based on the following practical limitations, namely :

- (1) That it is not advisable in practice to offer acceptance of a sub-standard risk subject to a debt which, during the first few years, is so large as to deprive the assured of the fundamental feature of life assurance, namely, protection against premature death. If the defect is such that the initial debt would have to be so large as say £75 per-cent, it is better simply to quote the extra premium.
- (2) That at no time during the currency of the policy shall the nominal sum assured, less the current debt, be less than the total amount of the premiums already paid.
- (3) That the annual reduction of the debt be deferred until after the first 5, 7 or 10 years: this with the object of keeping down the magnitude of the initial debt.
- (4) That the annual reduction after the period of deferment shall be at least the amount of the renewal premium paid. This follows, I believe, an American system.

The headings in the tables indicate the endowment periods and debts, and the columns show :

- (1) The EXTRA PURE PREMIUMS, calculated at  $3\frac{1}{2}$  per-cent interest, on the assumption that the extra mortality is so distributed as to produce policy-values equalling normal  $O^M$  3 per-cent policy-values at the end of each year, bonus excluded. (If the extra premium, which is payable for the whole endowment period, is paid in lieu of the debt, 10 per-cent might be added for expenses). Formula adopted

$$q_{[x]+n-1}(1 - {}_nV_{xt}) = q'_{[x]+n-1}(1 - nth \text{ debt} - {}_nV_{xt})$$

the select values of  $q$  and  $q'$  being assumed to be in the same ratio as the  $O^M$  values.

- (2) The number of YEARS RATING-UP which will produce, on the basis of  $O^{[M]}$  Mortality,  $3\frac{1}{2}$  per-cent interest, premiums for the increasing assurance approximately equal to premiums for a level £100 at true ages.
- (3)  $n$ , where  $100(P_{[x+n]t} - P_{[x]t})$  by  $O^{[M]}$   $3\frac{1}{2}$  per-cent, equals the tabulated extra pure premiums.
- (4) The CONSTANT ADDITIONS to  $O^{[M]}$  Rates of Mortality which will produce premiums for the increasing assurance approximately equal to the normal  $O^{[M]}$  premiums for £100. Interest  $3\frac{1}{2}$  per-cent.

In the tables various types of extra mortality which are covered by the specified debts are indicated. Four diagrams are appended to illustrate the mortality curves involved in respect of age at entry 35.

The *thick* curve is the standard for comparison,  $O^{[M]}$ .

The *circles* curve represents rates of mortality which produce, under the debt system, claims each year approximately equal to the  $O^{[M]}$  claims under normal policies.

The *thin* curve represents Hardy's method of assuming sub-standard mortality such as to give normal reserves, and is seen to be applicable to sub-standard lives which may be expected to show increasing extra mortality during the first few years, thereafter dropping to the normal. Though I cannot think that any class of sub-standard risks would show such a pronounced maximum mortality as this "thin" curve, I gather from enquiries amongst medical officers that curves with that tendency (more probably the "circles" curve) are quite likely, and a recent history of typhoid fever or of abscess in the abdomen are suggested as examples. Mr. Burn presumes such a type on p. 207 of the 5th Congress Transactions, and the Associated Scottish Life Offices' Investigation found that among publicans the extra risk increases with duration of assurance to a maximum and then diminishes. It must not be forgotten that among standard lives there are many who develop progressive diseases, and the reduction of sub-standard lives through earlier death must surely leave the residue nearer to the standard.

The *dotted* curve represents the too-popular method of assessing a life by stating that it is equivalent to one a certain number of years older. The way such a curve soars after a few years suggests to me its rarity in practical experience. Nevertheless, impaired lives are accepted which undoubtedly

annum.

Age at Entry	INITIAL		—			—				Age at Entry
	Extra pure prem. on de- scribed basis	extra pure prem. on de- scribed basis	Years rat- ing up	<i>n</i>	Constant addition to $q^{(M)}$ covered by the debt	Extra pure prem. on de- scribed basis	Years rat- ing up	<i>n</i>	Constant addition to $q^{(M)}$ covered by the debt	
	(1)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	
20	<i>s. d.</i> 8 11	...	...	...	...	...	...	...	...	20
25	9 7	...	...	...	...	...	...	...	...	25
30	10 9	...	...	...	...	...	...	...	...	30
35	12 4	...	...	...	...	...	...	...	...	35
40	14 11	...	...	...	...	...	...	...	...	40
45	19 2	...	...	...	...	...	...	...	...	45
50	25 10	...	...	...	...	...	...	...	...	50
55	36 5	...	...	...	...	...	...	...	...	55

annum.

	INITIAL				—				
	<i>s.</i>	<i>d.</i>							
20	8	9	...	...	...	...	...	...	20
25	9	7	...	...	...	...	...	...	25
30	10	10	...	...	...	...	...	...	30
35	12	8	...	...	...	...	...	...	35
40	15	8	...	...	...	...	...	...	40
45	20	6	...	...	...	...	...	...	45
50	28	3	...	...	...	...	...	...	50

er annum.

	INITIAL		INITIAL DEBT £22 10s.				—				
	<i>s.</i>	<i>d.</i>	<i>d.</i>								
20	11	2	1	...	...	...	...	...	...	20	
25	12	4	4	7	7	·002	...	...	...	25	
30	14	0	7	...	...	...	...	...	...	30	
35	16	10	2	4	5	·002	...	...	...	35	
40	21	6	0	..	...	...	...	...	...	40	
45	28	9	3	3	4	·004	...	...	...	45	

er annum.

	INIT		INITIAL DEBT £28				INITIAL DEBT £24 10s.					
	<i>s.</i>	<i>d.</i>	<i>d.</i>				<i>s.</i>	<i>d.</i>				
20	9	8	10	...	...	...	2	4	...	...	...	20
25	10	9	2	7	8	·002	2	6	7	7	·002	25
30	12	5	7	...		...	2	10	...	...	...	30
35	15	2	5	4	5	·003	3	6	4	4	·002	35
40	19	8	7	..	...	...	4	5	...	...	...	40

*15-year Endowment Assurance. Initial Debts fixed during first 5 years, then reducing by £7 10s. per annum*

Age at Entry	INITIAL DEBT £52 10s.				INITIAL DEBT £45				INITIAL DEBT £37 10s.				INITIAL DEBT £30				INITIAL DEBT £22 10s.																Age at Exit
	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	
20	s. d.				s. d.				s. d.				s. d.				s. d.																20
25	8 11				5 7				2 2				3 3				1 3																25
30	9 7	20	23	'007	6 0	16	19	'005	3 10	12	15	'003	2 5	9	11	'002	1 5	6	8	'001													30
35	10 9				6 8				4 3				2 8				1 7																35
40	12 4	15	17	'009	7 8	11	13	'006	4 10	8	10	'004	3 0	6	7	'003	1 10	4	5	'002													40
45	14 11				9 3				5 10				3 7				2 2																45
50	19 2	12	13	'013	11 10	8	10	'009	7 6	6	7	'006	4 8	4	5	'004	2 10	3	3	'002													50
55	25 10				16 0				10 2				6 5				3 10																55
55	36 5	10	11	'028	22 8	7	8	'018	14 4	5	6	'011	9 0	4	4	'007	5 4	2	3	'005	...												55

*20-year Endowment Assurance. Initial Debts fixed during first 7 years, then reducing by £5 10s. per annum*

Age at Entry	INITIAL DEBT £19 10s.				INITIAL DEBT £14				INITIAL DEBT £8 10s.				INITIAL DEBT £3				INITIAL DEBT £27 10s.				INITIAL DEBT £22												Age at Exit
	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	
20	s. d.				s. d.				s. d.				s. d.				s. d.				s. d.												20
25	8 8				6 6				4 11	13	14	'001	3 3				2 7	9	10	'002	1 7	7	7	'001									25
30	9 7	18		'007	6 10	15	17	'005	4 11	13	14	'001	3 3				2 7	9	10	'002	1 7	7	7	'001									30
35	10 10				7 5				5 7				4 0				2 10				1 11												35
40	12 8	13	15	'004	9 0	11	12	'006	6 6	9	10	'005	4 8	7	8	'004	3 4	5	6	'003	2 3	4	4	'002									40
45	15 8				11 2				8 0				5 9				4 1				2 9												45
50	20 6	10	12	'015	11 7	8	9	'011	10 6	6	7	'008	7 6	5	6	'006	5 3	4	4	'004	3 7	3	3	'003									50
50	28 3				20 1				14 5				10 3				7 3				4 10												50

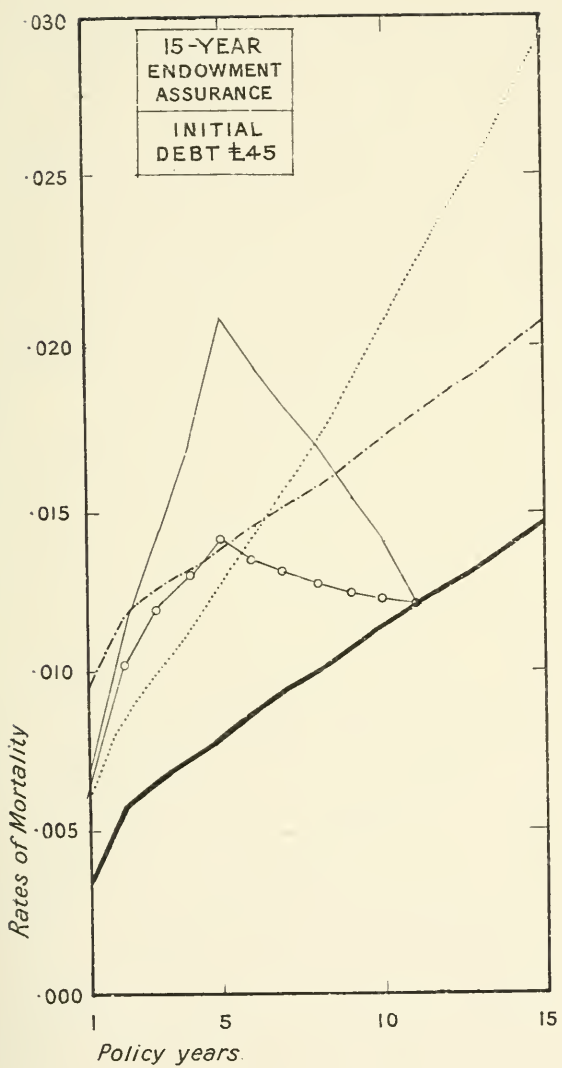
*25-year Endowment Assurance. Initial Debts fixed during first 10 years, then reducing by £4 10s. per annum*

Age at Entry	INITIAL DEBT £49 10s.				INITIAL DEBT £15				INITIAL DEBT £10 10s.				INITIAL DEBT £36				INITIAL DEBT £31 10s.				INITIAL DEBT £27				INITIAL DEBT £22 10s.								Age at Exit
	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	
20	s. d.				s. d.				s. d.				s. d.				s. d.				s. d.				s. d.								20
25	11 2				4 5				6 5				1 11				3 9				2 10				2 1								25
30	12 4	18	20	'008	5 3	16	18	'007	7 1	11	15	'005	2 5	12	13	'004	2 9	10	11	'003	2 7	8	9	'002	2 1	7	7	'002					30
35	14 0				10 7				8 1				4 8				3 6				3 6				2 7								35
40	16 10	14	15	'011	12 9	12	13	'009	9 8	10	11	'007	7 5	8	9	'005	5 8	7	8	'004	4 3	5	6	'003	3 2	4	5	'002					40
45	21 6				16 2				12 4				8 5				7 3				5 5				4 0								45
50	28 9	12	13	'019	21 8	10	11	'015	16 6	8	9	'012	12 7	6	7	'009	10 6	5	6	'007	7 2	4	5	'005	5 3	3	4	'004					50

*30-year Endowment Assurance. Initial Debts fixed during first 10 years, then reducing by £3 10s. per annum*

Age at Entry	INITIAL DEBT £19				INITIAL DEBT £14 5 10s.				INITIAL DEBT £12				INITIAL DEBT £8 10s.				INITIAL DEBT £35				INITIAL DEBT £31 10s.				INITIAL DEBT £28				INITIAL DEBT £24 10s.				Age at Exit
	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	Extra pure prem. on de-scribed basis	Years rat-ing up	Years covered by the debt	Constant addition to g(w) covered by the debt	
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	
20	s. d.				s. d.				s. d.				s. d.				s. d.				s. d.				s. d.								20
25	9 8				7 11				6 6				5 11				4 5				3 7				2 1								25
30	10 9	15	17	'007	8 10	13	15	'006	7 3	12	13	'005	6 10	10	12	'004	4 10	9	11	'003	3 7	8	9	'003	2 10	7	8	'002	2 6	7	7	'002	30
35	12 5				10 2				8 4				7 6				5 0				4 0				2 10								35
40	15 2	12	13	'010	12 5	10	11	'008	10 2	9	10	'007	8 4	7	8	'006	6 9	6	7	'005	5 6	5	6	'004	4 5	4	5	'003	4 6	4	4	'002	45
40	19 8				16 1				14 2				10 8				8 8				7 0				5 7								40

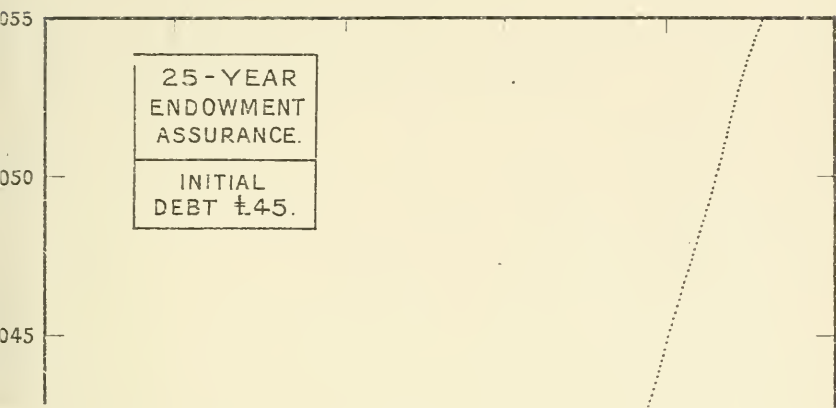


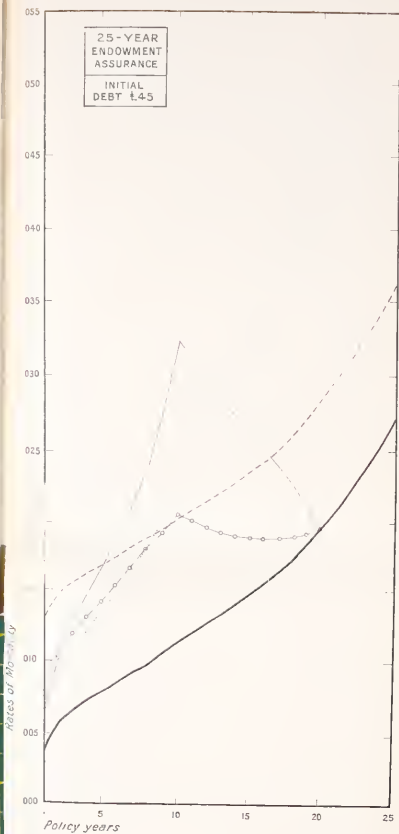






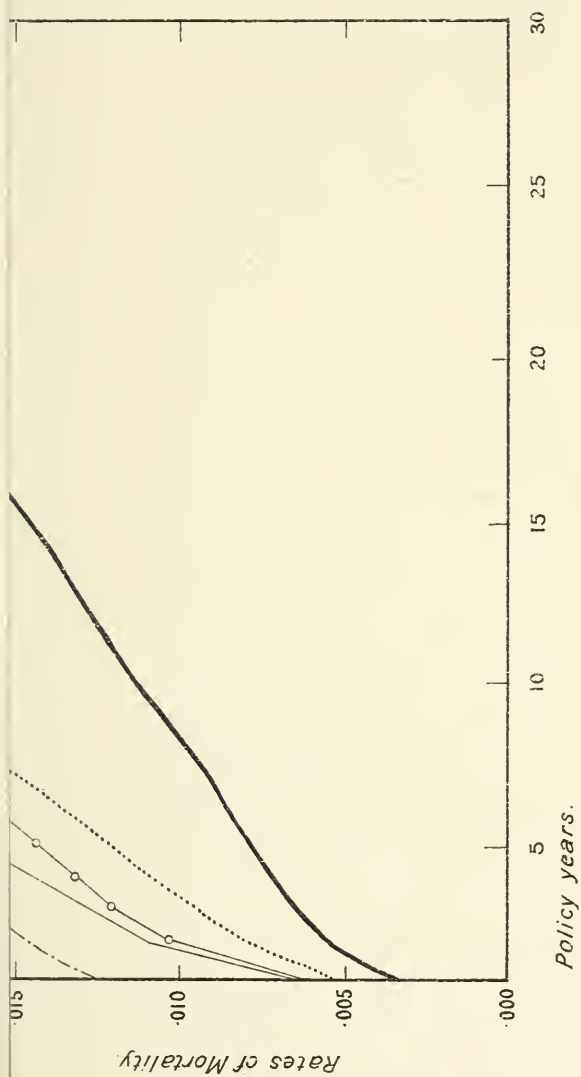


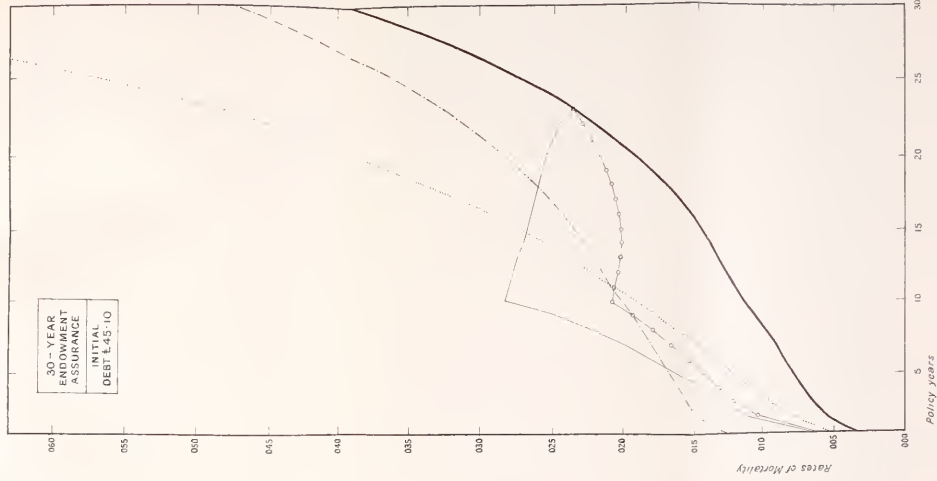






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show an increasing extra mortality, though less steeply than that represented by a substantial addition to the age.

The *broken* curve is perpendicularly equidistant from the standard and represents a constant addition to the normal rate of mortality such as will justify the specified debt. Probably a large number of lives will be put into this class as a result of injuries received in the war.

The debt system can, of course, be used for any particular incidence of extra risk because the consideration is received at the outset. In other words, the assured "pays" in the early years for the extra risk, although that extra risk may not mainly arise till the later years. On the other hand, the system is limited in its usefulness to the lighter extra risks. No additional labour need be incurred at the periodical valuations, since all types of extra mortality will be operating and we shall get a good average result by valuing all the endowment assurances as normal for the full sums assured.

Industrial offices get large numbers of endowment assurance proposals on the lives of unmarried young men to whom the investment element appeals more strongly than the immediate death benefit. To those who are found to be sub-standard lives, a decreasing debt ought to be more acceptable than the payment of an extra premium. Indeed, in such cases, it seems to me that we can avoid any "moral hazard" by not quoting an extra premium. In other cases the assured should always be given the option of paying the corresponding extra premium.

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### *Mathematical Symbols.*

STUDENTS of Lewis Carroll's "Pillow Problems" may recall the use in that sleep-dispelling work—and the interesting explanation in the preface to the Fourth Edition—of two geometrical symbols for the sine and cosine. These particular symbols have never, so far as we know, been used by other mathematicians. But they are illustrative of an evolutionary process that has been going on from the earliest times—or at any rate since the introduction or systematization of algebraical symbolism by Diophantus. With the continual development of mathematical analysis, the necessity arises for new symbols, not only to save the frequent repetition of verbal expressions,

but also to convey the idea of an operation or relation to the mind more rapidly than is possible by the use of words. Of the symbols tentatively introduced to meet this need many—such as the above-mentioned symbols for the sine and cosine—do not survive their birth; others—as in the case of the symbol for equality, first introduced by Recorde\* in 1557, but not generally adopted until more than a century later—became established only after a long struggle for existence; others again attain a limited acceptance in some special branch of research; a few are almost immediately accepted as supplying a long-felt want.† This process of evolution in mathematical symbolism has been very active during the last 20 or 30 years, and symbols which are unfamiliar to students brought up on the older books are now commonly employed, without explanation or definition, in modern textbooks and tracts even of an elementary nature. A symbol of which the meaning is not accurately understood is as serious an obstacle to study as if the work in which it is used were written in an unknown language, and it is hoped, therefore, that the appended list of a few modern symbols may assist actuarial students to pursue their reading both of current actuarial work and of analytical research bearing thereon.

The principle governing the use of mathematical symbols is clear. A “recognized” symbol may be employed without explanation; any other—whether one that although frequently used cannot be held to have been generally accepted by mathematicians, or a special symbol employed for a special

\* “to avoide the tedious repetition of these woordes : is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe” [*i.e.*, twin] “lines of one lengthe, thus :  $\equiv$  : bicause noe 2 thynges can be moare equalle.” —The Whetstone of Witte, 1557.

† For some information as to the origin of the more common symbols in algebra reference may be made to W. W. Rouse Ball’s *Short Account of the History of Mathematics*, and to the article on *Algebra* in the *Encyclopædia Britannica*.

We are indebted to Mr. G. J. Lidstone for the following interesting extract from Arbogast’s *Du Calcul des Dérivations* :

“Il a fallu introduire des signes nouveaux; j’ai donné une attention particulière à cet objet, persuadé que le secret de la puissance de l’Analyse consiste de la choix et l’emploi heureux des signes simples et caractéristiques de la chose qu’ils doivent représenter. Je me suis prescrit à cet égard les règles suivantes : (1) de rendre les notations le plus qu’il était possible analogues à des notations reçues; (2) de ne point introduire des notations inutiles et que j’aurais pu remplacer sans confusion par des notations déjà en usage; (3) de les choisir très-simples, en y faisant entrer cependant toutes les variétés qu’exigent les différences des opérations.”

purpose—should be defined. The difficulty consists, in the absence of any final authority, in determining what constitutes “recognition”, and it is increased by the fact that in certain instances two symbols seem to have attained recognition, or at any rate are employed by different writers without explanation, for the same purpose. Our list deals only with symbols (of comparatively recent introduction) which are certainly recognized or may be considered to be on the border line. It is not suggested that the list is complete, and some of the symbols have been in use among mathematicians for many years but have been included because they may be unfamiliar to many actuarial students.

A consideration of some practical importance is that it is desirable, when possible, to avoid the use of symbols that entail the necessity for the process known to printers as “justification”, and thus to reduce the expense both of composition and of corrections. Several of the symbols in our list—(1) and (4) for example—are instances of alternative symbols that have been devised with this object, and such expressions as “arctan”, “sinh”, “cosh”, &c., belong to the same category. It may be remembered that a good many years ago the British Association appointed a committee of eminent mathematicians to report “on Mathematical Notation and Printing, with the view of leading mathematicians to prefer, in optional cases, such forms as are more easily put into type, and of promoting uniformity of notation.” The resulting report (see *J.I.A.*, vol. xx, p. 355) would not appear to have had so much effect as might be desired on the practice of mathematical authors, and this may perhaps have been due to the fact that the committee, although expressing a pious opinion that uniformity of notation would “tend towards a common language in mathematics”, confined their attention almost entirely to the first part of their reference and recommended some forms of notation in preference to others “from the printing, and not from the scientific point of view.” It may be suggested that the scientific point of view should take priority, that recommendations made from that point of view by a strong committee of leading mathematicians might do much to promote uniformity, and that while some forms of notation must always necessitate justification, printers would probably be able to a very considerable extent to meet the requirements of mathematicians, if those

requirements were based on uniformity of practice, by stocking certain symbols that have at present to be specially composed.

### SOME MODERN MATHEMATICAL SYMBOLS AND THEIR MEANINGS.

/ ... .. Used instead of the horizontal line to represent a fraction. Thus  $a/b = \frac{a}{b}$ .

This is not generally used when the fraction itself is the subject of transformation. Thus we should write

$$e^x = 1 + x + x^2/2! + x^3/3! + \dots$$

but less commonly

$$(x-a)/(x^3-a^3) = 1/(x^2+ax+a^2).$$

! ... .. Used instead of  $\perp$  to represent a factorial. Thus  $n! = n \cdot (n-1) \cdot (n-2) \dots 1$ , where  $n$  is supposed positive and integral.

$\equiv$  ... .. Denotes an *identity* which is true for all values of the variables as distinguished from an *equation* which is true only when some of the variables have particular values; thus  $(a+b)^2 \equiv a^2 + 2ab + b^2$ , i.e., "is transformable into"  $a^2 + 2ab + b^2$  "by applying the laws of algebra, without any assumption regarding the operands involved" (Chrystal Algebra, Part I, 5th Edition, p. 22). Sometimes used in a more restricted sense to indicate merely that one (usually shorter) expression *represents* another; thus

$$v_q \equiv a_0 + a_1q + a_2q^2 + \dots + a_nq^n;$$

$$x^3 \equiv x \times x \times x.$$

$\sim$  ... .. Formerly placed between two quantities to indicate that the smaller (algebraically) is to be subtracted from the larger; thus  $a \sim b$  denotes  $a-b$  or  $b-a$  according as  $a$  or  $b$  is the larger. But now practically



superseded in this sense by the symbol  $\sim$  (*see below*), and used in the theory of infinite series to denote a certain kind of approximate equality called "asymptotic."

D ... Used to denote differentiation. Thus  $Df(x) \equiv df(x)/dx$  or  $f'(x)$ . Note that in Boole's *Finite Differences* (1st Edition) and in *Text-Book*, Part II, this symbol is used to denote the operator  $(1 + \Delta)$  or  $e^{d/dx}$ , now more generally represented by E.

$\binom{m}{r}$  ... Used instead of  ${}_rC_m$ ,  $(m, r)$  or  $\frac{m!}{r!(m-r)!}$ .  
Thus, for example,

$$(1+x)^m = 1 + mx + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \dots$$

exp. ( ) ... Used to represent  $e$  raised to the power whose index is the quantity in the brackets. Thus

$$\exp. (a+bx+cx^2+\dots) = e^{a+bx+cx^2+\dots}$$

$| \quad |$  ... The modulus of; thus  $|z|$  is the modulus of  $z$ . If  $z$  is a complex quantity equal to  $x+yi$  [where  $i$  is the imaginary quantity  $\sqrt{-1}$ ], the modulus of  $z$  is the positive value of  $\sqrt{x^2+y^2}$ . Thus  $|x+yi| = +\sqrt{x^2+y^2}$ . Hence when  $y=0$  and  $z$  is wholly real we have  $|x| = +\sqrt{x^2}$ , *i.e.*, the modulus of a real quantity is its absolute value irrespective of sign, or its numerical value taken positively; the symbol is consequently used to denote the absolute value, without reference to sign, of a magnitude which may be either positive or negative. For many purposes the modulus of a complex quantity takes the place of the numerical value of a real quantity, especially in questions relating to convergency. *Note.*—Care must be taken not to confuse the symbol for the *modulus* with a similar symbol sometimes used as

an abbreviated notation for a *determinant*. For example, the determinant of the third order whose successive rows are  $a_1, b_1, c_1$ ;  $a_2, b_2, c_2$ ;  $a_3, b_3, c_3$ , where the constituents of the principal diagonal are  $a_1, b_2, c_3$ , is represented in this notation by  $|a_1 b_2 c_3|$ , which in another connection would mean the modulus of the product  $a_1 b_2 c_3$ .

$\neq$  ... Is not equal to; has any other value than. (In continental works the symbol  $\lesseqgtr$  is often used with this meaning.) *Example.*

$$\int x^m dx = x^{m+1}/(m+1) \text{ if } m \neq -1.$$

If  $m = -1$ , the function  $x^{m+1}/(m+1)$  becomes infinite, but the indefinite integral  $= \log_e x$ .

$\nlessgtr$ , or  $\leq$  or  $\leq$  Is not greater than; is either equal to or less than.

$\nlessgtr$ , or  $\geq$  or  $\geq$  Is not less than; is either equal to or greater than.

*Note.*—Where an actual inequality is in question the form  $\nlessgtr$  or  $\nlessgtr$  is usually the more convenient,\* but the later forms are much used to denote the limits of a variable. Thus to express that  $x$  may take any value over the range  $-1$  to  $+1$  except  $+1$  itself, we should write  $-1 \leq x < +1$ . For example, if this be the case the series  $x - x^2/2 + x^3/3 \dots ad \text{ inf.}$  converges to the sum  $\log_e(1+x)$ .

$\approx$  or  $\doteq$  ... Is approximately equal to. Thus  $\delta \doteq i + d$ . Such a symbol is much required in actuarial work. Both forms are used without special definition in the *Phil. Mag.* The latter form, proposed by Lord Kelvin, is more easily distinguished from the symbol  $=$ , and is more easily written.

\* Rouse Ball, however, remarks (*loc. cit.* p. 242) that the symbols  $\neq$ ,  $\nlessgtr$ ,  $\nlessgtr$ , "are, I believe, now rarely used outside Great Britain; they were employed, if not invented, by Euler. The symbols  $\geq$  and  $\leq$  were introduced by P. Bouguer "in 1734."

$\rightarrow$  ... Tends to the value ; has the limiting value.

*Examples.*

As  $x \rightarrow \infty$ ,  $\left(1 + \frac{n}{x}\right)^x \rightarrow e^n$ , and  $e^{-x} \rightarrow 0$ .

As  $h \rightarrow 0$ ,  $(u_{x+h} - u_x)/h \rightarrow \frac{d}{dx} u_x$ .

$O( )$  ... Is of the order of ; ultimately preserves a finite ratio to (*i.e.*, as the argument or independent variable  $\rightarrow \infty$ ). Thus

$$\frac{15n+19}{n^3+1} \text{ is } O\left(\frac{1}{n^2}\right).$$

The strict analytical definition is as follows.

If  $u_x$  and  $v_x$  are two functions of  $x$  such that there is a number, say  $n$ , such that the modulus  $|u_x : v_x|$  is  $< K$  whenever  $x > n$ , where  $K$  is finite and independent of  $n$ , we say that  $u_x$  is  $O(v_x)$ .

$o( )$  ... Ultimately vanishes in comparison with (*i.e.*, as the argument or independent variable  $\rightarrow \infty$ ). Thus

$$\frac{\log x}{x} \rightarrow 0 \text{ as } x \rightarrow \infty ; \text{ i.e., } \log x \text{ is } o(x).$$

## LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law*.

Relief from  
Super-tax  
in respect of  
life premium  
paid under  
Contract.

UNDER Sec. 36 (1) of the Finance Act, 1916, it is enacted that relief from super-tax shall not be given in respect of premiums payable under life assurances and deferred annuities.

An interesting point in this connection arose in the case of *Earl Howe v. Commissioners of Inland Revenue*, which will be found reported in *The Times* of 10 July 1918, but had not appeared in the official reports at the time when these Notes were written.

The case was an appeal from a decision of the Commissioners for the Special Purposes of the Income Tax Acts in respect

of a claim by Lord Howe to deduct amounts paid by him as premiums on life assurance policies from his assessment to super tax.

The facts are as follows:

For the purpose of raising certain sums of money, Lord Howe granted his interest in certain estates and also assigned a certain policy of life assurance on his own life to an assurance company by way of mortgage. He covenanted to pay (1) interest on the sums advanced by him; (2) the premiums on the life policy. He also gave a right to the assurance company to pay the premiums if he himself neglected to do so, and to charge them on the mortgaged property. It was contended on behalf of Lord Howe that there was no distinction in principle between the annual premium payable and the mortgage interest. The Commissioners decided that Lord Howe was not entitled to deduct the amounts of such premiums from his assessment.

In giving judgment in favour of the appellant, Mr. Justice Sankey said: "If the case came before me unembarrassed by authority, I am bound to say that a careful consideration of the sections of the Income Tax Acts would lead me to the conclusion that the contention of the Crown is correct. The matter has, however, recently come before the Irish Courts in the case of *Commissioners of Inland Revenue v. Lord Massy*, and two out of three judges decided that the contention as advanced by Lord Howe in the present case is correct. An appeal was brought before the Irish Court of Appeal, but unfortunately, before judgment was given, Lord Massy died, and his executors did not continue the case; I think, therefore, in these circumstances I ought to follow the majority judgment in the Irish Courts, and, as a result, the appeal succeeds."

I understand an appeal has been lodged against the decision and the case will shortly come before the Court of Appeal.

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The question whether certain forms of endowment policies could be correctly termed policies of life assurance, again came before the Court in the case of *In re National Standard Life Assurance Corporation*, 118 L.T.R. 621.

The decision in this case is not inconsistent with previous decisions in regard to somewhat similar kinds of assurance reported in these Notes under the names of *Flood v. Irish*

When is an endowment policy a policy of life assurance?

*Provident Assurance Company, Limited* (*J.I.A.*, vol. xlvii, p. 298), and *Joseph v. Law Integrity Insurance Company, Limited* (*J.I.A.*, vol. xlvii, p. 300).

The National Standard Life Assurance Corporation was incorporated in 1906, and £20,000 was deposited in the Court of Chancery in respect of the life assurance business of the company.

On 6 July 1916, an order to wind up the company compulsorily was made, and the liquidator took out a summons to have it determined whether certain assurances were "policies on human life" within the meaning of the Assurance Companies Act, 1909, so as to entitle the holders thereof to participate in the statutory deposit.

The assurances were divided into four classes, as follows : In consideration of certain premiums, the Corporation bound itself

(1) to pay a fixed sum to the assured at a fixed date if he should be then living, with provision for the payment of a smaller amount to the legal personal representatives in the case of death ;

(2) to pay a fixed sum at a fixed date without any reference to the death of the assured. In this case the premiums were payable during the whole period, and it was open to the legal personal representatives to continue to pay the premiums ;

(3) to pay a sum at a fixed date, such sum varying according to whether certain options were exercised, the legal personal representatives having the right to surrender the policy within six months of the death of the holder, and to receive the total amount of premiums paid ;

(4) to pay a fixed sum at a fixed date with a provision that should the holder die before the date of maturity, the legal personal representatives should be entitled to a return of the premiums paid.

In the course of his judgment, Neville, J., said : " In my opinion  
" there is only one class of policy mentioned in the summons  
" which raises any special difficulty, and that is the policy  
" falling under the third class, whereby the company assures  
" payment of a sum of money at a fixed date, and gives an  
" option to the legal personal representatives to determine the  
" contract on the death of the assured, on the exercise of which  
" option all premiums are to be repaid. The policies falling  
" within the fourth class clearly come within the statutory  
" definitions, because in that case the money is payable upon

“ the death of the assured unless something else intervenes in the shape of a new contract between the legal personal representatives of the assured and the company. Unless that happens, the premiums are to be returned on the death of the assured. In the case of policies falling under the third class, however, the premiums are not to be returned to the legal personal representatives unless they choose to determine the contract, which they have power to do; but in my opinion, the contention was right that the determination of the contract by them is a ‘ contingency dependent on human life ’, and falls within the statutory definition contained in Sect. 30 (a) of the Assurance Companies Act, 1909. The result, therefore, is that all the policies, with the exception of those falling under the second class, are in my opinion ‘ policies on human life ’, and entitle the holders to rank against the statutory deposit.”

Legal decisions on this point were formerly of some importance owing to their bearing on the question as to what constituted a life insurance policy for the purpose of obtaining relief from income tax. The restrictions contained in recent Finance Acts have, however, considerably modified the importance of this question. At the same time it is interesting to note the distinction drawn between a pure endowment policy with return of premiums in the event of death, and a capital redemption or sinking fund policy.

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Annuity payable  
free of  
deductions  
does not avoid  
deduction of  
Income Tax.

If it be desired to grant an annuity payable free of income tax, it is necessary that words clearly indicating this intention should be used, and the statement that the annuity is to be paid free of any deduction is not sufficient to avoid deduction of income tax. In the case of *In re Loveless*, 34 T.L.R. 356, the Court of Appeal held that a direction in a will to pay a clear annuity is not to be construed as giving the annuity free of tax. The facts are as follows: The testator, Thomas Henry Loveless, by his will dated 18 July 1902, after giving various legacies and annuities—to be paid free of estate, settlement estate, legacy or succession duty—directed that his trustees should stand possessed of two equal third parts of the residuary trust fund upon trust out of the income thereof to pay to his wife, Edith Jane Loveless, “ a clear annuity of £2,000 during her widowhood commencing



“from my death”, and in the event of her remarriage then for the remainder of her life “a clear reduced annuity of £1,000 “for her separate use without power of anticipation”, such annuities to be paid by equal quarterly payments.

The will contained a power to appropriate out of the fund a sufficient part to answer the annuity payable to his wife, with power to resort to the capital in case of a deficiency. The testator died on 12 November 1916. The trustees took out an originating summons to determine, amongst other questions, whether, according to the true construction of the will, the widow was entitled to receive the annuity free from income tax, or whether she had to bear the income tax on it.

Mr. Justice Eve held, on the authorities, that the word “clear” was properly applicable to the instalments of legacy duty payable in respect of the annuity, and that the widow was not entitled to receive the annuity free of income tax.

The Court of Appeal dismissed the appeal of the widow against this decision.

In giving judgment Lord Justice Swinfen Eady said that the learned judge below had held that the direction to pay a clear annuity did not give an annuity free from income tax, and in his (his Lordship’s) opinion he was right in so holding. The Income Tax Act, 1842, provided (Section 102) that on annuities or other annual payments there should be charged yearly for every 20s. of the amount thereof the tax payable without deduction, and a proviso contained a power enabling the person authorized to make the annual payment to deduct the tax payable, and the person entitled to receive the annual payment must allow the deduction, as if the amount deducted had actually been paid to him. The Income Tax Act, 1853, Section 40, extended that power, providing that “the person liable to such payment shall be acquitted “and discharged of so much money as the deduction shall “amount unto, as if the amount thereof had actually been “paid unto the person to whom such payment shall have been “due and payable.” The Customs and Inland Revenue Act, 1888, Section 24, Sub section 3, contained a positive direction that the person paying the annuity should deduct the income tax payable thereon.

In his opinion a clear annuity was paid to an annuitant where there was paid, first the income tax payable thereon, and then the balance payable direct to the annuitant, the

two sums making together the full sum payable to the annuitant. The tax was paid out of the annuity, and it was as much paid to the annuitant as if the whole annuity had been paid to her direct without deduction, in which case she would have to pay the income tax thereout herself. In all cases where there was a direction in a will to pay an annuity clear of or free from all deductions, the will ought not by reason of those words to be construed as giving the annuity free of income tax.

Lord Justice Banks and Mr. Justice Neville concurred.

Ambiguous  
answers to  
questions in  
proposal.

The case of *Yorke v. Yorkshire Insurance Company, Limited* (1918), 1 K.B. 662, raised some interesting points with regard to the questions contained in a proposal for life assurance.

The facts are as follows: The insurance company granted a policy of insurance for £1,000, dated 6 January 1917, to one Robert Smith, payable in the event of death occurring before 4 December 1917. The policy was assigned to the plaintiff on 28 February 1917. Robert Smith died on 25 March 1917, and the insurance company refused payment on the ground that certain answers in the proposal were untrue, and that the assured had failed to disclose that he suffered from heart trouble and insomnia, and was addicted to the veronal habit.

The proposal form contained (*inter alia*) the following questions: What illnesses have you suffered? Answer: None of consequence. Do you ordinarily enjoy good health? Answer: Yes. Are you now and have you always been of sober and temperate habits? Answer: Yes.

In giving judgment in favour of the insurance company, Mr. Justice McCardie held that the question as to what illnesses the assured had suffered was not ambiguous and that the answer thereto was not a mere expression of opinion and that the untruth of the answer rendered the policy void. The words "sober and temperate" in a proposal for life insurance must be held, however, to refer only to the use and abuse of alcohol, and not to drug habits. Unless the question be expressly asked, insurance companies must, with respect to the use of drugs by a proposer, rely on the rule of law which requires the disclosure of all material facts known to the proposer which might lead the insurer to refuse the risk.

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*The United States Moratorium in respect of the Life Assurance Premiums of Officers, Men and Nurses on Active Service.*

[On pp. 54-56 of the current volume we gave the more material sections of the Act of the United States Legislature providing for the grant by the Treasury War Risk Insurance Bureau of new convertible term assurances to officers, men and nurses on active service. Our attention has since been drawn by Mr. J. Douglas Watson to the Soldiers' and Sailors' Civil Rights Act, signed by President Wilson on March 8, 1918, by which provision is made for the non-lapsing of existing assurances effected before 1 September 1917 by persons now in military or naval service. *The Commercial and Financial Chronicle* of New York comments as follows on the assurance provisions of the Act: "The bill of the special session" (a Moratorium Bill which was not passed by Congress) "was especially objectionable in the portion relating to life insurance, because it proposed to stop lapsing on policies which had been in force a year and in practice would have undertaken to confiscate the reserve on some policies for the benefit of holders of other policies . . . . When the insurance experts pointed this out to the sub-committee of the Senate, whose members had proposed to rush ahead and forbid lapsing and had not stopped to consider how that might work, they were able to see the point and candid enough to confess it. Then came a proposition that the Government guarantee the companies against loss and protect itself, in turn, by taking a lien on the policies. This proposition, with its crudeness removed, has been followed in the Act as passed. . . . The Article" (*i.e.*, the Insurance Article of the Act) "has far smaller application than was proposed at the special session, and its terms are far different from the broad bar upon lapsing first written into the bill. Any loss occurring will now fall upon the country, as it should; the companies will be subjected to considerable trouble and some expense, but they will be protected, and the scheme is apparently workable. In respect to justice it is far better than the" (British) "Emergency Powers Act which was hastily enacted by Parliament soon after the war began . . . ." The following extracts from the Act will sufficiently indicate the nature of the scheme.—EDS. *J.I.A.*] :

SEC. 401. That the benefits of this Article shall apply to any person in military service\* who is the holder of a policy of life insurance when such holder shall apply for such benefits on a form prepared in accordance with regulations which shall be prescribed by the Secretary of the Treasury. . . . The original of such application shall be sent by the insured to the insurer, and a copy thereof to the Bureau of War Risk Insurance. . . .

SEC. 402. That the benefits of this Act shall be available to any person in military service in respect of contracts of insurance in force under their terms up to but not exceeding a face value of \$5,000, irrespective of the number of policies held by such person whether in one or more companies when such contracts were made and a premium was paid thereon before September 1 1917. . . .

SEC. 404. That when one or more applications are made under this Article by any one person in military service in respect of insurance exceeding a total face value of \$5,000, whether on one or more policies or in one or more companies, and the insured shall not in his application indicate an order of preference, the Bureau of War Risk Insurance shall reject such policies as have the inferior cash surrender value, so as to reduce the total benefits conferred within the face value of \$5,000, and where necessary for this purpose shall direct the insurer to divide any policy into two separate policies. . . .

SEC. 405. That no policy which has not lapsed for the non-payment of premium before the commencement of the period of military service of the insured, and which has been brought within the benefits of this Article, shall lapse or be forfeited for the non-payment of premium during the period of such service or during one year after the expiration of such period; *Provided* that in no case shall this prohibition extend for more than one year after the termination of the war.

SEC. 406. That within the first fifteen days of each calendar month after the date of approval of this Act until the expiration of one year after the termination of the war, every insurance corporation or association to which application has been made as herein provided, for the benefits of this Article, shall render to the Bureau of War Risk Insurance a report, duly verified, setting forth the following facts:

First. The names of the persons who have applied for such benefits, and the face value of the policies in respect of which such benefits have been applied for by such persons, during the preceding calendar month;

Second. A list as far as practicable of the premiums in respect of policies entitled to the benefits of this Article which remain unpaid on the last day of the preceding calendar month . . . .

Fourth. A computation of the difference between the total amount of defaulted premiums . . . . and the total amount of premiums paid after having been previously reported as in

\* Including, broadly, as defined by Sec. 101, all officers, men and nurses on active service in the Army or Navy.

default. . . . The final sum so arrived at shall be denominated the monthly difference.

SEC. 408. That the Secretary of the Treasury shall, within ten days thereafter, deliver each month to the proper officer of each insurer, bonds of the United States to the amount of that multiple of \$100 nearest to the monthly difference certified in respect of each insurer. . . .

SEC. 409. That the bonds so delivered shall be held by the respective insurers as security for the payment of the defaulted premiums with interest. To indemnify it against loss the United States shall have a first lien upon any policy receiving the benefits of this Article, subject to any lien existing at the time the policy became subject to this Act. . . .

SEC. 410. That in the event that the military service of any person being the holder of a policy receiving the benefits of this Article shall be terminated by death, the amount of any unpaid premiums, with interest at the rate provided for in the policy for policy loans, shall be deducted from the proceeds of the policy and shall be included in the next monthly report of the insurer as premiums paid.

SEC. 411. That if the insured does not within one year after the termination of his period of military service pay to the insurer all past-due premiums with interest thereon from their several due dates at the rate provided for in the policy for policy loans, the policy shall at the end of the year immediately lapse and become void, and the insurer shall thereupon become liable to pay the cash surrender value thereof if any: *Provided* that if the insured is in the military service at the termination of the war, such lapse shall occur and surrender value be payable at the expiration of one year after the termination of the war.

SEC. 412. That at the expiration of one year after the termination of the war there shall be an account stated between each insurer and the United States, in which the following items shall be credited to the insurer:

- (1) The total amount of the monthly differences reported under this Article;
- (2) The difference between the total interest received by the insurer upon the bonds held by it as security and the total interest upon such monthly differences at the rate of 5 per-cent per annum;

and in which there shall be credited to the United States the amount of the cash surrender value of each policy lapsed or forfeited as provided in Section 411, but not in any case a greater amount on any policy than the total of the unpaid premiums with interest thereon at the rate provided for in the policy for policy loans.

SEC. 413. That the balance in favour of the insurer shall, in each case, be paid to it by the United States upon the surrender by the insurer of the bonds delivered to it from time to time by the Secretary of the Treasury under the provisions of this Article.

SEC. 414. That this Article shall not apply to any policy which



is void or which may at the option of the insurer be voidable, if the insured is in military service, either in this country or abroad, nor to any policy which as a result of being in military service, either in this country or abroad, provides for the payment of any sum less than the face amount thereof or for the payment of an additional amount as premium.

## ACTUARIAL NOTE.

*A Comprehensive Table for the calculation of yields (gross and net) on Redeemable Securities. By J. R. ARMSTRONG, F.F.A., of the Scottish Provident Institution.*

THE tables of net yields on redeemable securities after allowing for income tax, as published in *J.I.A.*, vol. li, pp. 28-31, are admirably ingenious and compact. They apply, however, only to bonds bought at a discount, and moreover the double interpolation generally required in their use may involve an accumulation of errors which, while not serious in practice, throws us back on a troublesome calculation in cases where greater exactitude may be desired. The present writer feels that there is room for further treatment of the subject, and, acting on a suggestion from Mr. Lidstone, ventures to submit the appended table, based on the first-difference formula discussed in *J.I.A.*, vol. l, pp. 247-250, by means of which both the gross and the net yield on a bond bought at either a premium or a discount may be calculated very easily and with close accuracy.

The editorial comments on the note just referred to include an alternative formula of Mr. Lidstone's, which the present writer recognizes as a distinct and characteristic improvement on his own. The improvement consists in starting, not from  $g$ , the nominal rate, as the writer suggested, but from a *trial* rate  $g+x_1$  nearer to the required rate  $g+x$  and thus involving interpolation for a smaller interval, with, as a result, a higher degree of accuracy in cases where  $x$  is large.

The derivation of the "trial rate" formula from the well known expression for the discount in the price, namely,

$$(g+x-g)a^{g+x}=K$$



consists in substituting  $x_1 + (x - x_1)$  for  $x$ , where  $g + x_1$  is a trial rate, whence we have

$$x = K \div a^{g+x_1+(x-x_1)} = K [R^{g+x_1} + \frac{x-x_1}{h} \Delta R^{g+x_1}]$$

$$i.e., x(1 - \frac{K}{h} \Delta R^{g+x_1}) = K(R^{g+x_1} - \frac{x_1}{h} \Delta R^{g+x_1})$$

whence, taking the difference-interval  $h = .01$ , we have

$$x = \frac{K(R^{g+x_1} - 100x_1 \Delta R^{g+x_1})}{1 - 100K \Delta R^{g+x_1}} \quad (1)$$

or adding  $g$  so as to obtain the yield direct, we have

$$g + x = \frac{g + K[R^{g+x_1} - 100(g+x_1) \Delta R^{g+x_1}]}{1 - 100K \Delta R^{g+x_1}} \quad (2)$$

The other essential feature of Mr. Lidstone's suggestion consists in substituting, for the variable trial rate  $g + x_1$  a "standard" trial rate  $I$  which would probably vary only gradually from time to time and which, if taken at present at say .05, would give good results so far as gross yields are concerned.

For net yields after allowing for tax, a lower "standard" rate—say 4 per-cent—would be more suitable.

Passing to percentages we thus have, in Mr. Lidstone's notation

$$\begin{aligned} 100i \text{ (gross)} &= \frac{100g \pm 100K(R^I - 100I \Delta R^I)}{1 \mp 100K \Delta R^I} \\ &= \frac{100g \pm 100KA}{1 \mp 100KB} \quad (3) \end{aligned}$$

$$\begin{aligned} \text{and } 100i \text{ (net)} &= \frac{100g(1-t) \pm 100K(R^{I'} - 100I' \Delta R^{I'})}{1 \mp 100K \Delta R^{I'}} \\ &= \frac{100g(1-t) \pm 100KA'}{1 \mp 100KB'} \quad (4) \end{aligned}$$

the upper or lower signs being taken according as  $K$  is the discount or the premium per unit of the price, and  $t$  being the rate of income tax per £.

The functions tabulated in the appended table are  $100A$  and  $100B$  at 5 per-cent and  $100A'$  and  $100B'$  at 4 per-cent convertible half-yearly. The process of calculation is obvious; it need only be noted that the functions, being tabulated as

percentages, are to be used with  $K$ , the discount or premium per unit of price.

As an example take a 3 per-cent debenture redeemable in 30 years, price 58.49 per-cent. The discount being .4151 per unit, we have for the gross yield

$$\frac{3 + .4151 \times 2.691}{1 - .4151 \times .756} = \frac{4.117}{.6862} = 5.999 \text{ per-cent,}$$

the true gross yield being 6 per-cent.

For the net yield after allowing for tax at 6s. per £, we have, substituting 2.1 for 3 in the numerator and taking  $A'$  and  $B'$  at 4 per-cent

$$\frac{2.1 + .4151 \times 2.885}{1 - .4151 \times .717} = \frac{3.298}{.7024} = 4.696 \text{ per-cent,}$$

the true net yield being 4.694 per-cent.

Had the mean of each of the tabulated functions been used in the calculation of both the gross and the net yield the results would have been 5.987 and 4.690 respectively. The example chosen being a very severe test in view of the unusual disparity between the gross rate and the nominal and trial rates, these last results indicate that for most cases in practice a single set of functions, say at  $4\frac{1}{2}$  per-cent, would serve for calculating both the gross and the net yield. The difference between the expressions for the gross and net yields would then be simply  $\frac{100gt}{1 \mp 100K\Delta R}$ , a very convenient expression for the tax deduction from the gross yield when the latter is ascertainable from prepared tables without requiring to be calculated.

While the quantities  $A$  and  $A'$  are tabulated with a view to the direct calculation of the *full* rate, gross or net, the writer may add that personally he prefers the more compact formula (1), giving  $x$ , the addition to the nominal rate. In this case the quantities required are simply  $100R$  at a convenient rate, say 4 per-cent (convertible half-yearly), and the values of  $100\Delta R$  for successive 1 per-cent intervals up to, say, 6 per-cent. The trial rate can then be easily varied at pleasure.

The general formula

$$\frac{g \pm K(R^i - 100i\Delta R^i)}{1 \mp 100K\Delta R^i}$$

invites comparison with Mr. Todhunter's well known approximate expression for the yield, namely,

$$\frac{g \pm \frac{K}{n}}{1 \mp K \frac{n+1}{2n}}.$$

On the assumption of constant first differences

$$K(R^i - 100i\Delta R^i) = KR^0 = K/n$$

as in Mr. Todhunter's formula. Similarly  $\frac{n+1}{2n}$  in the latter is a first approximation to  $100\Delta R$ . It will be seen from the values tabulated that the errors of the factors of  $K$  in Mr. Todhunter's formula are positive and negative in the numerator and the denominator respectively. Hence the values of the numerator and denominator err in the same direction, with the result that, within limits, the expression gives a fair approximation. Closely approximate expressions for  $R^i$  and  $\Delta R^i$  for periods from 10 to 50 years and for rates of interest from 3 per-cent to 6 per-cent may however be obtained as follows: Taking selected values of  $R$  within these limits and deducting from them  $\frac{1}{n} + .0053(100i)$  we are left with a residue the quotient of which by  $n(100i)^2$  lies, for the most part, between 6 and 7 in the sixth decimal place. We may therefore write

$$100R^i = 100n^{-1} + .53(100i) + 2n(100i)^2 \div 3000 \text{ approx.} \quad (5)$$

$$100\Delta R^i = .53 + 2n(200i + 1) \div 3000 \quad ,, \quad (6)$$

$$100(R^i - 100i\Delta R^i) = 100n^{-1} - 2n100i(100i + 1) \div 3000 \quad ,, \quad (7)$$

$2 \div 3000 = .0006\bar{6}$  being taken as the most convenient factor nearly midway between .0006 and .0007. The substitution of

(7) and (6) for  $100n^{-1}$  and  $\frac{n+1}{2n}$  respectively in Mr.

Todhunter's formula would not only extend its range considerably, but make it practically as accurate in most cases as if the actual tabular values of the functions were used, especially with a trial rate.

As an illustration, reverting to the above example and using formula (1), we have, at trial rate 5 per-cent :

$$100R = 3.333 + 2.65 + .5 = 6.483$$

$$100\Delta R = .53 + .22 = .75$$

$$(5 - 3) \times 100\Delta R = 1.5$$


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$$4.983$$


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and 
$$\frac{.4151 \times 4.983}{1 - .4151 \times .75} = 3.004 = 100x$$

whence we have as the gross yield  $100(g + x) = 6.004$  per-cent an error of .004.

Again, at 4 per-cent,

$$100R = 3.333 + 2.12 + .32 = 5.773$$

$$100\Delta R = .53 + .18 = .71$$

$$(4 - 2.1) \times 100\Delta R = 1.349$$


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$$4.424$$


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and 
$$\frac{.4151 \times 4.424}{1 - .4151 \times .71} = 2.604$$

which added to 2.1, the net nominal rate, gives 4.704 per-cent as the net yield, an error of .010.

It may be added that  $n^{-1}$ , *i.e.*,  $R^0$ , is the first term in the expansion of  $a_n^{-1}$ , while  $.53i$  is the mean value of the second term in the expansion for  $n = 10$  to 50.

The above expressions are not, of course, put forward with any idea of their being generally used in place of the actual values of the functions in question. But they are so nearly accurate, are so easily remembered, and lend themselves so readily to an occasional rapid calculation independently of tables that the writer thinks it worth while to mention them.

Years	4 PER-CENT		5 PER-CENT	
	100A'	100B'	100A	100B
2	49·969	·639	49·954	·642
2½	39·961	·618	39·942	·622
3	33·286	·605	33·262	·610
3½	28·516	·597	28·488	·602
4	24·936	·592	24·904	·598
4½	22·150	·588	22·114	·595
5	19·920	·586	19·880	·594
5½	18·093	·586	18·050	·594
6	16·570	·586	16·522	·595
6½	15·280	·586	15·228	·596
7	14·173	·587	14·118	·598
7½	13·212	·588	13·153	·600
8	12·371	·590	12·308	·602
8½	11·628	·592	11·561	·605
9	10·966	·594	10·895	·608
9½	10·373	·596	10·299	·611
10	9·839	·598	9·761	·614
11	8·914	·603	8·829	·620
12	8·141	·608	8·048	·627
13	7·484	·614	7·384	·634
14	6·919	·620	6·812	·641
15	6·428	·626	6·314	·648
16	5·996	·632	5·875	·656
17	5·613	·638	5·486	·663
18	5·272	·644	5·138	·670
19	4·964	·650	4·825	·678
20	4·687	·656	4·541	·685
21	4·434	·662	4·283	·693
22	4·204	·668	4·047	·700
23	3·992	·675	3·830	·707
24	3·797	·681	3·629	·714
25	3·617	·687	3·444	·721
26	3·450	·693	3·272	·729
27	3·294	·699	3·112	·736
28	3·149	·705	2·962	·742
29	3·013	·711	2·822	·749
30	2·885	·717	2·691	·756
31	2·765	·723	2·568	·763
32	2·653	·729	2·451	·769
33	2·546	·735	2·342	·775
34	2·445	·740	2·238	·782
35	2·350	·746	2·140	·788
36	2·260	·751	2·047	·794
37	2·174	·757	1·959	·800
38	2·092	·762	1·875	·806
39	2·014	·768	1·795	·812
40	1·940	·773	1·720	·817
41	1·869	·778	1·647	·823
42	1·801	·783	1·578	·828
43	1·737	·789	1·513	·833
44	1·675	·794	1·450	·838
45	1·616	·798	1·390	·843
46	1·559	·803	1·333	·848
47	1·504	·808	1·278	·853
48	1·452	·813	1·226	·858
49	1·402	·817	1·176	·862
50	1·353	·822	1·128	·867

## CORRESPONDENCE.

## ORIGINAL TABLES.

*To the Editors of the Journal of the Institute of Actuaries.*

DEAR SIRS,—Having on several occasions had to estimate the value of functions by the  $O^{[M]}$  Table at 5 per-cent interest, I have prepared a table of annuities on this basis, and, as it may be of some service to other members of the profession, I send you the table referred to herewith.

In constructing this table the ultimate value of  $D_x$  was made up by a continuous process from the relation

$$\log D_{x+1} = \log D_x + \log vp_x$$

the value of  $\log p_x$  being taken from the table published in the official volume to five places. Throughout the whole construction, therefore, five place logarithms only were used.

The value of  $D_{[x]}$  was then constructed—again by five-place logarithms—directly from the value of  $(\log v^x + \log l_{[x]})$  and thereafter the tables of both  $D_{[x]}$  and  $D_{[x]+10}$  were checked at decennial ages by the arithmometer.

From the relation  $\log D_{[x]+n+1} = \log D_{[x]+n} + \log vp_{[x]+n}$  complete tables of  $D_{[x]+n}$  were made up—the values of  $\log D_{[x]+10}$  so found agreeing with the values of  $\log D$  ultimate first obtained.

Having by summation constructed tables of  $N_{[x]}$  and  $N$  and thence constructed—in duplicate by the arithmometer—tables of  $a_{[x]}$ ,  $a_{[x]:\overline{60}}$ ,  $a_{[x]:\overline{40}}$  and  $a_{[x]:\overline{20}}$  to serve as check values: the tables were then completed by the continuous process

$$a_{[x]:\overline{n}} = a_{[x]:\overline{n-1}} + D_{[x]+n} \cdot D_{[x]}^{-1}$$

In using the values of  $\log vp$  the necessary corrections were made to counteract the ignoring of the sixth and seventh places of decimals in the value of  $\log v$ : but as  $\log p$  was, as has been said, taken only to five places of decimals, the resultant annuity-values may differ by one in the third place from values derived from  $\log p$  as tabulated to seven places in the "Account of Principles and Methods."

I am,

Yours faithfully,

C. KEITH GRANGER.

30, Renfield Street, Glasgow.

16 July 1918.



$0^M$  $\alpha_{[x]n}$ 

5 per-cent.

Duration	10	11	12	13	14	15	16	17	18	19	Duration
	16-916	16-863	16-808	16-750	16-689	16-626	16-559	16-490	16-417	16-341	
0	0000	0000	0000	0000	0000	0000	0000	0000	0000	0000	0
1	0950	0950	0950	0950	0950	0950	0950	0950	0950	0950	1
2	1851	1851	1851	1851	1851	1851	1851	1851	1851	1851	2
3	2705	2705	2705	2705	2705	2705	2705	2705	2705	2704	3
4	3515	3515	3515	3514	3514	3514	3514	3513	3513	3513	4
5	4282	4281	4281	4281	4281	4280	4280	4280	4279	4279	5
6	5008	5008	5007	5007	5007	5006	5005	5005	5004	5004	6
7	5696	5695	5695	5694	5694	5693	5692	5692	5691	5690	7
8	6347	6346	6346	6345	6344	6343	6342	6341	6340	6339	8
9	6963	6962	6961	6961	6960	6958	6957	6956	6955	6953	9
10	7546	7545	7544	7543	7542	7540	7539	7537	7537	7534	10
1	8098	8097	8095	8094	8092	8091	8089	8087	8085	8082	1
2	8620	8618	8617	8615	8613	8611	8609	8606	8604	8601	2
3	9113	9112	9110	9108	9106	9103	9100	9098	9095	9091	3
4	9580	9578	9576	9574	9571	9568	9565	9562	9558	9555	4
15	10-022	10-019	10-017	10-014	10-011	10-008	10-004	10-001	9-997	9-992	15
6	10-439	10-437	10-434	10-431	10-428	10-424	10-420	10-415	10-411	10-406	6
7	10-834	10-831	10-828	10-825	10-821	10-817	10-812	10-807	10-802	10-796	7
8	11-203	11-204	11-201	11-197	11-193	11-188	11-182	11-177	11-171	11-164	8
9	11-561	11-557	11-553	11-548	11-544	11-538	11-532	11-526	11-519	11-512	9
20	11-894	11-890	11-886	11-881	11-876	11-870	11-863	11-856	11-848	11-840	20
1	12-210	12-205	12-200	12-195	12-189	12-182	12-175	12-167	12-159	12-150	1
2	12-508	12-503	12-497	12-491	12-485	12-478	12-470	12-461	12-452	12-442	2
3	12-789	12-784	12-778	12-771	12-764	12-756	12-747	12-738	12-728	12-717	3
4	13-055	13-049	13-043	13-036	13-028	13-019	13-010	12-999	12-988	12-976	4
25	13-307	13-300	13-293	13-285	13-277	13-267	13-257	13-246	13-234	13-220	25
6	13-544	13-537	13-529	13-521	13-511	13-501	13-490	13-478	13-465	13-450	6
7	13-768	13-760	13-752	13-743	13-733	13-722	13-709	13-696	13-682	13-667	7
8	13-980	13-971	13-962	13-952	13-941	13-930	13-916	13-902	13-887	13-870	8
9	14-179	14-170	14-160	14-150	14-138	14-125	14-111	14-096	14-079	14-061	9
30	14-367	14-358	14-347	14-336	14-323	14-309	14-294	14-278	14-260	14-241	30
1	14-545	14-535	14-523	14-511	14-498	14-483	14-466	14-449	14-430	14-409	1
2	14-712	14-701	14-689	14-676	14-662	14-646	14-628	14-610	14-589	14-567	2
3	14-870	14-858	14-845	14-831	14-816	14-799	14-780	14-760	14-739	14-715	3
4	15-018	15-006	14-992	14-977	14-961	14-943	14-923	14-902	14-879	14-854	4
35	15-158	15-145	15-130	15-114	15-097	15-078	15-057	15-034	15-010	14-983	35
6	15-290	15-275	15-260	15-243	15-225	15-204	15-182	15-158	15-132	15-104	6
7	15-413	15-398	15-382	15-364	15-344	15-323	15-299	15-274	15-247	15-217	7
8	15-530	15-514	15-496	15-477	15-457	15-434	15-409	15-382	15-353	15-322	8
9	15-639	15-622	15-603	15-583	15-562	15-538	15-511	15-483	15-453	15-420	9
40	15-741	15-723	15-704	15-683	15-660	15-635	15-607	15-577	15-545	15-510	40
1	15-837	15-818	15-798	15-775	15-751	15-725	15-696	15-665	15-631	15-594	1
2	15-927	15-907	15-886	15-862	15-837	15-809	15-778	15-746	15-710	15-672	2
3	16-011	15-990	15-968	15-943	15-917	15-887	15-855	15-821	15-784	15-744	3
4	16-090	16-068	16-044	16-018	15-991	15-960	15-926	15-890	15-851	15-809	4
45	16-164	16-141	16-116	16-088	16-059	16-027	15-992	15-955	15-914	15-870	45
6	16-232	16-208	16-182	16-154	16-123	16-089	16-053	16-014	15-971	15-925	6
7	16-296	16-271	16-244	16-214	16-182	16-147	16-109	16-068	16-024	15-976	7
8	16-355	16-329	16-301	16-269	16-236	16-200	16-160	16-117	16-071	16-022	8
9	16-411	16-383	16-353	16-321	16-286	16-248	16-207	16-163	16-115	16-064	9
	10	11	12	13	14	15	16	17	18	19	

$0^{[M]}$  $\alpha_{[x]n}$ 

5 per-cent.

Duration	20	21	22	23	24	25	26	27	28	29	Duration
	16·262	16·179	16·092	16·002	15·907	15·810	15·707	15·601	15·490	15·375	
0	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	0
1	·950	·950	·950	·950	·950	·950	·950	·950	·950	·949	1
2	1·851	1·850	1·850	1·850	1·850	1·850	1·850	1·850	1·849	1·849	2
3	2·704	2·704	2·704	2·703	2·703	2·703	2·703	2·702	2·702	2·701	3
4	3·513	3·512	3·512	3·511	3·511	3·510	3·510	3·509	3·508	3·508	4
5	4·278	4·278	4·277	4·276	4·276	4·275	4·274	4·273	4·272	4·271	5
6	5·003	5·002	5·001	5·000	4·999	4·998	4·997	4·996	4·994	4·992	6
7	5·689	5·688	5·686	5·685	5·684	5·682	5·680	5·679	5·676	5·674	7
8	6·338	6·336	6·335	6·333	6·331	6·329	6·327	6·324	6·321	6·319	8
9	6·951	6·949	6·947	6·945	6·943	6·940	6·937	6·934	6·930	6·927	9
10	7·531	7·529	7·527	7·524	7·521	7·518	7·514	7·510	7·505	7·501	10
1	8·080	8·077	8·074	8·071	8·067	8·063	8·058	8·054	8·048	8·042	1
2	8·598	8·594	8·591	8·587	8·582	8·578	8·572	8·567	8·560	8·553	2
3	9·088	9·083	9·079	9·074	9·069	9·063	9·057	9·050	9·042	9·035	3
4	9·550	9·545	9·540	9·535	9·528	9·522	9·514	9·507	9·497	9·488	4
15	9·987	9·981	9·976	9·969	9·962	9·954	9·946	9·937	9·926	9·915	15
6	10·400	10·393	10·387	10·379	10·371	10·362	10·352	10·342	10·330	10·318	6
7	10·789	10·782	10·774	10·766	10·756	10·747	10·735	10·724	10·710	10·696	7
8	11·157	11·149	11·140	11·131	11·120	11·109	11·096	11·083	11·068	11·052	8
9	11·504	11·495	11·485	11·475	11·462	11·450	11·436	11·421	11·404	11·386	9
20	11·831	11·821	11·810	11·798	11·785	11·771	11·756	11·739	11·720	11·700	20
1	12·139	12·128	12·116	12·104	12·089	12·073	12·056	12·038	12·017	11·995	1
2	12·430	12·418	12·405	12·391	12·374	12·358	12·338	12·318	12·295	12·271	2
3	12·704	12·691	12·677	12·661	12·643	12·625	12·604	12·582	12·557	12·530	3
4	12·963	12·948	12·932	12·915	12·896	12·875	12·853	12·828	12·801	12·772	4
25	13·206	13·190	13·172	13·154	13·133	13·111	13·086	13·060	13·030	12·999	25
6	13·434	13·417	13·398	13·378	13·355	13·331	13·305	13·276	13·244	13·210	6
7	13·649	13·631	13·610	13·589	13·564	13·538	13·509	13·478	13·444	13·408	7
8	13·851	13·831	13·809	13·786	13·759	13·732	13·701	13·667	13·631	13·591	8
9	14·041	14·020	13·996	13·971	13·942	13·913	13·879	13·844	13·804	13·762	9
30	14·219	14·196	14·171	14·144	14·113	14·081	14·046	14·008	13·966	13·921	30
1	14·386	14·362	14·335	14·306	14·273	14·239	14·201	14·160	14·115	14·068	1
2	14·543	14·516	14·488	14·457	14·422	14·386	14·345	14·302	14·254	14·203	2
3	14·689	14·661	14·630	14·598	14·561	14·522	14·479	14·433	14·383	14·329	3
4	14·826	14·796	14·764	14·729	14·690	14·649	14·603	14·555	14·501	14·444	4
35	14·954	14·922	14·888	14·851	14·810	14·766	14·718	14·667	14·610	14·550	35
6	15·073	15·039	15·003	14·964	14·920	14·874	14·824	14·770	14·710	14·647	6
7	15·184	15·149	15·110	15·069	15·023	14·975	14·921	14·864	14·802	14·735	7
8	15·287	15·250	15·209	15·166	15·118	15·067	15·011	14·951	14·886	14·816	8
9	15·383	15·344	15·301	15·255	15·205	15·151	15·093	15·030	14·962	14·889	9
40	15·472	15·431	15·386	15·338	15·285	15·229	15·167	15·102	15·030	14·955	40
1	15·554	15·511	15·464	15·413	15·358	15·299	15·235	15·167	15·093	15·014	1
2	15·630	15·584	15·535	15·483	15·425	15·364	15·297	15·226	15·148	15·066	2
3	15·699	15·652	15·601	15·546	15·486	15·422	15·352	15·278	15·198	15·113	3
4	15·763	15·714	15·660	15·603	15·541	15·474	15·402	15·326	15·243	15·155	4
45	15·822	15·770	15·715	15·655	15·590	15·521	15·447	15·367	15·282	15·191	45
6	15·875	15·822	15·764	15·702	15·635	15·564	15·486	15·404	15·316	15·223	6
7	15·924	15·868	15·808	15·744	15·675	15·601	15·521	15·437	15·346	15·250	7
8	15·968	15·910	15·848	15·782	15·710	15·634	15·552	15·465	15·372	15·274	8
9	16·008	15·948	15·884	15·815	15·741	15·663	15·579	15·490	15·394	15·294	9
	20	21	22	23	24	25	26	27	28	29	

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5 per-cent.

Duration	30	31	32	33	34	35	36	37	38	39	Duration
	15.255	15.130	15.001	14.866	14.728	14.583	14.434	14.279	14.120	13.955	
0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	0
1	.949	.949	.949	.949	.949	.949	.949	.949	.949	.948	1
2	1.849	1.849	1.849	1.848	1.848	1.848	1.847	1.847	1.846	1.846	2
3	2.701	2.700	2.700	2.699	2.699	2.698	2.697	2.696	2.695	2.694	3
4	3.507	3.506	3.505	3.504	3.503	3.502	3.500	3.499	3.497	3.495	4
5	4.269	4.268	4.267	4.265	4.263	4.261	4.259	4.257	4.254	4.252	5
6	4.990	4.989	4.987	4.984	4.982	4.979	4.976	4.972	4.969	4.965	6
7	5.672	5.669	5.666	5.663	5.660	5.656	5.651	5.647	5.642	5.636	7
8	6.315	6.312	6.308	6.303	6.299	6.294	6.288	6.282	6.276	6.269	8
9	6.923	6.918	6.913	6.908	6.902	6.896	6.888	6.881	6.872	6.863	9
10	7.496	7.490	7.484	7.477	7.470	7.462	7.453	7.443	7.433	7.421	10
1	8.036	8.029	8.022	8.013	8.005	7.995	7.984	7.972	7.959	7.945	1
2	8.546	8.537	8.528	8.518	8.508	8.496	8.483	8.469	8.454	8.437	2
3	9.026	9.016	9.005	8.993	8.981	8.967	8.951	8.935	8.917	8.897	3
4	9.478	9.466	9.454	9.440	9.425	9.409	9.391	9.372	9.351	9.328	4
15	9.903	9.890	9.876	9.860	9.843	9.824	9.803	9.781	9.757	9.731	15
6	10.304	10.288	10.272	10.254	10.235	10.213	10.190	10.164	10.137	10.107	6
7	10.680	10.663	10.645	10.624	10.602	10.578	10.551	10.523	10.491	10.458	7
8	11.034	11.015	10.994	10.971	10.946	10.919	10.889	10.857	10.822	10.784	8
9	11.366	11.345	11.322	11.296	11.268	11.238	11.205	11.169	11.130	11.088	9
20	11.678	11.654	11.629	11.600	11.569	11.536	11.499	11.459	11.416	11.370	20
1	11.971	11.944	11.916	11.884	11.851	11.814	11.773	11.730	11.682	11.631	1
2	12.245	12.216	12.184	12.150	12.113	12.072	12.028	11.980	11.929	11.873	2
3	12.501	12.469	12.435	12.397	12.357	12.313	12.264	12.213	12.156	12.095	3
4	12.741	12.706	12.669	12.628	12.584	12.536	12.484	12.427	12.366	12.301	4
25	12.964	12.927	12.887	12.842	12.795	12.743	12.686	12.626	12.560	12.489	25
6	13.173	13.133	13.089	13.041	12.990	12.934	12.873	12.808	12.737	12.661	6
7	13.368	13.324	13.277	13.226	13.171	13.111	13.045	12.976	12.900	12.819	7
8	13.548	13.502	13.452	13.396	13.337	13.273	13.203	13.129	13.048	12.962	8
9	13.716	13.666	13.613	13.553	13.491	13.422	13.348	13.269	13.183	13.091	9
30	13.872	13.818	13.761	13.698	13.632	13.559	13.480	13.396	13.305	13.208	30
1	14.015	13.959	13.898	13.831	13.761	13.683	13.600	13.511	13.416	13.313	1
2	14.148	14.088	14.024	13.953	13.878	13.797	13.709	13.615	13.514	13.407	2
3	14.270	14.207	14.138	14.064	13.985	13.899	13.807	13.708	13.603	13.490	3
4	14.382	14.315	14.243	14.165	14.082	13.992	13.895	13.792	13.682	13.564	4
35	14.485	14.414	14.339	14.256	14.169	14.075	13.974	13.866	13.751	13.628	35
6	14.578	14.504	14.425	14.339	14.248	14.149	14.044	13.932	13.812	13.685	6
7	14.663	14.586	14.503	14.413	14.318	14.215	14.106	13.989	13.865	13.734	7
8	14.741	14.659	14.573	14.479	14.380	14.274	14.160	14.039	13.911	13.775	8
9	14.810	14.726	14.636	14.538	14.435	14.325	14.207	14.083	13.950	13.811	9
40	14.873	14.785	14.691	14.590	14.484	14.370	14.248	14.120	13.984	13.841	40
1	14.928	14.837	14.740	14.636	14.526	14.408	14.283	14.151	14.012	13.866	1
2	14.978	14.884	14.783	14.676	14.562	14.441	14.313	14.178	14.036	13.886	2
3	15.022	14.925	14.821	14.710	14.594	14.469	14.338	14.200	14.055	13.903	3
4	15.060	14.960	14.853	14.739	14.620	14.493	14.359	14.218	14.070	13.916	4
45	15.094	14.990	14.881	14.765	14.642	14.513	14.376	14.233	14.083	13.926	45
6	15.123	15.017	14.905	14.786	14.661	14.529	14.390	14.245	14.093	13.934	6
7	15.148	15.039	14.925	14.803	14.676	14.542	14.401	14.254	14.100	13.940	7
8	15.169	15.058	14.942	14.818	14.689	14.552	14.410	14.261	14.106	13.945	8
9	15.187	15.074	14.955	14.829	14.699	14.561	14.417	14.266	14.110	13.948	9
	30	31	32	33	34	35	36	37	38	39	



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5 per-cent.

Duration	40	41	42	43	44	45	46	47	48	49	Duration
	13.784	13.610	13.429	13.242	13.051	12.854	12.652	12.445	12.233	12.017	
0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	0
1	.948	.948	.948	.948	.947	.947	.947	.946	.946	.946	1
2	1.845	1.845	1.844	1.843	1.843	1.842	1.841	1.840	1.839	1.837	2
3	2.693	2.692	2.691	2.689	2.687	2.686	2.684	2.682	2.679	2.677	3
4	3.493	3.491	3.489	3.486	3.484	3.481	3.477	3.474	3.470	3.465	4
5	4.248	4.245	4.242	4.238	4.233	4.229	4.224	4.218	4.212	4.205	5
6	4.960	4.956	4.950	4.945	4.938	4.932	4.924	4.916	4.907	4.898	6
7	5.630	5.624	5.617	5.609	5.600	5.591	5.581	5.570	5.558	5.545	7
8	6.261	6.252	6.243	6.233	6.221	6.209	6.196	6.182	6.166	6.149	8
9	6.853	6.842	6.830	6.817	6.803	6.788	6.771	6.752	6.732	6.711	9
10	7.409	7.396	7.381	7.365	7.347	7.328	7.307	7.284	7.259	7.232	10
1	7.930	7.914	7.896	7.876	7.855	7.831	7.806	7.778	7.748	7.716	1
2	8.418	8.399	8.378	8.354	8.328	8.300	8.270	8.237	8.202	8.163	2
3	8.875	8.853	8.827	8.799	8.769	8.737	8.701	8.662	8.621	8.575	3
4	9.303	9.276	9.247	9.214	9.179	9.142	9.100	9.056	9.007	8.955	4
15	9.702	9.671	9.637	9.600	9.560	9.517	9.470	9.418	9.363	9.303	15
6	10.074	10.039	10.001	9.958	9.913	9.864	9.810	9.752	9.690	9.622	6
7	10.420	10.381	10.338	10.290	10.239	10.184	10.124	10.059	9.989	9.914	7
8	10.743	10.699	10.650	10.597	10.540	10.478	10.412	10.339	10.262	10.178	8
9	11.042	10.993	10.939	10.880	10.817	10.749	10.675	10.596	10.510	10.418	9
20	11.319	11.265	11.205	11.141	11.071	10.996	10.915	10.828	10.735	10.634	20
1	11.575	11.516	11.451	11.380	11.304	11.222	11.134	11.039	10.938	10.829	1
2	11.812	11.747	11.676	11.599	11.516	11.428	11.332	11.229	11.120	11.002	2
3	12.029	11.959	11.882	11.799	11.709	11.614	11.511	11.400	11.282	11.156	3
4	12.229	12.153	12.070	11.980	11.884	11.781	11.671	11.553	11.427	11.293	4
25	12.412	12.330	12.241	12.145	12.042	11.932	11.814	11.689	11.555	11.412	25
6	12.579	12.491	12.396	12.293	12.184	12.067	11.942	11.809	11.667	11.517	6
7	12.730	12.637	12.536	12.427	12.311	12.187	12.055	11.914	11.765	11.607	7
8	12.868	12.769	12.661	12.546	12.423	12.292	12.153	12.006	11.849	11.684	8
9	12.992	12.887	12.773	12.652	12.523	12.385	12.239	12.085	11.922	11.750	9
30	13.103	12.992	12.873	12.745	12.610	12.466	12.314	12.153	11.984	11.805	30
1	13.203	13.086	12.961	12.828	12.686	12.536	12.378	12.211	12.035	11.851	1
2	13.291	13.169	13.039	12.899	12.752	12.596	12.432	12.259	12.079	11.889	2
3	13.369	13.242	13.106	12.961	12.808	12.647	12.478	12.300	12.114	11.920	3
4	13.438	13.305	13.164	13.014	12.856	12.690	12.516	12.333	12.143	11.944	4
35	13.498	13.360	13.214	13.059	12.897	12.726	12.547	12.360	12.166	11.963	35
6	13.549	13.407	13.257	13.097	12.930	12.755	12.572	12.382	12.184	11.978	6
7	13.594	13.447	13.292	13.129	12.958	12.779	12.593	12.399	12.198	11.990	7
8	13.631	13.481	13.322	13.155	12.980	12.798	12.608	12.412	12.208	11.998	8
9	13.663	13.509	13.347	13.176	12.998	12.812	12.621	12.422	12.216	12.004	9
40	13.690	13.532	13.366	13.193	13.012	12.824	12.630	12.429	12.222	12.008	40
1	13.711	13.551	13.382	13.206	13.023	12.833	12.637	12.434	12.226	12.011	1
2	13.729	13.566	13.395	13.216	13.031	12.840	12.642	12.438	12.229	12.013	2
3	13.743	13.577	13.404	13.224	13.037	12.844	12.646	12.441	12.230	12.015	3
4	13.754	13.586	13.412	13.230	13.042	12.848	12.648	12.442	12.232	12.015	4
45	13.762	13.593	13.417	13.234	13.045	12.850	12.650	12.444	12.232	12.016	45
6	13.769	13.598	13.421	13.237	13.047	12.852	12.651	12.444	12.233	12.016	6
7	13.774	13.602	13.424	13.239	13.048	12.853	12.651	12.445	12.233	12.016	7
8	13.777	13.605	13.426	13.240	13.049	12.853	12.652	12.445	12.233	12.016	8
9	13.780	13.606	13.427	13.241	13.050	12.854	12.652	12.445	12.233	12.016	9
	40	41	42	43	44	45	46	47	48	49	

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5 per-cent.

Duration	50	51	52	53	54	55	56	57	58	59	Duration
	11.795	11.569	11.338	11.104	10.865	10.623	10.377	10.128	9.876	9.622	
0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	0
1	.945	.945	.944	.944	.943	.942	.942	.941	.940	.939	1
2	1.836	1.835	1.833	1.831	1.829	1.827	1.825	1.823	1.820	1.817	2
3	2.674	2.671	2.668	2.664	2.660	2.656	2.651	2.646	2.640	2.634	3
4	3.461	3.455	3.450	3.444	3.437	3.429	3.421	3.413	3.403	3.393	4
5	4.198	4.190	4.181	4.171	4.161	4.150	4.138	4.124	4.109	4.094	5
6	4.887	4.876	4.863	4.850	4.835	4.819	4.801	4.782	4.761	4.739	6
7	5.531	5.515	5.498	5.480	5.460	5.438	5.414	5.388	5.360	5.330	7
8	6.130	6.110	6.088	6.063	6.037	6.009	5.978	5.944	5.908	5.869	8
9	6.687	6.661	6.633	6.603	6.570	6.534	6.495	6.453	6.407	6.358	9
10	7.203	7.171	7.137	7.099	7.059	7.014	6.967	6.915	6.860	6.800	10
1	7.680	7.642	7.600	7.555	7.506	7.453	7.396	7.334	7.268	7.196	1
2	8.121	8.075	8.026	7.972	7.915	7.852	7.785	7.712	7.635	7.551	2
3	8.526	8.473	8.415	8.353	8.286	8.213	8.136	8.052	7.962	7.866	3
4	8.898	8.837	8.771	8.699	8.622	8.539	8.451	8.355	8.253	8.144	4
15	9.239	9.169	9.094	9.013	8.926	8.832	8.732	8.625	8.510	8.388	15
6	9.550	9.471	9.387	9.296	9.198	9.094	8.982	8.863	8.736	8.601	6
7	9.832	9.745	9.651	9.550	9.442	9.326	9.203	9.072	8.933	8.786	7
8	10.088	9.992	9.888	9.777	9.658	9.532	9.398	9.254	9.103	8.944	8
9	10.319	10.213	10.100	9.979	9.850	9.712	9.567	9.412	9.249	9.078	9
20	10.527	10.411	10.288	10.157	10.018	9.869	9.713	9.547	9.373	9.191	20
1	10.712	10.587	10.455	10.314	10.164	10.005	9.838	9.662	9.478	9.285	1
2	10.877	10.743	10.601	10.450	10.290	10.122	9.945	9.759	9.564	9.362	2
3	11.022	10.880	10.728	10.568	10.399	10.221	10.035	9.839	9.636	9.424	3
4	11.150	10.999	10.838	10.669	10.491	10.304	10.109	9.905	9.694	9.474	4
25	11.261	11.101	10.933	10.755	10.569	10.374	10.170	9.959	9.740	9.514	25
6	11.358	11.190	11.013	10.827	10.633	10.431	10.220	10.002	9.776	9.544	6
7	11.440	11.265	11.081	10.888	10.686	10.477	10.260	10.035	9.804	9.567	7
8	11.511	11.328	11.137	10.937	10.729	10.514	10.291	10.061	9.826	9.584	8
9	11.570	11.380	11.183	10.977	10.764	10.543	10.315	10.081	9.841	9.597	9
30	11.619	11.423	11.220	11.009	10.791	10.565	10.333	10.096	9.853	9.605	30
1	11.659	11.458	11.250	11.034	10.812	10.582	10.347	10.106	9.861	9.611	1
2	11.692	11.486	11.274	11.054	10.827	10.595	10.356	10.114	9.867	9.616	2
3	11.718	11.508	11.292	11.069	10.839	10.604	10.363	10.119	9.870	9.618	3
4	11.738	11.525	11.306	11.080	10.848	10.610	10.368	10.122	9.873	9.620	4
35	11.754	11.538	11.316	11.087	10.854	10.615	10.371	10.125	9.874	9.621	35
6	11.766	11.548	11.323	11.093	10.858	10.618	10.374	10.126	9.875	9.621	6
7	11.775	11.555	11.329	11.097	10.861	10.620	10.375	10.127	9.876	9.622	7
8	11.782	11.560	11.332	11.100	10.862	10.621	10.376	10.127	9.876	9.622	8
9	11.786	11.563	11.335	11.101	10.864	10.622	10.376	10.128	9.876	9.622	9
40	11.790	11.565	11.336	11.102	10.864	10.622	10.376	10.128	9.876	9.622	40
1	11.792	11.567	11.337	11.103	10.865	10.622	10.376	10.128	9.876	9.622	1
2	11.793	11.568	11.338	11.103	10.865	10.623	10.377	10.128	9.876	9.622	2
3	11.794	11.568	11.338	11.104	10.865	10.623	10.377	10.128	9.876	9.622	3
4	11.794	11.569	11.338	11.104	10.865	10.623	10.377	10.128	9.876	9.622	4
45	11.795	11.569	11.338	11.104	10.865	10.623	10.377	10.128	9.876	9.622	45
6	11.795	11.569	11.338	11.104	10.865	10.623	10.377	10.128	9.876	9.622	6
7	11.795	11.569	11.338	11.104	10.865	10.623	10.377	10.128	9.876	...	7
8	11.795	11.569	11.338	11.104	10.865	10.623	10.377	10.128	...	...	8
9	11.795	11.569	11.338	11.104	10.865	10.623	10.377	...	...	...	9
	50	51	52	53	54	55	56	57	58	59	

$0^{[M]}$  $a_{[x]n}$ 

5 per-cent.

Duration	60	61	62	63	64	65	66	67	68	69	Duration
	9.366	9.107	8.847	8.587	8.325	8.063	7.802	7.541	7.281	7.022	
0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	0
1	.938	.937	.936	.934	.933	.931	.930	.928	.926	.923	1
2	1.814	1.810	1.806	1.802	1.798	1.793	1.787	1.781	1.775	1.768	2
3	2.628	2.621	2.613	2.604	2.595	2.585	2.573	2.561	2.548	2.534	3
4	3.381	3.369	3.356	3.341	3.325	3.308	3.289	3.269	3.247	3.223	4
5	4.076	4.058	4.037	4.015	3.991	3.965	3.936	3.906	3.873	3.837	5
6	4.714	4.688	4.658	4.627	4.593	4.557	4.517	4.474	4.428	4.378	6
7	5.297	5.261	5.222	5.180	5.135	5.086	5.034	4.977	4.917	4.851	7
8	5.826	5.780	5.730	5.677	5.619	5.557	5.490	5.418	5.342	5.259	8
9	6.305	6.248	6.186	6.119	6.048	5.971	5.889	5.801	5.708	5.608	9
10	6.735	6.666	6.590	6.510	6.424	6.332	6.234	6.130	6.019	5.901	10
1	7.119	7.037	6.948	6.853	6.752	6.645	6.530	6.409	6.281	6.145	1
2	7.461	7.365	7.262	7.153	7.036	6.913	6.782	6.644	6.498	6.345	2
3	7.763	7.653	7.536	7.411	7.280	7.140	6.993	6.839	6.677	6.508	3
4	8.028	7.904	7.773	7.633	7.487	7.332	7.169	7.000	6.822	6.638	4
15	8.259	8.121	7.976	7.822	7.661	7.491	7.314	7.130	6.938	6.740	15
6	8.459	8.308	8.148	7.981	7.815	7.622	7.431	7.234	7.030	6.819	6
7	8.630	8.466	8.293	8.113	7.924	7.728	7.525	7.315	7.100	6.879	7
8	8.776	8.599	8.414	8.221	8.021	7.813	7.599	7.379	7.154	6.923	8
9	8.898	8.710	8.513	8.309	8.098	7.880	7.656	7.427	7.193	6.956	9
20	9.000	8.801	8.593	8.379	8.158	7.931	7.699	7.462	7.222	6.979	20
1	9.083	8.874	8.658	8.434	8.205	7.970	7.731	7.488	7.243	6.994	1
2	9.151	8.933	8.708	8.477	8.241	7.999	7.754	7.506	7.257	7.005	2
3	9.205	8.980	8.747	8.509	8.267	8.020	7.771	7.519	7.266	7.012	3
4	9.248	9.016	8.777	8.534	8.286	8.035	7.782	7.527	7.272	7.016	4
25	9.281	9.042	8.799	8.551	8.300	8.046	7.790	7.533	7.276	7.018	25
6	9.306	9.063	8.815	8.563	8.309	8.053	7.795	7.536	7.278	7.020	6
7	9.325	9.077	8.826	8.572	8.315	8.057	7.798	7.538	7.279	7.021	7
8	9.338	9.088	8.834	8.578	8.319	8.060	7.800	7.540	7.280	7.021	8
9	9.348	9.095	8.839	8.581	8.322	8.061	7.801	7.540	7.281	7.021	9
30	9.354	9.099	8.843	8.584	8.324	8.062	7.801	7.541	7.281	7.022	30
1	9.359	9.103	8.845	8.585	8.324	8.063	7.802	7.541	7.281	7.022	1
2	9.362	9.105	8.846	8.586	8.325	8.063	7.802	7.541	7.281	7.022	2
3	9.363	9.106	8.847	8.586	8.325	8.063	7.802	7.541	7.281	7.022	3
4	9.364	9.106	8.847	8.586	8.325	8.063	7.802	7.541	7.281	7.022	4
35	9.365	9.107	8.847	8.587	8.325	8.063	7.802	7.541	7.281	7.022	35
6	9.365	9.107	8.847	8.587	8.325	8.063	7.802	7.541	7.281	7.022	6
7	9.365	9.107	8.847	8.587	8.325	8.063	7.802	7.541	7.281		
8	9.366	9.107	8.847	8.587	8.325	8.063	7.802	7.541			
9	9.366	9.107	8.847	8.587	8.325	8.063	7.802				
40	9.366	9.107	8.847	8.587	8.325	8.063		49	48	12.445	
1	9.366	9.107	8.847	8.587	8.325		50	12.017	12.233	12.445	58
2	9.366	9.107	8.847	8.587		51		12.017	12.233	12.445	7
3	9.366	9.107	8.847		52	11.569	11.795	12.017	12.233	12.445	6
4	9.366	9.107		53	11.338		11.795	12.017	12.233	12.445	5
45	9.366		54	11.104	11.338	11.569	11.795	12.017	12.233	12.445	54
	...	55	10.865	11.104	11.338	11.569	11.795	12.017	12.233	12.445	3
	...	10.623		11.104	11.338	11.569	11.795	12.017	12.233	12.445	2
	...		10.865	11.104	11.338	11.569	11.795	12.017	12.233	12.445	1
	...	10.623	10.865	11.104	11.338	11.569	11.795	12.016	12.233	12.445	0
	60	55	54	53	52	51	50	49	48	47	



0<sup>[M]</sup> $\alpha_{[x]n}$ 

5 per-cent.

Duration	70	71	72	73	74	75					Duration
	6'765	6'510	6'257	6'008	5'762	5'519					
0	·000	·000	·000	·000	·000	·000					0
1	·921	·918	·915	·912	·908	·904					1
2	1'760	1'752	1'743	1'733	1'722	1'711					2
3	2'519	2'502	2'483	2'464	2'442	2'419					3
4	3'197	3'169	3'139	3'106	3'071	3'033					4
5	3'798	3'756	3'712	3'663	3'612	3'556					5
6	4'325	4'267	4'206	4'140	4'070	3'995					6
7	4'781	4'707	4'627	4'542	4'452	4'356					7
8	5'172	5'079	4'980	4'875	4'765	4'649					8
9	5'502	5'390	5'272	5'147	5'017	4'880					9
10	5'777	5'646	5'509	5'365	5'215	5'059					10
1	6'003	5'854	5'698	5'536	5'368	5'195					1
2	6'186	6'020	5'847	5'669	5'485	5'296					2
3	6'333	6'150	5'962	5'769	5'571	5'369					3
4	6'448	6'251	6'049	5'843	5'634	5'421					4
15	6'537	6'328	6'114	5'898	5'678	5'457					15
6	6'604	6'384	6'161	5'936	5'709	5'481					6
7	6'654	6'426	6'195	5'962	5'729	5'496					7
8	6'691	6'455	6'218	5'980	5'742	5'506					8
9	6'716	6'475	6'233	5'991	5'751	5'512					9
20	6'734	6'489	6'243	5'998	5'756	5'515					20
1	6'746	6'497	6'249	6'003	5'758	5'517					1
2	6'754	6'503	6'253	6'005	5'760	5'518					2
3	6'759	6'506	6'255	6'007	5'761	5'518					3
4	6'761	6'508	6'256	6'007	5'761	5'519					4
25	6'763	6'509	6'257	6'008	5'761	5'519					25
6	6'764	6'509	6'257	6'008	5'762	5'519					6
7	6'764	6'510	6'257	6'008	5'762	5'519					7
8	6'765	6'510	6'257	6'008	5'762	5'519					
9	6'765	6'510	6'257	6'008	5'762	5'519			38	37	
30	6'765	6'510	6'257	6'008	5'762	5'519		39	38	14'279	
1	6'765	6'510	6'257	6'008	5'762		40	13'955	14'120	14'279	68
2	6'765	6'510	6'257	6'008		41	13'784	13'955	14'120	14'279	7
3	6'765	6'510	6'257		42	13'610	13'784	13'955	14'120	14'279	6
4	6'765	6'510		43	13'429	13'610	13'784	13'955	14'120	14'279	5
35	6'765		44	13'242	13'429	13'610	13'784	13'955	14'120	14'279	64
		45	13'051	13'242	13'429	13'610	13'784	13'955	14'120	14'279	3
	46	12'854	13'051	13'242	13'429	13'610	13'784	13'955	14'120	14'279	2
	12'652	12'854	13'051	13'242	13'429	13'610	13'784	13'955	14'120	14'279	1
59	12'652	12'854	13'051	13'242	13'429	13'610	13'784	13'955	14'120	14'279	59
8	12'652	12'854	13'051	13'242	13'429	13'610	13'784	13'955	14'120	14'279	8
7	12'652	12'854	13'051	13'242	13'429	13'610	13'784	13'955	14'120	14'279	7
6	12'652	12'854	13'051	13'242	13'429	13'610	13'784	13'955	14'119	14'279	6
5	12'652	12'854	13'051	13'242	13'429	13'610	13'784	13'955	14'119	14'278	5
54	12'652	12'854	13'051	13'242	13'429	13'609	13'784	13'954	14'119	14'278	54
3	12'652	12'854	13'051	13'242	13'429	13'609	13'784	13'954	14'118	14'277	3
2	12'652	12'854	13'051	13'242	13'429	13'609	13'783	13'953	14'117	14'275	2
1	12'652	12'854	13'051	13'242	13'428	13'608	13'782	13'952	14'116	14'273	1
0	12'652	12'854	13'050	13'242	13'428	13'608	13'781	13'951	14'113	14'270	0
	46	45	44	43	42	41	40	39	38	37	

$O^{[M]}$ 
$$a_{[x] \overline{n}}$$

5 per-cent.

[illegible]





## THE INSTITUTE OF ACTUARIES.

## PROCEEDINGS OF THE INSTITUTE.—SESSION 1917-1918.

*First Ordinary Meeting, 4 December 1917.*

The President (Mr. S. G. WARNER) in the Chair.

The President delivered an Inaugural Address.

*Second Ordinary Meeting, 14 January 1918.*

The President (Mr. S. G. WARNER) in the Chair.

Mr. William Maxwell Gunn Wilson, F.F.A., was duly elected an Associate of the Institute.

A discussion on "The Financial Management of Life Assurance Companies" was opened by Mr. O. T. Falk, in which the following gentlemen also took part, namely, Sir Gerald Ryan, Sir Alfred Watson, Messrs. W. P. Phelps, G. Marks, A. G. Mackenzie, C. R. V. Coutts, J. Burn, and the President.

*The Seventy-first Annual General Meeting, 3 June 1918.*

The President (Mr. S. G. WARNER) in the Chair.

The proceedings at the Annual General Meeting will be found on page 166.

## REPORT, 1917-1918.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1917-1918, the seventieth year of its existence.

There has been a *decrease* of 36 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-two years later, at 31 March 1907, it was 956. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1908	253	313	421	22	1,009
1909	254	325	400	19	998
1910	259	335	348	21	963
1911	267	339	308	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896
1918	295	332	215	18	860



The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

*Schedule of Membership, 31 March 1918.*

	Fellows	Associates	Students	Corresponding Members	Total
i. Number of Members in each class on 31 March 1917 .	303	344	231	18	896
ii. Withdrawals by					
(1) Death . . . . .	8	11	6	...	41
(2) Resignation or otherwise . . . . .	...	6	10	...	
	295	327	215	18	855
iii. Additions to Membership					
(1) By Election . . . . .	...	2	...	...	5
(2) By Examination . . . . .	...	...	...	...	
(3) By Re-instatement . . . . .	...	2	1	...	
	295	331	216	18	860
iv. Transfers					
(1) By Examination:					
<i>from Associates</i> . . . . .	...	...	...	...	...
<i>to Fellows</i> . . . . .	...	...	...	...	...
	295	331	216	18	860
(2) By Examination:					
<i>from Students</i> . . . . .	...	...	1	...	...
<i>to Associates</i> . . . . .	...	1	...	...	...
v. Number of Members in each class on 31 March 1918 .	295	332	215	18	860

There are also 156 candidates admitted as Probationers, and 70 as Students conditionally on their passing Part 1 of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes since 31 March 1912 have been as follows:

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1913	197	55	1916	172	73
1914	200	67	1917	173	67
1915	188	72	1918	156	70

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of eight Fellows, Messrs. H. W. Andras, D. A. Bumsted, G. D. Doucet, J. Graham, E. A. Newton, W. F. Somerville, G. Todd, and A. H. Turnbull; seven Associates, Messrs. F. S. Blake, G. A. Brown, R. A. Craig, G. S. Fielden, J. Hogg, E. Litchfield, and J. J. Stuckey; and five Students, Messrs. T. N. Askwith, H. J. Hammond, T. Holgate, B. Needell, and H. Orrell. Nine of these Members, namely, Captains F. S. Blake, H. J. Hammond, and J. Hogg, Lieutenants T. N. Askwith, G. D. Doucet, and G. S. Fielden, Sergeant B. Needell, and Privates



T. Holgate and H. Orrell have fallen in the service of their King and Country; as also have three Probationers of the Institute, Captain W. E. Smith, and Lieutenants W. Askham and S. G. Weatherdon. The Council have sent letters of sympathy to the relatives of all who have thus sacrificed their lives in the nation's cause.

Messrs. H. W. Andras and G. Todd had both filled the office of Vice-President, and had for many years been Members of the Council. Mr. Todd had also served the Institute in the capacity of Joint Honorary Secretary, and Mr. Andras rendered valuable assistance for many years as Joint Honorary Librarian.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £1,376. 0s. 6d., as compared with £1,476. 6s. 0d. in the previous year. The Income and Expenditure for the year were £1,812. 2s. 11d. and £1,609. 5s. 2d. respectively.

The number of Members and Probationers on the roll of service with the Army and Navy has, since last year, increased to 403. The Council have to deplore the loss of 47 who have been killed in action or died of wounds.

The position of candidates whose preparation for the Examinations has been affected by the War has been receiving the sympathetic consideration of the Council, and with the view of giving candidates every possible assistance, having regard to the special circumstances, the Council have decided for the present to revise the Syllabus so as to limit the amount of the necessary reading; to request the Board of Examiners to afford some guidance as to such reading; and to hold Examinations more frequently than has hitherto been the practice. The first Examinations under the new Syllabus will be held as soon as possible after the termination of the War.

The revised Regulations and Syllabus of Examinations have already been published in the April Number of the *Journal*.

Mr. Arthur James Cook, who died on the 25 February 1917, bequeathed to the Institute a share in the ultimate residue of his estate for the furtherance of the objects of the G. F. Hardy Memorial Fund, and in remembrance of his long connection with the Institute as an Associate. An intimation has been received to the effect that the Memorial Fund thus benefits to the extent of £135. 11s. 3d., and the Council desire to take this opportunity of placing on record their appreciation of the bequest.

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
23,825 . . . .	Parts of <i>Journal</i> .
730 . . . .	Index to Vols. 1 to 40.
1,708 . . . .	<i>Text-Book</i> , Part I (Revised Edition).
491 . . . .	<i>Text-Book</i> , Part II (Second Edition).
633 . . . .	Government Joint-Life Annuity Tables.
733 . . . .	Select Life Tables.
33 . . . .	A Short Collection of Actuarial Tables (New Edition).
922 . . . .	Frequency-Curves and Correlation (W. P. Elderton).
172 . . . .	Ditto ditto ditto (Addendum and Errata).
38 in cloth } . . .	{ Lectures on Finance and Law (Clare and
2,314 in paper } . . .	{ Wood Hill).

[Continued on page 166.]



year ending 31 March 1918.

Cr.

1917.			1918.		
£	s.	d.	£	s.	d.
220	9	5	260	3	4
48	8	0	43	2	6
268	17	5	303	5	10
112	16	9	102	5	2
156	0	8			
19	8	6			
41	14	9			
18	11	3			
600	0	0	600	0	0
488	17	2	520	9	0
66	19	3	49	17	2
39	14	5	30	7	2
69	4	9	78	17	11
23	17	9	20	5	0
11	7	10	57	13	8
1,300	1	2			
456	8	1			
9,645	13	1			

*Examined and found correct, 25 April 1918.*

11,637	17	6	GEORGE H. LAWTON, E. W. HUMPHRY, STANLEY HAZELL,	Auditors.	£11,457	16	0
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year ending 31 March 1918.

£	s.	d.	£	s.	d.
186	14	0	140	16	11
249	0	2	323	19	2

*Examined and found correct, 25 April 1918.*

£435	14	2	GEORGE H. LAWTON, E. W. HUMPHRY, STANLEY HAZELL,	Auditors.	£464	16	1
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31 March 1918.

£	s.	d.	ASSETS.	£	s.	d.
1,875	0	0	£3,000 Natal 3 per-cent Inscribed Stock . . . . .	1,875	0	0
726	5	0	£1,000 Dominion of Canada 3½ per-cent Registered 1930-50 Stock . . . . .	726	5	0
700	0	0	£1,000 New South Wales 3½ per-cent Inscribed 1930-50 Stock . . . . .	700	0	0
357	15	0	£600 Belgian Government 3 per-cent Sterling Loan of 1914 . . . . .	357	15	0
4,275	0	0	£4,500 5 per-cent War Stock, 1929-47 . . . . .	4,275	0	0
...			£400 National War Bonds, 4 per-cent, 1927 . . . . .	400	0	0
249	0	2	Stock of Publications (excluding <i>Journal</i> ) in hand . . . . .	323	19	2
200	0	0	Cash on Deposit Account . . . . .	...		
441	16	4	Cash on Current Account and in hand . . . . .	355	18	10
70	7	0	Subscriptions in Arrear . . . . .	87	3	0
767	15	9	£810. 5s. 8d. 5 per-cent War Stock, 1929-47—4½ per-cent Converted— (G. F. Hardy Fund) . . . . .	767	15	9

*The Stock Exchange Securities are taken at the values at which they stood on 31 March 1916, and at cost price in the case of any acquired since that date.*

*Examined and found correct, 25 April 1918.*

£9,662	19	3	GEORGE H. LAWTON, E. W. HUMPHRY, STANLEY HAZELL,	Auditors.	£9,868	16	9
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No. of Copies	Description of Work
1,526 . . . .	Lectures on the Companies Acts (A. C. Clauson).
1,189 . . . .	Lectures on the Law of Mortgage (W. G. Hayter).
703 . . . .	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,400 . . . .	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
884 . . . .	Lectures on Stock Exchange Investments (J. Burn).
1,500 . . . .	Lectures on Friendly Society Finance (Sir A. W. Watson).
315 . . . .	South African War Mortality (F. Schooling and E. A. Rusher).
248 . . . .	Life Assurance Law (A. R. Barrand).
654 . . . .	British Offices' Valuation Tables.
652 . . . .	British Offices' 2½ per-cent Temporary Annuity Values.
138 . . . .	Transactions of the Second International Congress of Actuaries.
785 . . . .	Index to Transactions of Seven International Actuarial Congresses.
1,500 . . . .	Examination Questions, 1912-15.

7 May 1918.

#### PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventy-First Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 4 June 1918, Mr. SAMUEL G. WARNER (the President) in the Chair.

Mr. J. DOUGLAS WATSON (Honorary Secretary), read the notice convening the Meeting. The Minutes of the preceding General Meeting were read and confirmed and the Report and Accounts were taken as read.

The PRESIDENT, in moving the adoption of the Report and Accounts, said that there was not much in the Report to which special attention need be drawn. There was a slight decrease in the number of members, due to the present conditions, and the examinations were at a standstill. There was, he regretted to say, a long obituary list. The place of honour on that list, should, he thought, be given to those who had fallen in the service of their country. As they would have seen from the report the total number connected with the Institute who had entered military service was 403, and of those 47 had fallen on the field. Five had suffered that fate since he last addressed the Institute, and he would like to say something about one of them, Mr. H. T. Kay Robinson. Mr. Robinson became a Fellow of the Institute by examination in 1905, joined the army at the outbreak of war and rose to the rank of Lieut.-Colonel and was awarded the Distinguished Service Order and bar for conspicuous gallantry in the field. This, he thought, was a record of which the Institute should be proud. And the Institute was proud of all who had made this supreme sacrifice. They were too young to have gained that professional distinction of which many of them gave promise, but they had attained a distinction which nothing could take from them, a distinction which would live imperishably in the memories of their kindred and their friends.

There would be noticed in the report an expression of regret at the loss of a number of valued members of the profession. Mr. Henry Walsingham Andras was well known to all. He took a keen interest in the Institute, and though his failing health prevented him in recent years from taking an active part in its affairs he had served it as vice-president and librarian, and

was for 11 years a member of the Council. Everybody who knew him appreciated the geniality of his character, the value of his friendship and his service to the Institute. Following in a brief space of time came the death of Mr. George Todd. His actuarial career had much in common with that of Mr. Andras. They were contemporaries, both taking a great interest in the Institute's affairs, and held in high esteem by its members. Mr. Todd was for 23 years a member of the Council and for four years honorary secretary. The remaining two losses by death had a special, and what might be called a historic, interest. In the preamble to their Charter there was a reference to 15 gentlemen there named, of whom it was stated that "being members of a collective body established in the year 1848 under the name of the Actuaries' Club and exercising the calling or profession of actuary," they "are desirous of becoming entitled to such privileges as may arise from the incorporation of the members of the Institute of Actuaries under our Royal Charter." Those fifteen names represent what might be called the actuarial profession of the past; and by the deaths, within the last month or two, of Mr. Priestley and Mr. Stevens the last of them disappeared. A link with the past was thus severed, and the tribute of respectful memory was due to the last survivors of the men who, in long bygone days, did the work and maintained the honour of the profession.

An important subject which required a few words of notice was the question of examinations. A movement which had been on foot for some little time among members of the Council to deal with this subject in view of the immediate future after the war had during the present year taken a practical form, and the revised syllabus approved by the Council was now in their hands in the current number of the *Journal*.

This had been decided upon by the Council after much careful consideration. Its object was to make it easier for those of their members who had not passed all the examinations necessary for the Fellowship when they went to the war to complete their curriculum on their return with a minimum expenditure of time and labour. He might add that the close attention which had thus been given to the general subject of the qualifying study required for their profession might have a value beyond its immediate use, and that some of the simplifications now adopted might assume permanent form.

Another subject about which a few words were necessary was that of the recent alterations in the Institute's bye-laws, agreed to at a general meeting held at the Institute a week or two previously. The object of these alterations was to give by practical means a properly exercisable freedom of choice to the members in the annual selection of new members of Council, and it was hoped that this power would be freely used.

In conclusion, he would only say one word about the national position in which they found themselves that day. Each year since the war began they had met together thus in the hope that before another such gathering took place it would have ended; and each succeeding year had brought disappointment of that hope. To-day the conflict still raged, and the outlook was as stern as at any time since its outbreak.

None the less, their resolution remained unbroken and their faith unshaken. As a part of a great nation, engaged in a branch of its work which they knew to be important, they would go forward into the future sharing to the full the national spirit, with "courage never to submit or yield" till the objects in view had been attained.

Mr. R. R. TILT, in seconding the motion, said that on looking at the prosaic details of the accounts the members would see that since the war there had been a reduction in subscriptions by £600, owing mainly to subscriptions having been remitted by the Council in the case of those members taking part in the war. On the other hand they had savings in tutorial and examination fees. With regard to the issue of the *Journal*, he thought the editors ought to be congratulated on the high standard of



that publication during the period when they had no ordinary sessional meetings to record, and he thought that their work was harder than it had ever been. He wished to call special attention to that.

The motion was unanimously agreed to.

#### ELECTION OF OFFICERS.

A ballot was then taken for the election of the President, Vice-Presidents, Council and Officers for the ensuing year; and the Scrutineers subsequently reported that the following Fellows recommended by the Council had been elected:

#### *President.*

GEOFFREY MARKS, O.B.E.

#### *Vice-Presidents.*

RALPH TODHUNTER, M.A.		JOSEPH BURN, C.B.E.
ARTHUR DIGBY BESANT, B.A.		JAMES DOUGLAS WATSON.

#### *Council.*

SAMUEL JOHN HENRY WALLIS ALLIN, C.B.E.		HENRY EDWARD MELVILLE.
ARTHUR DIGBY BESANT, B.A.		WILLIAM PEYTON PHELPS, M.A.
JOSEPH BURN, C.B.E.		WILLIAM CHARLES SHARMAN.
CHARLES RONALD VAWDREY COUTTS.		JOHN SPENCER.
WILLIAM PALIN ELDERTON.		EDWARD ROBERT STRAKER.
DUNCAN CUMMING FRASER, M.A.		ALFRED CHARLES THORNE.
*LEWIS FREDERICK HOVIL.		RALPH TODHUNTER, M.A.
CHARLES WILLIAM KENCHINGTON.		*EDWARD WILLIAM TOWNLEY.
OWEN KENTISH.		*HAROLD MOLTKE TROUNCER, M.A.
ABRAHAM LEVINE, M.A.		SAMUEL GEORGE WARNER.
*GEORGE JAMES LIDSTONE, F.R.S.E.		SIR ALFRED WILLIAM WATSON.
GEOFFREY MARKS, O.B.E.		JAMES DOUGLAS WATSON.
*REGINALD GEORGE MAUDLING.		ARTHUR THOMAS WINTER.
SIR GEORGE ERNEST MAY, K.B.E.		ERNEST WOODS.
		WILLIAM ARTHUR WORKMAN.
		FRANK BERTRAND WYATT.

#### *Treasurer.*

WILLIAM PEYTON PHELPS, M.A.

#### *Honorary Secretaries.*

ABRAHAM LEVINE, M.A.		WILLIAM PALIN ELDERTON.
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\* New Members of the Council.

Mr. J. C. WARDROP proposed, and Mr. W. G. TITMUS seconded, the election of Messrs. W. Mouat Jones, E. W. Humphry, and Stanley Hazell as auditors for the ensuing year.

The motion was carried unanimously.

Mr. J. G. FRASER moved that a vote of thanks be accorded the auditors for their work during the past year, and added that he would like to take that opportunity of saying that as one of the distant members of the Institute he greatly appreciated the work done in London by the members in the difficult task of carrying on the work of the Institute.

Mr. TARN seconded the motion, and it was carried.

Mr. H. J. BAKER, in proposing a vote of thanks to the President, Vice-Presidents, the Council, and officers for their services during the past year, said that the Institute was to be congratulated on its good fortune in having Mr. Warner as its President during the past two years. The Vice-Presidents, Council, and officers had done very useful work and had



sacrificed for them some of their leisure from their professional duties. Mr. Jarvis, too, had accomplished his task in a most admirable manner. The affairs of the Institute had in a time of great difficulty been carried on in a manner which added materially to its national usefulness and interest, and the least they could do for those who had borne the heat and burden of the day was to accord them a hearty vote of thanks.

Mr. V. MARR, in seconding the vote of thanks, added his testimony of commendation to the work which had been accomplished, and the motion was carried with applause.

Mr. WARNER, in acknowledging the vote on behalf of his colleagues and himself, said that personally he had found great happiness in the service of the Institute during the past two years. To have filled the highest office the profession had to offer was a thing to be marked with a white stone in memory for the years to come. He was very sensible also of all the help he had received during the period from the members of the Council. In vacating the chair, he would just add how gladly he resigned it to so old a friend and so fit a President as Mr. Geoffrey Marks. Mr. Marks had rendered good service to his profession, to the business of life assurance, and to the needs of national service. In his hands they felt sure that the interests of the Institute would be absolutely safe, and in him they would have a worthy representative, of whom they might be proud.

Mr. GEOFFREY MARKS said he need hardly say how grateful he felt at their electing him to the highest honour which members of the profession could possibly achieve in the world. He had the interests of the Institute at heart, and he hoped the occasion would arise when he would be able to prove to them that whatever he had done in the past was but an earnest of what was to follow for the well-being of the Institute. On occasions like these it was usual for the new President to say how difficult it would be for him to do as much as the ex-President had done, and he thought that no one could say that with more force than he. They had had the benefit of Mr. Warner's guidance for the past two years. Mr. Warner had added a new grace and charm to the Presidential office, and he felt that he could hardly hope to live up to the ideal which he had set. Before he sat down, he hoped that they would forgive him for referring to a personal matter. He found that by an accident, which for him was a very happy one, there were on the Council four Fellows who at one part of their career and for the whole of his, had been his Assistant-Actuaries. Not only that, but one of them was a Vice-President and another was a Honorary Secretary. Those four Fellows could have no illusion as to his qualities and defects, and he knew that he could rely on them for the loyal help which they had always given him in other circumstances. He relied on the help of the Council, because he felt that in these days they must stand together and consider the questions which would come before them in a spirit of good fellowship and solidarity.

The proceedings then terminated.

### *Additions to the Library.*

The following works have been added to the Library since the publication of the *Journal* for October 1917 :

Actuaries, Faculty of.

Transactions, Vol. VIII.

"Notes on Compound Interest Formulas and Tables", by the late Sir G. F. Hardy, K.C.B.  
1918.

*By whom presented  
(when not purchased).*

*The Faculty.*

*By whom presented  
(when not purchased).*

**Actuarial Society of America.**

Transactions, 1917-18.

*The Society.*

Containing, *inter alia*—

“The War Revenue Act of 3 October 1917, as it affects Life Insurance Companies”, by E. E. Rhodes.

“The effect of Glycosuria and of Albuminuria on Mortality”, by Dr. Oscar A. Rogers and A. Hunter.

“Reinstatement of Policies”, by R. D. Murphy.

“Observations on the methods and publications of the United States Census Bureau”, by H. H. Wolfenden.

“Beneficial and Relief Associations: Sickness, Accident and Death Benefits to employees and their Dependents”, by A. Hunter.

“Mortality Graphs”, by H. Moir.

“Graduation by Symmetrical Coefficients”, by J. R. Larns, Jr.

“Joint Mortality Experience of the Aetna Life and Travellers’ Insurance Companies on Group Policies”, by E. E. Cammack and E. B. Morris.

“Mortality by order of Birth”, by P. H. Evans.

“Determination of Makeham Graduation Constants by means of Equivalent Ages”, by P. C. H. Papps.

“An Investigation of the mortality prevailing among the American Clergy in its relation to other Classes of the Population and its bearing on a new Standard Table of Mortality”, by W. S. Nichols.

“Note on Double Indemnity Clauses in Life Insurance Contracts”, by W. A. Hutcheson.

**Actuarial Society of New South Wales.**

Transactions, 1918.

*The Society.*

**American Mathematical Society.**

Transactions, 1917-18.

*The Society.*

**American Statistical Association.**

Transactions, 1917-18.

*The Association.*

**Argentine Republic.**

Boletín de la Dirección General de Estadística y Departamento Provincial del Trabajo. Svo. 1918.)

*The Department.*

**Ball (W. W. Rouse).**

A short account of the History of Mathematics. }  
6th edit. Svo. 1915. }

*Purchased.*

**Cantelli (F. P.).**

Sull’ aumento di mortalità dovuto alla guerra. }  
Riflessioni critiche di metodologia statistica. }  
Rome. 1917. }

*The Author.*

**Casualty Actuarial and Statistical Society of America.**

Proceedings, 1917-18.

*The Society.*

*By whom presented  
(when not purchased).*

**Central Control Board (Liquor Traffic).**

Alcohol. Its action on the Human System. Svo. } *Purchased.*  
1918.

**Charlier (C. V. L.).**

Über das Fehlergesetz. 1905.  
Die zweite Form des Fehlergesetzes. 1905.  
Über die Darstellung willkürlicher Funktionen. 1905.  
Weiters über das Fehlergesetz. 1908.  
Die Strenge Form des Bernoulli'schen Theorems. 1909.  
Contributions to the Mathematical Theory of Statistics.  
5. Frequency Curves of type A in Heterograde  
Statistics. 6. The Correlation Function of type A.  
7. Frequency Curves of Compound Functions.  
8. On Multiple Correlation. 1914-15.  
The Radiation Law applied to stellar photometry.  
1915.  
Studies in Stellar Statistics. 3 Parts. 4to. 1912-16. } *Dr. A. Lindstedt.*

**Chartered Insurance Institute, Journal of the**

Vol. XX. Svo. 1917. *The Institute.*

**De Morgan (A.).**

A Budget of Paradoxes. Second Edition by D. Eugene } *Purchased.*  
Smith. 2 vols. Svo. 1915.

**Denmark.**

Beretning fra Forsikringsraadet, 1917.  
Statsanstalten for Livsforsikrings. Den 9de } *The Danish*  
femaarige Risiko-Opgorelse. Copenhagen. 1917. } *Government.*

**Downing (E. R.).**

The Third and Fourth Generation. An Introduction } *Purchased.*  
to Heredity. Svo. Chicago. 1918.

**Durell (Col. A. J. V.).**

The Principles and Practice of the System of Control } *Purchased.*  
over Parliamentary Grants. Svo. 1917.

**Economic Society (Royal).**

Journal of the, 1917-18. *Purchased.*

**Fisher (A.).**

Note on the Construction of Mortality Tables by means } *Dr. F. L. Hoffman.*  
of Compound Frequency Curves. Boston, Mass. }  
1917.

**Hoffman (Dr. F. L.).**

The Malaria Problem in Peace and War. Newark, } *The Author.*  
N.J. 1918.

**Holland.**

Archief voor de Verzekeerings-Wetenschap. 1917-18. *The Society.*  
Nationale Levensverzekering-Bank. Verslag over } *The Company.*  
het jaar 1917.

**Huygens (Christiaan).**

Du Calcul dans les Jeux de Hasard. Rédigé par D. J. } *D. J. Korteweg.*  
Korteweg. Extrait des Oeuvres complètes. }  
Tome XIV. 4to. N.D.

*By whom presented  
(when not purchased).*

- Institute of Bankers.**  
Journal of the, 1917-18. *The Institute.*
- Insurance Institute of New South Wales.**  
Transactions, 1916. *The Institute.*
- Insurance Institute of Toronto.**  
Proceedings, 1917-18. *The Institute.*
- Joffe (S. A.).**  
Interpolation Formulæ and Central Difference Notation. } *The Author.*  
1917. }
- Knibbs (G. H.).**  
Australian Joint Life Tables, 1901-1910. Svo. } *The Author.*  
Melbourne. 1917. }
- Life Offices' Association.**  
Minutes, etc., of the Meetings of the Standing } *The Association.*  
Committee, 1917-18. }
- London Mathematical Society.**  
Proceedings, 1917-18. *The Society.*
- National Health Insurance.**  
Report on the Administration of National Health } *The Government*  
Insurance during the years 1914-17. Svo. 1917. } *Actuary.*  
Reports of the Medical Research Committee, 1915-18. } *Purchased.*  
Svo. }  
Summary of the Principal Provisions of the Act of } *The Government*  
1918. } *Actuary.*
- Nicholson (J. S.).**  
War Finance. Svo. 1917. *Purchased.*
- Parliamentary Papers.**  
Assurance Companies. Returns to the Board of } *The Board*  
Trade. 1917. } *of Trade.*  
Censuses.  
1911. England and Wales. Vol. XIII. Fertility } *The Registrar-*  
of Marriage. Part I. Fol. 1917. } *General.*  
Colonies.  
Canada.  
Report of the Superintendent of Insurance for } *The Government*  
the year 1916. } *Insurance Dept.*  
Insurance Companies. Abstract of Statements }  
for the year 1917. }  
New South Wales.  
Friendly Societies, &c. Report of the Registrar }  
for 1917. } *The Government*  
Official Year Book, 1917. } *of N.S.W.*  
Statistical Register for 1916-17 and previous }  
years. }  
Vital Statistics. Report for 1917 and previous }  
years. }

*By whom presented  
(when not purchased).*

**Parliamentary Papers—continued.**

*Colonies—continued.*

*New Zealand.*

- |  |                                     |
|--|-------------------------------------|
| Friendly Societies. Forty-first Annual Report<br>of the Registrar, 1917.                                   | } <i>The Government<br/>of N.Z.</i> |
| Official Year Book, 1917.  |                                     |
| Statistics of the Dominion for the years<br>1916-17.   |                                     |
| Results of a Census of the Dominion, 1916.<br>Part I—Population. Appendices A-D.<br>Fol. Wellington. 1918. |                                     |

*Victoria.*

- |  |  |
|--|--|
| Friendly Societies. Thirty-ninth Annual<br>Report of the Government Statist, 1916. | } <i>The Government<br/>of Victoria.</i> |
| Twenty-seventh Annual Report of the<br>Registrar, 1916.                            |  |

*Western Australia.*

- |   |                                     |
|---|-------------------------------------|
| Friendly Societies. Report of Proceedings<br>by the Registrar for the year ended<br>30 June 1917. | } <i>The Government<br/>of W.A.</i> |
| Statistical Register for 1916 and previous<br>years.  |                                     |

*India.*

- |   |                           |
|---|---------------------------|
| Life Assurance Companies. Returns of<br>Companies doing business in British<br>India, 1916. Fol. Simla. 1917. | } <i>H. G. W. Meikle.</i> |
|---|---------------------------|

*Registrar-General. England.*

- |  |                                      |
|--|--------------------------------------|
| Seventy-ninth Annual Report of Births, Deaths,<br>and Marriages, 1916. | } <i>The Registrar-<br/>General.</i> |
|--|--------------------------------------|

**Pearson (James).**

- |   |                     |
|---|---------------------|
| The Elements of the Calculus of Finite Differences,<br>created on the method of Separation of Symbols.<br>8vo. Cambridge. 1849. | } <i>Purchased.</i> |
|---|---------------------|

**Periodicals.**

- |  |  |
|--|--|
| Accountants' Magazine.                   | } <i>Purchased.</i>                    |
| American Bankers' Magazine.              |  |
| American Economic Review.                | } <i>The Editor.</i>                   |
| Bankers' Magazine.                       |  |
| Economist.                               | } <i>Purchased.</i>                    |
| Insurance Record.                        |  |
| Journal of Political Economy. (Chicago). | } <i>The Editor.</i>                   |
| Post Magazine.                           |  |
| Post Magazine Almanack.                  |  |
| The "Secretary."                         | } <i>Institute of<br/>Secretaries.</i> |

**Ross (Lt.-Col. Sir Ronald).**

- |   |                     |
|---|---------------------|
| An application of the Theory of Probabilities to the<br>study of Pathometry. 8vo. 1916. | } <i>Purchased.</i> |
|---|---------------------|

**Royal Astronomical Society.**

- |   |                     |
|---|---------------------|
| Memoirs of the. Vol. LIII, Part III. 4to. 1918. | <i>The Society.</i> |
|---|---------------------|

**Scandinavia.**

- |                                       |                    |
|---------------------------------------|--------------------|
| Skandinavisk Aktuarietidskrift, 1918. | <i>The Editor.</i> |
|---------------------------------------|--------------------|

**Schooten (Franz van).**

- |   |                   |
|---|-------------------|
| Exercitationum Mathematicarum. 4to. 1657. | <i>Purchased.</i> |
|---|-------------------|

*By whom presented  
(when not purchased).*

**"Scientific Monthly."**

New York. April, 1917.

*Purchased.*

Containing, *inter alia*—

"Thrift from the Life Insurance view-point", by  
E. E. Rittenhouse.

"Some fallacies of Compulsory Health Insurance",  
by Dr. F. L. Hoffman.

"Insurance as a basis of Social Economy", by  
H. Fiske.

"Recent developments in the Life Insurance  
field", by R. L. Cox.

"Life Insurance and Life Conservation", by  
E. L. Fisk.

"Life Insurance and the War", by S. S. Huebner.

**Secrist (H.).**

An Introduction to Statistical Methods. 8vo. New York. 1917.

*Purchased.*

**Sellar (A. S.).**

Tables for computing Income Tax at 6/- in the £,  
showing Income Tax and also net amount at  
one inspection. 8vo. 1918.

*The Author.*

**Spain.**

Conferencia de Seguros Sociales, 24-31 October 1917.  
8vo. Madrid. 1917.

*F. Lopez-  
Valencia.*

**Statistical Society (Royal).**

Journal of the, 1917-18.

*The Society.*

**Steffensen (Dr. J. F.).**

On certain inequalities between mean values, and their  
application to actuarial problems. 1918.

*The Author.*

**Stirling (James).**

The Differential Method; or a Treatise concerning  
Summation and Interpolation of Infinite Series.  
Translated by Francis Holliday. 4to. 1749-64.

*Purchased.*

**Sweden.**

Enskilda Försäkringsanstalter år 1916, av Kungl.  
Försäkringsinspektionen.

*The Swedish  
Government.*

Utgjämning av sjutton Svenska Livförsäkringsbolags  
Dödlighetstabeller för Lakareundersökta Livförsäkrade.  
By G. Stoltz. 8vo. Stockholm. 1917.

*The Author.*

**Switzerland.**

Rapport du Brucan Fédéral des Assurances sur les  
Entreprises privées en matière d'Assurances en  
Suisse. 1915.

*The Swiss  
Government.*

**United States of America.**

Official Publications.

First Annual Report on Birth Statistics for the  
Registration Area of the United States, 1915.  
Annual Report of the Bureau of the Census on  
Mortality Statistics, 1915.

*U.S Census  
Bureau.*

Connecticut.

Fifty-third Annual Report of the Insurance  
Commissioner, 1918.

*The Commissioner.*



*By whom presented  
(when not purchased).*

United States of America—*continued.*

Official Publications—*continued.*

Illinois.

Report of Pension Laws Commission. Svo. 1916. } *D. F. Campbell.*

Massachusetts.

Sixty-second Annual Report of the Insurance Commissioner (Life and Miscellaneous), 1916. } *The Commissioner.*

New Jersey.

Reports of the Pension and Retirement Fund Commission. Newark, N.J. 1917-18. } *Dr. F. L. Hoffman.*

New York.

Report on the Pension Funds of the City of New York. Part III. Svo. 1918. } *George B. Buck.*

Proceedings of the Conference on Social Insurance, 5-9 December 1916. Svo. Washington. 1917. } *U.S. Dept. of Labour.*

Wicksell (S. D.).

Some Theorems in the Theory of Probability, with special reference to their importance in the Theory of Homograde Correlation. 1916.  
On Logarithmic Correlation, with an application to the distribution of ages at first marriage. 1917.  
On the Genetic Theory of Frequency. 1917.  
On the Correlation of Acting Probabilities. 1918.  
The Correlation Function of type A, and the Regression of its characteristics. 4to. 1917. } *Dr. A. Lindstedt.*

Withers (H.).

The Business of Finance. Svo. 1918. *Purchased.*

## NEW BYE-LAWS.

At a Special General Meeting of the Fellows and Associates of the Institute of Actuaries, held on 23 May 1918, it was moved that certain alterations be made in the Bye-Laws. The motion was adopted by the Meeting, and the resolutions were duly confirmed at a further Special General Meeting, held on 17 June 1918.

The proposed amendments, as follows, have now received the sanction of the Privy Council :

AT THE COUNCIL CHAMBER, WHITEHALL,

THE 28 JUNE 1918.

By the Lords of His Majesty's Most Honourable Privy Council.

WHEREAS the Institute of Actuaries, in exercise of the powers in that behalf conferred on it by the Charter of the said Institute, did, by Resolution of a General Meeting of the Fellows and Asso-

ciates of the said Institute, duly confirmed at a subsequent General Meeting of the said Fellows and Associates, amend certain of the Bye-Laws of the said Institute, and did rescind certain others of the said Bye-Laws and make others in their stead :

And whereas the said amended and substituted Bye-Laws have in compliance with Article 45 of the Charter been submitted to the Lords of the Council :

NOW, THEREFORE, Their Lordships, having taken the said Bye-laws (a copy whereof is hereunto annexed) into consideration, are pleased to allow the same.

*Almeric FitzRoy.*

#### BYE-LAWS REFERRED TO IN THE FOREGOING ORDER.

At a Special General Meeting of the Fellows and Associates of the Institute of Actuaries, duly convened and held at Staple Inn Hall, on Thursday, the 23 May 1918, and at a subsequent Special General Meeting of the said Fellows and Associates, also duly convened and held at the same place, on Monday, the 17 June 1918, the following portions of Bye-Laws 30, 32, 34, and 35 were rescinded, namely :

30. The portion comprising the words "*shall be elected every year at the Annual General Meeting from the Fellows of the Institute. He*"

32. The portion comprising the words "*and shall be elected every year at the Annual General Meeting from the Fellows of the Institute.*"

34. The portion comprising the words "*shall be elected every year at the Annual General Meeting from the Fellows of the Institute. He*"

35. The portion comprising the words "*shall be elected every year at the Annual General Meeting from the Fellows of the Institute. They*";

and Bye-Laws 4, 5 and 73 were also rescinded, and in substitution thereof the following Bye-Laws, to be numbered respectively 4, 5 and 73, were duly made, passed and confirmed, namely :

"4. (a) The President, the four Vice-Presidents, the Treasurer and the Honorary Secretaries shall go out of office as such at the close of the Annual General Meeting of the year for which they were elected, and, except as provided by Bye-Laws 30, 32, 34 and 35, shall be eligible for re-election.

"(b) The Council shall, at a Council Meeting to be held not less than fifty days before the Annual General Meeting in each year, elect from the then Members of the Council the President, the four Vice-Presidents, the Treasurer and the Honorary Secretaries for the ensuing year. Such election shall be conducted in such manner as the Council may, from time to time, prescribe."

“ 5. (A) At every Annual General Meeting five Members of  
“ the Council (which number shall include any ceasing to hold  
“ office as Members of the Council under Bye-Law 7) shall retire  
“ and five Fellows shall be elected in manner hereinafter  
“ provided to fill the vacancies so arising. The Members so  
“ to retire (apart from any ceasing to hold office as Members  
“ of the Council under Bye-Law 7) shall be chosen by the  
“ Council at a Meeting to be held not less than fifty days before  
“ such Annual General Meeting, and the Members so chosen  
“ shall not be eligible for re-election before the Annual General  
“ Meeting next following that at which they go out of office,  
“ but any Members ceasing to hold office as Members of the  
“ Council under Bye-Law 7 shall be eligible for re-election.

“ (B) The Council shall, not less than forty days before each  
“ Annual General Meeting, send to each Fellow and Associate,  
“ at his usual or last known address, a list containing the  
“ names of those Members of the Council who will retire under  
“ Section A of this Bye-Law and under Bye-Law 7, distinguishing  
“ those eligible and those not eligible for re-election.

“ (C) The Fellows and Associates shall be entitled to nomi-  
“ nate any Fellow or Fellows for election to the vacancies in  
“ the Council arising by retirement under Section A of this  
“ Bye-Law, or under Bye-Law 7 or otherwise, but no nomination  
“ shall be valid unless it be signed by not less than seven  
“ Fellows or Associates (of whom not fewer than four shall be  
“ Fellows) and lodged at the Offices of the Institute not less  
“ than twenty-one days before the date of the Annual General  
“ Meeting.

“ (D) If sufficient nominations to fill up all the vacancies  
“ in the Council, arising as aforesaid, are not so lodged the  
“ Council shall nominate a Fellow or Fellows to fill up any  
“ remaining vacancies.

“ (E) The Council shall, not less than seven days before  
“ each Annual General Meeting, send to each Fellow and  
“ Associate, at his usual or last known address :

“ (a) A list containing the names of those Fellows elected  
“ under Bye-Law 4 to the offices above mentioned,

“ and

“ (b) A list containing the names of the Fellows duly  
“ nominated as aforesaid to fill the vacancies in the Council  
“ arising as aforesaid.

“ (F) If the names in the last-mentioned list shall exceed  
“ the number of vacancies as aforesaid such list shall be the  
“ balloting list for the election at the Annual General Meeting  
“ of Fellows to fill the vacancies in the Council, and at such  
“ Meeting each Fellow and Associate may vote for as many  
“ candidates as there are vacancies to be filled, and, if he shall  
“ vote for more candidates than there are vacancies to be  
“ filled, his balloting paper shall be rejected by the Scrutineers.

“(g) The Scrutineers shall be two or more in number to be chosen previous to the balloting for election from those present by those present at the Meeting, and they shall receive the votes and report the result of their scrutiny to the Chairman before the close of the Meeting.”

“73. At the Annual General Meeting there shall be elected in manner provided by the Bye-Laws the Members to fill the vacancies in the Council arising under section A of Bye-Law 5 or under Bye-Law 7 or otherwise, and the Auditors for the ensuing year.”

The Common Seal of the Institute of Actuaries was hereunto affixed by order of the Council this 17 June 1918, in the presence of



L. S.

GEOFFREY MARKS, *President.*

A. LEVINE, *Joint Hon. Secretary.*

Members of the Council.

## COURSE OF READING

recommended by the Board of Examiners for the guidance of Students in connection with the revised Examination Syllabus (see p. 72 of the present volume of the *Journal*).

*The latest editions of Text-Books are referred to in each case. The references to Transactions of Actuarial Societies have been restricted with the object of directing special attention to particular phases of the subjects. The Board accept no responsibility for any opinions expressed in the papers recommended or in the discussions following the papers, which should also be studied.*

*Questions in Parts III and IV will be as far as possible of a practical nature, involving the application of the principles discussed in the suggested course of reading, and candidates should also be familiar with current actuarial and insurance topics.*

### PART I.

#### SECTION A.

H. S. Hall and S. R. Knight: “Higher Algebra.” (Macmillan & Co.).

*No questions will be set on Interest and Annuities, Advanced Convergency and Divergency of Series, Continued Fractions, Indeterminate Equations of the Second Degree, Theory of Numbers, Inverse Probabilities, Determinants, Elimination, or Cubic and Biquadratic Equations.*

J. Burn and E. H. Brown: "Elements of Finite Differences", Part I. (C. & E. Layton).

Institute *Text-Book*, Part II, chapters xxii, xxiii, and xxiv, sections 1-20.

J. Edwards: "Differential Calculus for Beginners." (Macmillan & Co.).

*Chapters 1-7 and 13, omitting Trigonometrical references.*

J. Edwards: "Integral Calculus for Beginners." (Macmillan and Co.).

*Chapters 1-6 and the general propositions in Chapter 8, omitting Trigonometrical references.*

Institute *Text-Book*, Part I, chapter ix.

*J.I.A.*, vol. xl, p. 116, section 1 (W. P. Elderton); vol. xlv, p. 402, sections 1-5 (G. J. Lidstone and S. E. Macnaghten).

#### SECTION B.

Institute *Text-Book*, Part I (excluding chapters ix and x).

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#### PART II.

Institute *Text-Book*, Part II (excluding chapters xix, xx, xxii, xxiii, and xxiv, sections 1-20).

*J.I.A.*, vol. xxii, p. 407 (T. B. Sprague); vol. xl, p. 302 (Actuarial Note, No. 3); vol. xli, p. 97 (G. J. Lidstone); vol. xliii, p. 99 (W. P. Elderton); vol. xlv, p. 402 (G. J. Lidstone and S. E. Macnaghten).

Transactions of the Faculty of Actuaries, vol. v, p. 130 (W. Borland).

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#### PART III.

##### SECTION A.

W. P. Elderton and R. C. Fippard: "The Construction of Mortality and Sickness Tables." (A. & C. Black).

Supplement to the 75th Annual Report of the Registrar-General, Part I, Life Tables (Cd. 7512, 1914). (The review of this Report, *J.I.A.*, vol. xlix, p. 96, might conveniently be read as an introduction to it.)

Sir George F. Hardy: "The Theory of the Construction of Tables of Mortality, &c.", chapters i and ii. (C. & E. Layton).

*J.I.A.*, vol. xxi, p. 406 (T. B. Sprague); vol. xxvi, p. 77 (T. B. Sprague); vol. xxxii, p. 371 (Sir G. F. Hardy); vol. xxxviii, p. 11 (G. J. Lidstone); vol. xxxviii, p. 501 (Sir G. F. Hardy); vol. xli, p. 348 (G. J. Lidstone); vol. xlvii, p. 548 (Extract from Cd. 6907, 1913).

Journal of the Students' Society, vol. i, No. 3, p. 43. (Editorial); vol. i, No. 2, p. 44 (W. P. Elderton).

#### SECTION B.

*Students should, if possible, study some system of Valuation in actual use, and should possess a general knowledge of current financial conditions.*

*J.I.A.*, vol. xxxvii, p. 57\* (S. G. Warner); vol. xxxvii, p. 453\* (G. King); vol. xxxviii, p. 1 (G. J. Lidstone); vol. xxxviii, p. 69\* (Sir G. H. Ryan); vol. xxxviii, p. 385\* (D. C. Fraser); vol. xl, p. 122 (D. C. Fraser); vol. xli, p. 18\* (J. Buchanan); vol. xlii, p. 145 (G. King); vol. xlii, p. 161 (C. R. V. Coutts); vol. xlii, p. 409 (G. J. Lidstone); vol. xliv, p. 261 (G. J. Lidstone); vol. xlviii, p. 1 (W. P. Elderton); vol. xlviii, p. 121 (A. E. King); vol. l, p. 231 (P. H. McCormack).

*Transactions* of the Faculty of Actuaries, vol. vi, p. 93 (A. Fraser).

Journal of the Students' Society, Special Number on "Valuation of Liabilities and Distribution of Surplus", vol. i, No. 2, p. 24 (E. A. Woodall) and p. 31 (R. C. Fippard); vol. i, No. 4, p. 5 (R. C. Simmonds).

Schedules IV and V of the Assurance Companies Act, 1909, and the Returns relating to Life and Employers' Liability business to the Board of Trade under that Act.

\* These papers, in conjunction with Returns to the Board of Trade, should be studied in the first instance.



## PART IV.

## SECTION A.

*Students should possess a general knowledge of current financial conditions, especially as affecting Trustee and gilt-edged securities.*

- (1) General provisions of the Acts relating to Life Assurance Companies, &c. :

Assurance Companies Act, 1909.

*J.I.A.*, vol. xlv, p. 462 ; vol. xlv, p. 257 (A. R. Barrand).

Friendly Societies Acts, 1896 and 1908, and Collecting Societies and Industrial Assurance Companies Act, 1896 (actuarial and financial aspects only).

Legal Notes in *J.I.A.*, on the above Acts.

Bunyon on the Law of Life Assurance (Fifth Edition), chapters vi-x. (C. & E. Layton).

- (2) Life Interests and Reversions, &c. :

Institute *Text-Book*, Part II, chapter xix.

*J.I.A.*, vol. xxvii, p. 107 (T. B. Sprague) ; vol. xxxvi, p. 81 (correspondence with Inland Revenue) ; vol. xl, p. 317 (C. R. V. Coutts) ; vol. xlvi, p. 406 (G. J. Lidstone).

*Transactions* of the Faculty of Actuaries, vol. i, p. 79 (N. Campbell) ; vol. iii, p. 201 (A. E. Sprague).

Journal of the Students' Society, vol. i, No. 4, p. 50 (Editorial) and No. 5, p. 24 (Editorial).

## SECTION B.

*Students should, if possible, study the practice of Offices with regard to the subjects in this section, including some system of Life Assurance Book-keeping in actual use, and forms of Returns to the Board of Trade.*

- (1) Law relating to Life Assurance Contracts :

Bunyon on the Law of Life Assurance (Fifth Edition), chapters i to v, and chapter xi to end. (C. & E. Layton).

*J.I.A.*, vol. xxxiii, p. 205 (A. R. Barrand); vol. xxxiii, p. 373 (T. B. Sprague); vol. xxxv, p. 371 (G. J. Lidstone); vol. xli, p. 109 (A. R. Barrand).

Legal Notes in *J.I.A.*, vol. xli, p. 409, *et seq.*

*Transactions* of the Faculty of Actuaries, vol. ii, p. 343 (A. E. Sprague).

(2) Life Assurance Accounts, &c. :

A. E. Sprague: "Insurance Companies' Accounts." (C. & E. Layton).

Journal of the Students' Society, vol. i, No. 1, p. 36 (S. E. Macnaghten); vol. i, No. 5, p. 22 (Editorial).

Income Tax Act, 1918:

Sections 32, 33 and 46.

Schedule D, Cases I and II, Rule 15; Case III, Rule 3; Case IV, Rules 1 and 2; Case V, Rules 1, 2 and 3.

General Rules applicable to Schedules A, B, C, D and E: Nos. 19 and 21.

(3) Office Premiums, &c. :

*Transactions* of the Faculty of Actuaries, vol. ii, p. 207 (H. Moir).

Insurance Guide and Handbook, vol. i, chapters v and vi. (C. & E. Layton).

(4) Extra Premiums:

Journal of the Students' Society, Special Number on "Extra Risks."

### SECTION C.

*Students are expected to be familiar with the principal provisions of the National Health Insurance Acts.*

(1) Friendly Societies, &c. :

*J.I.A.*, vol. xxvii, p. 245 (Sir G. F. Hardy).

Sir A. W. Watson: "Lectures on Friendly Societies." (C. & E. Layton).

Sir A. W. Watson : "Account of an Investigation of the Sickness and Mortality Experience of the I.O.O.F. Manchester Unity, 1893-1897." (C. & E. Layton).

Journal of the Students' Society, vol. i, No. 5, p. 7 (R. C. Simmonds), and p. 18 (Editorial).

(2) National Health Insurance :

Actuaries' Report on the Schemes embodied in the National Insurance Bill, 1911 (*J.I.A.*, vol. xlv, p. 406).

\*Reports on the Administration of National Health Insurance (1912-13, Cd. 6907, pp. 16-35 and 552-601 ; 1913-14, Cd. 7496, pp. 29-64 ; 1914-17, Cd. 8890, pp. 7-15). Review, *J.I.A.*, vol. xlviii, p. 109.

\*Reports of the Departmental Committee on Approved Society Finance and Administration (Cd. 8251, Cd. 8396, Cd. 8451). Reviews, *J.I.A.*, vol. l, pp. 99 and 221.

(3) Statistics :

W. P. and E. M. Elderton : "Primer of Statistics." (A. & C. Black).

\*Supplement to the 65th Annual Report of the Registrar-General. Review, *J.I.A.*, vol. xliii, p. 230.

Report on the Administration of National Health Insurance, 1912-13, Cd. 6907, pp. 552-601.

*Transactions* of the Faculty of Actuaries, vol. v, p. 1 (J. C. Dunlop) ; vol. vii, p. 357 (J. C. Dunlop and R. M. Hunter).

\* The Reviews of these Reports might conveniently be read as introductions to them.

## Obituary.

FREDERICK GEORGE GOODYEAR, M.C., Probationer of the Institute,  
2nd Lieutenant, London Regiment.

*Died of Wounds 9 December 1917.*

EDWARD MADDISON DOVE, M.C., Probationer of the Institute,  
Lieutenant, 8th Battalion East Surrey Regiment.

*Killed in Action 23 March 1918.*

HUGH THOMAS KAY ROBINSON, D.S.O. with Bar, Fellow of the Institute, Lieut.-Colonel, Royal Sussex Regiment.

*Killed in Action 26 April 1918.*

ALAN DAVIDSON STEED, Probationer of the Institute, Private, 15th London Rifles (Queen's Westminsters).

*Killed in Action 30 April 1918.*

GEORGE LESLIE LEWIS CARTER, Student of the Institute, Naval Instructor, Royal Naval College, Dartmouth.

*Died on Service 29 July 1918.*

EDWIN CHARLES KAYE CLARKE, M.C., Probationer of the Institute, Captain, London Regiment.

*Killed in Action 31 August 1918.*

HENRY BATTEN KEABLE, Associate of the Institute, Lieut., R.N.V.R., H.M.S. "Manxman."

*Died on Service 11 September 1918.*

ARTHUR SKELTON GREGORY, Probationer of the Institute, 2nd Lieutenant, 6th Battalion Loyal North Lancashire Regiment (attached 2nd Batt. East Lanes.).

*Killed in Action 21 September 1918.*

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# JOURNAL

OF THE

## INSTITUTE OF ACTUARIES.

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*Opening Address by the President, GEOFFREY MARKS, Esq., O.B.E.*

[Delivered 16 December 1918.]

GENTLEMEN,—It is customary on these occasions to refer briefly to those Fellows of the Institute who have died since the last Presidential address. In the ordinary course of nature there have gone Mr. Chas. Stevens and Mr. J. G. Priestley, the last surviving members of the Actuaries' Club who became Fellows under the Charter. They had attained great ages—one rather less and one rather more than 90—and their names recall little to us of this generation except a tradition of courtliness and general culture which were attractive features of the older generation to which they belonged. Two other Fellows, Mr. James Graham and Mr. W. F. Somerville, spent their professional lives largely outside our immediate circle, the one in Australia and the other in the North of England. They were therefore little known to us in London, but they were both loyal members of our body so far as their opportunities allowed them to give proof of it. Mr. H. W. Andras and Mr. Geo. Todd, on the other hand, were closely identified with us and with our work throughout their careers. They had both filled almost every one of our high offices, and their careers and characters showed much in common, especially a personal charm and genius for

friendship, which will ensure their being held in continued and affectionate remembrance. A few weeks ago all who knew him here and at the Actuaries' Club were saddened by the announcement of Vyvyan Marr's sudden death. He too was one of our most loyal sons, always ready to devote to the service of the Institute abilities which had gained for him a considerable reputation in connection with pension-fund work, that difficult branch of our activities which chiefly appealed to him.

In our War record we may properly feel and express a pride mingled with regret. According to the latest returns 424 of our members, a very large proportion of our total number, have enlisted or been mobilized. Of these 51 have been killed in action, 10 have died of wounds, and 6 from other causes while on active service. With all others who have served and died for their country they will always be held in grateful remembrance. Honour and Peace to them!

Many of our members have won distinctions in the War, but I think that without being invidious I may mention especially two, both Fellows—one a soldier and the other a sailor—H. T. Kay Robinson, who unhappily is dead, and W. R. Ashton, whom, having regard to the nature of his services, we may well congratulate on being alive. Robinson at the time of his death had attained the rank of Lieut.-Colonel. He won his first D.S.O. for gallantry at the Schwaben Redoubt, a bar to it for "fine leadership and courage" at "Tower Hamlets," while a second bar was awarded to him after his death. His Brigadier wrote of him that he was a "born leader of men and quite fearless in action," while one of his officers describes him as "the bravest man I have ever met." If there is any consolation for the death of a man so gallant and so honoured it lies in such tributes as these.

We have read recently something of the work of the "Q" boats and of the cool resource, the calm and sustained courage of those who manned them. There is no more noble record of bravery or endurance in the annals even of the Navy. Ashton is one of those who have earned undying fame in this perilous work. He was a member of the R.N.V.R. before the War and was mobilized in 1914, but did not begin his "Q" ship career until 1917. In the course of it he earned the D.S.C. with two bars and the D.S.O., and was promoted



out of course to Staff-Paymaster for services in action. His last action was on the 24 December 1917, when his ship went down after sinking its opponent.

Another Fellow who has earned high distinction is Lieut.-Colonel H. J. Wenyon, who has won the D.S.O. with a bar. Yet another Fellow, Capt. E. H. Lever, on service with the British Forces in Italy, has received the following French decorations: The Legion of Honour, and the Croix de Guerre twice—"with palm" and "with gold star"—the first and second orders. An Associate, T. P. Wansbrough, was also a Lieut.-Colonel when I saw him last. He rapidly attained his rank in the comparatively peaceful but arduous and responsible work of the R.A.S.C., in which his powers of organization were quickly recognized and rewarded.

These are a few outstanding examples of the honours gained by our men, but our records are very incomplete and I earnestly beg that all who have any knowledge of further distinctions will bring them to the notice of the Council. It is due to the Institute, but still more to the men, that this should be done.

Much has been written and much said in the attempt to reconcile what an actuary is to-day with the historical origin of the distinctive name under which he is classified among the professions, and to trace the steps by which the shorthand writer of B.C. 40 developed into the G. F. Hardy, the Lidstone, or the Elderton of our time.

With this interesting but somewhat futile speculation I do not propose to concern myself much to-night.

Nevertheless a definition is necessary to my purpose to-night because I wish:

- (A) To find a comprehensive description of the actuary as he might be under present conditions, and would be if he were to attain the ideal which few or none can reach.
- (B) To see how far the modified training which the Institute now proposes to give to its students will fit them to become, not of course the ideal actuary, but the sound and reliable one, capable of dealing with all the ordinary problems which are likely to come before him.
- (C) To indicate some directions in which the usefulness of our members may be enlarged for the advantage

of the State and the benefit of the Institute and of themselves; and

- (D) To suggest methods by which students may obtain the necessary training and experience for this further work, while still remaining members of the Institute and still regarding it as their *Alma Mater*.

Only thus do I conceive that in these days, and in those which are to come, can the Institute maintain its right to the high place which it has now reached, and fulfil its duties towards its Members and the Commonwealth. It is my belief, and indeed my hope, that one result of the War will be that there will be far less room for the half-hearted or the wholly ineffectual in the future than there has been in the past, whether they be institutions or individuals. I guard myself against misconception by saying that in my view at no previous time has the reputation of the Institute and of the actuary stood higher than it does now, largely because of the conspicuous ability of those members of our body who in the past few years have been associated in their professional capacity with the Government and Government Departments.

(A) What is an Actuary? The best dictionary definition which I know is in Murray's Oxford Dictionary, quoted in the *Journal* (vol. xxvi, p. 389):

"An official in an insurance office, whose duty it is to compile statistical tables of mortality, and estimate therefrom the necessary rates of premium, &c., or one whose profession it is to solve for insurance companies or the public, all monetary questions that involve a consideration of the separate or combined effect of interest and probability in connection with the duration of human life, the average proportion of losses due to fire or other accidents, &c.

My own definition would be rather more general:

"One whose profession it is to devise means to solve all questions involving the application of the theory of probability to human affairs, whether in conjunction with the rate of interest or not, and to apply them to the solution of practical problems."

This definition is intentionally wide, and the ideal which it presents is probably impossible of attainment by any human being. It implies a depth and breadth of knowledge to acquire which would leave no time or room for the practical

experience of men and affairs, without which the practising actuary at least is little or nothing worth.

As we have lately been reminded, a former President said : "An actuary should be a man of general culture, with a knowledge both of books and men, and the more he has of both the better." In this I heartily concur, but in the same paper the President committed himself to the definition of an actuary as a "scientific financier" and in this, as things were then and are at present, I can hardly follow him. The description might serve to satisfy the lukewarm interest of one's neighbour at dinner, and has, in fact, been useful in that connection, but as a practical definition it seems to me to exaggerate in one direction the possibilities of our training, while limiting them in others.

Before I leave this point perhaps I may make one other quotation from an extract printed in the *Journal* (vol. xxx, p. 344.) It may interest you, as it did me, because I think that it shows a wise prevision of the possibilities of actuarial training, and of the true character of the Institute, all the more remarkable because it was written so long ago as 1848, on the occasion of the Institute's foundation, and by a man who was not an actuary. He speaks of :

"The science of which an actuary practises the application, and which, though at present it principally relates to life contingencies, yet must be held to include all contingencies to which calculation of probability can be applied."

Of the Institute, he says :

"Such a society should be as republican in its constitution and as liberal in its principles as those which have preceded it. No man should have any rank except what he makes for himself out of the opinion of his comrades; no class of privileged members should exist."

The Institute has endeavoured during its 70 years of life faithfully to fulfil the conditions of its legitimate existence outlined here by John Francis. So I believe, and it is in that belief that I find the greatest comfort and satisfaction in occupying this chair and addressing you to-night.

(B) As you know under the pressure of the war the Institute has lately revised the syllabus of its examinations. The object of this revision was to modify the examination

standard to such an extent as would relieve those members and students who have spent the last four years in the arduous service of their country from the extended course of study necessary to success in pre-war examinations, while still maintaining a sufficient standard of general and actuarial knowledge to qualify them to act in all the capacities which an actuary is ordinarily called upon to fulfil. This modification is at present to be considered as an emergency measure, but my suggestions to-night are intended to direct your attention to the possibility that the present arrangement may be continued and extended, because I believe that it is along the lines proposed by the Examination Committee and adopted by the Council that the Institute is capable of attaining its widest and fullest development. I may add that this aspect of the case was fully in the minds of the Committee and the Sub-Committee which at the beginning of the year exhaustively examined the subject.

The modified syllabus which is now in your hands has been supplemented by a course of reading drawn up by the Board of Examiners which seems to me to meet admirably all the necessities of students within the limits laid down in the syllabus. In the past as you know unless the student were fortunate enough to have the benefit of the advice of experienced tutors or private coaches (a condition which in itself presents some dangers) his course of reading embraced numbers of papers, not all of equal merit, many of which duplicated his work to a great extent. In the courses of reading now prescribed for each section of the examination he will find all the equipment which is necessary to the ordinary actuary, and sufficient leisure to enable him to explore those byways in which he feels himself to be specially interested.

Certain branches of our work which I think might quite reasonably be made separate subjects of further development are still included in the revised syllabus. Nevertheless, inasmuch as Sections B and C of Part IV are alternative, and as I conceive that the majority of candidates will select Section B in preference to Section C, because the practical experience of most of them will lie in the subjects comprised in Section B, the effect will be that Section C and the subjects of which it treats will become purely voluntary.

I suggest therefore that Section C should be taken out of

the syllabus, and the subjects included in it (and possibly others) be dealt with on lines which I will describe hereafter. Sections A and B might be combined into one final section, and possibly three papers set in it, if it is thought desirable to include in it some questions on law, and on the elements of statistics in which a preliminary course of reading could easily be prescribed. Personally I think that it would not be desirable to exclude from the final examination all questions of law, because many of our every day experiences involve some knowledge of the law relating to life assurance companies and contracts, and of that governing mortgage transactions.

You will observe that I am not reducing the compulsory requirements for the final examination in the case of ordinary students more than they have been reduced by the Council. I think that the syllabus so compressed would provide adequate training for the ordinary actuary, and I am now considering the matter only with reference to him and to his needs.

You will however gather that although I feel that the reduced syllabus provides sufficient training for that class of man, the omission of Section C would prevent it meeting the needs of the young actuary who wishes to specialize in certain branches of our work, while all questions relating to Finance are omitted both from the revised syllabus and from my suggested modifications of it. To this aspect of the question I propose to refer later, and to offer some suggestions as to the mode in which the omission may be remedied on voluntary lines.

(c) In what directions useful to itself and others may the Institute hope to increase or develop its activities? These include in addition to certain aspects and varieties of insurance:

- (a) The great question of National Insurance, whether of health or against unemployment, and of Old Age Pensions.
- (b) Better and more complete treatment, on actuarial as distinguished from statistical lines, of the Census results.
- (c) Public Health and the investigation as a national question of the mortality of special classes, as for instance the Army and the Navy. (This branch of



investigation might be extended almost indefinitely, and with great advantage to the State, particularly if certain forms of industry are nationalized hereafter).

- (d) The question of State Pensions which for many years must be a point of anxious consideration for the State and for every citizen.

It will be seen that these are all public questions, covering a great many possible sub-divisions which I need not specify. All of us for instance can imagine many directions in which actuarial treatment might be more fully applied to the Census results. Great questions of national efficiency are involved in a complete examination of these results, including problems of trade and manufacture, taxation, national thrift, man power, whether for military or for other special purposes, and others which I will not indicate. These kinds of questions are what I had in mind when I suggested earlier in my address the possibility that the Institute and its members might prove of greater assistance to the Commonwealth in the future than they have had the opportunity of being in the past. As we know the Institute as a body and many of its individual members have given much valuable service to the State, but my present point is that if we lay ourselves out to take up these matters as part of an extended curriculum, there will naturally be submitted to us many more national questions of the kind indicated than have come before us in the past.

As regards Insurance there are directions in which actuarial science may still be applied to subjects with which we may be presumed to be more familiar already than with some of those enumerated above. In ordinary life assurance there is hardly room for much that is new, but it is convenient to refer to it here. On the purely technical side, variety and cheapness of policies, improvements in methods of valuation tending to greater expedition and accuracy, increased liberality in conditions, &c., there is probably little to be done. The general soundness of the lines on which life assurance institutions are conducted, including their finance, has, so far as we can tell at present, stood a test more severe than it was ever contemplated should be applied to them. Whatever further trials fall upon them they can hardly be so serious as those through which they have lately passed, let us hope



with complete success, although the final reckoning is yet to come.

But in one phase or incident of ordinary life assurance, the paper which shall be a classic remains to be written. The question of the proper terms on which two life assurance offices should carry out an amalgamation as distinct from a transfer, presents opportunities for wide differences of opinion and of treatment. When and why should the business of one office be kept open and the other closed? When is it advisable to keep both open and amalgamate them? In that case should existing surpluses be allocated to existing policy holders, and if not how should they be divided? On what principles should the future premiums to be charged to new entrants be determined? How should the proper method of allocating future profits between the existing policy holders of both offices and future entrants into the combined office be decided? These are questions which may quite possibly assume considerable importance in the future, and guidance upon them would be useful not only to actuaries but to others, such as directors and policy holders, who are directly interested.

In connection with one section of life assurance, industrial assurance, I think that there are possibilities of betterment into which careful enquiry should be made. The great objection to industrial assurance, in spite of the inestimable benefit which it has been to the cause of national thrift is its expense. On the lines on which this class of assurance is necessarily conducted at present there does not seem to be much hope of a reduction in cost. I would, however, commend to those who are particularly interested in it the system of "group insurance" which is making great progress in America.

This system is not to be confused with that of "collective insurance," on which the late R. P. Hardy read a paper here some years ago. His paper was designed to suggest for consideration, in connection with the question of Old Age Pensions, a system of endowment assurance or deferred annuities for the whole population. He submitted some interesting formulæ, but as a practical question the matter has lost its interest at least for the present. One thought is however forced upon me in reading the discussion, in which practically every speaker, possibly influenced by a solemn warning given in advance by the then President, felt

constrained to avoid anything which might be construed to have even a remote bearing on the political situation. The result was that the discussion consisted largely of ingenious evasions of political references or apologies for approaching them. I mention this only because I feel that if in the conditions which exist this Institute is to take its proper share in public life such an attitude towards political questions is impossible, and indeed childish. I suppose that one of the most burning political questions of the moment is that of Free Trade, but I can conceive some aspects of that matter on which actuarial investigation might throw light, while the whole question is one largely of the correct application of statistics. If my idea is correct some actuary in the near future might wish to read a paper on Free Trade versus Protection, and I sincerely trust that his enterprise would not be discouraged, and that, if it were, it would be on any grounds except the political one.

But to return to the question of "group insurance." Its central idea is that employers should provide out of their profits if possible (and in America apparently it is possible) death benefits for their employees by a premium which is calculated to provide for all the deaths taking place during one year amongst such employees. Only one policy is issued to cover the whole risk. It is a "renewable term" policy and so far as I can gather the soundness of the system depends on the fact that the numbers and age distribution of the employees in a large firm remain fairly constant. I have not seen any formulæ or indeed any reference to the mathematical basis of the system, but there is a non-technical description of it in "The Annals of the American Academy of Political and Social Science" for March 1917.

From this it appears that although the system has been introduced into America only since 1911 it has attained widespread favour amongst large employers. It is stated, for instance, that the Union Pacific Railroad has insured 35,000 employees each to the extent of one year's salary, the total insurance under this one policy amounting to \$30,000,000 while during 1916 one insurance company issued under this plan more than \$40,000,000 of insurance, and I have since heard that the Standard Oil Company has adopted the scheme for 500,000 employees. The unit of insurance recommended is one year's salary with a maximum of \$3,000 and on this

basis the monthly premium is said to be approximately 1 per 1,000 or 1·2 per-cent per annum.

The system has scarcely been in force long enough to warrant any confident statement as to its soundness in the form described, but inasmuch as under the American scheme the whole of the premium is paid by the employer (and large employers of labour in America are not the least shrewd of their race) it may safely be assumed that, at any rate from his point of view, the system is satisfactory. Whether this fact implies the opposite possibilities, namely, that it is unsatisfactory to the employed or to the insurance company, I cannot say, but it is worthy of note that according to the writer of the article to which I am referring the improved conditions which result from the adoption of this insurance scheme, and the closer and more confident relations which are thereby established between employer and employed, have made employers ready to make what at first sight must appear considerable sacrifices in order to introduce it amongst their men. As I know from recent experience one of the greatest difficulties with which large employers have to contend is the continual disturbance caused by the loss and replacement of the lower grades of employee, and this difficulty is that which it is asserted that group insurance largely meets.

I am aware that these are not actuarial considerations, but they have a direct bearing on social economy. Having regard to what I have already said as to the duty of the Institute towards society and also to the fact that industrial assurance would probably be the first object of attack by an ill informed democracy anxious to nationalize insurance, it seems well worth while for actuaries, who are interested in or identified with that form of assurance, to examine this scheme and if it is sound to fit it, or possibly some improvement on it, into their own organizations. At least it will be seen that industrial assurance on these lines is enormously cheapened. As already stated the policy is a yearly renewable one, which implies a very low premium rate, as no large reserve fund need be maintained. There are practically no funds to invest and no expense is incurred in their care. There are no medical fees, and small or no commissions. The cost of collection is infinitesimal and is really borne by the employer. Naturally there is a very large saving in stationery, in accounting, and other staff expenses.

On the other hand it is obvious to us that if the plan is open to the same objections as ordinary "assessmentism" it must be unsound, and should therefore be discouraged. Like all works schemes its solvency depends on that of the employer, but there are the special objections to this particular scheme :

- (1) That the first men to leave the firm which was tending towards insolvency would be the young and healthy. The renewable term premium would then be insufficient to meet claims, and would have to be raised, perhaps to an impossible figure, if it is to be adequate.
- (2) That in the eventual wreck there would be no surrender values, or other form of salvage.\*

As regards other forms of insurance I feel that there is still something to be done in the investigation of sickness experience with a view to the extension of sickness insurance. A large body of facts must have been accumulated in the last few years, and it seems reasonable to suppose, on *a priori* grounds, that an enquiry into these facts might permit such a cheapening of this form of insurance as would make it much more attractive than it is now to those large bodies of workers for whom a cessation of work means a cessation of earned income. To those outside National Health Insurance and particularly to the professional classes, sickness insurance should specially appeal. It would practically take the place of the unemployment insurance of the wage earner, and a more general use of it would fill one of the remaining gaps in a system of general protection against all the "bludgeonings of chance" which our present opportunities and knowledge should make complete.

That there are possibilities in the direction indicated appears to be recognized by the large accident companies and by certain life offices. The former have extended the limits within which their provision for sickness operates, while the latter provide for the suspension of premium payments during total disability from accident and certain forms of sickness, apparently without extra charge.

Accident insurance, including employer's liability, is yet another wide field for the actuary only partially opened up.

Some progress has been made in devising and enunciating correct methods of valuation, but I do not see in the suggested

\* See also J. Burn, *J.I.A.*, vol. xlix, p. 229.

course of reading which accompanies the new syllabus any reference to particular books or papers on the subject, although the student is referred, with sardonic humour, to the returns to the Board of Trade which relate to the employers' liability business. This omission will no doubt be remedied. There is at least one paper on the subject (vol. xlv, p. 101, W. Penman) and possibly there are other writings of which I have no knowledge.

There must be by this time an enormous mass of material a thorough examination of which would throw light on many points, and give valuable general information to the insurance manager, the actuary, and the state. I am aware that one of the objections to the application of the actuarial science to the business of employers' liability is that in this class of insurance the operation of the laws of probability is interrupted or diverted by the independent exercise of the human will on the part of the workman, and of the judges. It seems to me, however, that in a large mass and over long periods of time this objection is either negligible or if it is not, that then its effects must be traceable and measurable, even if they cannot be distinguished from other effects having a different origin but the same result. I confess that I should like to see a comprehensive investigation of the facts relating to all classes of accident insurance accumulated by the large composite companies. The data must be extensive enough to warrant their sub-division into many heads with results which would be both interesting and informing.

Unemployment insurance I do not see mentioned in the syllabus or course of reading. It is naturally still in its infancy, but it is part of the national scheme of insurance, the correct calculation of its rates of premium is certainly an actuarial matter, and therefore it must sooner or later come within the purview of the Institute, if it can be continued at all. Of this I am doubtful. It seems to me that the human and extra-human agencies—politics, strikes, &c., on the one hand, and trade cycles on the other—are disturbing agencies too uncertain for science to measure their effect and too powerful to be neglected. The aim of society should be to lessen unemployment, or to abolish it if possible, by social and industrial reforms or by increased production, rather than to pay for it by insurance. That course is a mere shifting of the burden.



There are various other forms of insurance against minor ills or accidents—the occurrence of twins, for instance—which are usually effected at Lloyd's. All these are susceptible of treatment either actuarially or statistically, and should therefore come within the scope of the Institute's activities. A more extended investigation into the question of insurance against issue is overdue, and should prove both valuable and interesting. But all these are better suited to be the actuary's recreation than to form part of his serious pursuits.

Fire and marine insurance are hardly susceptible of actuarial treatment in the strict sense, although a consideration of their methods and results from a statistical standpoint might very well prove a profitable undertaking for an office which was large enough to provide its own facts and enterprising enough to submit them to expert examination. But that suggestion is only part of a much larger one which I should like to submit to the responsible heads of the great composite companies. It has probably occurred already to the bold and imaginative minds which have made so striking a success of British insurance, but I do not know that it has ever been carried into effect. My proposal is that the large offices should create separate statistical departments, the basis of which would be their existing actuarial departments. These departments would need enlarging and strengthening, and to them would then be referred, not only the usual actuarial work of the life branch, but the statistical examination of all other branches and of the questions arising in them. That such a department would more than pay its way I have no doubt, and if it were found possible, as I think would be the case in time, to include among its duties the constant care and scrutiny, if not the actual management, of the investments, then I am sure that it would be very profitable. The present time or the immediate future, with their growing crop of new and difficult problems, seem very suitable for the initiation of something on these lines.

Before leaving the subject of insurance I should like to refer briefly to the possibility of a new investigation into the mortality of assured lives and annuitants. It is 25 years since data were last collected, and though a general investigation, even if it were necessary, is probably out of the question, and of doubtful permanent value owing to the events of the last four years, yet I feel that some early enquiry into the



effects of the war upon the mortality of the offices is both necessary and desirable.

I have specified certain directions in which lie wider openings for the activities of the Institute and its members in connection with government work. A general consideration of existing conditions prompts the suggestion which I shall make for dealing with the situation as a whole and meeting the need for more and, except in certain departments, better actuarial assistance than has hitherto been available for the service of the State. If we read through a list of government departments we shall see that there is a very large proportion of them in which actuarial knowledge and training must be either indispensable or advantageous. Such are the National Health Insurance Commission, the General Register Office, the Registry of Friendly Societies, the National Debt Office, the War Office, Admiralty, and any other department which deals with its own pensions, or other pensions on a large scale, and the Boards of Inland Revenue and of Trade. There are certainly others in which actuarial knowledge if not indispensable is nearly so, such as the Treasury, India Office, the Colonial Office and the Local Government Board, while the proposed Ministry of Health, if and when it comes into being, must also afford an appropriate field for actuarial work and methods.

Other smaller departments—Savings Banks, Paymaster General's Department, Public Works Loan Board—would probably be the better for it, and in any case actuarial or statistical questions requiring expert advice must arise in them from time to time.

How can the needs which I have indicated best be met? I had hoped, when the appointment of Sir Alfred Watson to the post of government actuary was announced, that that appointment would be a preliminary to the constitution of a special government department of which the government actuary would be the head. This department would be one, not only of actuarial work, but of general statistical enquiry. To it would be referred the preparation and examination of the statistical data collected by all other departments in the course of their particular work, as well as the various special points which would present themselves for consideration. Such a department would necessarily be a large one and the nucleus of it must be formed by the government actuary's

existing staff (at present I imagine largely if not wholly engaged in the National Health Insurance Commission) with the actuaries now attached (whether as actuaries or not) to other government departments—the War Office, National Debt Commission, Somerset House and others.

The new section of the administration would need to be a statistical as well as an actuarial department, having regard to the mass of statistical work which would or should be submitted to it by other departments, such, for instance, as the Board of Trade. But the bulk of its work would probably come from the National Health Insurance Commission, the Ministry of Pensions, and the General Register Office, and would be actuarial. However this may be, I feel that if such a department is to be instituted it should be started and controlled by the government actuary. No other official can have had equally good training for, or so wide an experience of, both kinds of work, the statistical and the purely actuarial.

That the necessity of a government statistical department is realized in other quarters is shown by the introduction into the recent report of the Committee on the Cost of Living of the following paragraph :

“ If it is not too far exceeding the bounds of our terms  
“ of reference, we would add that we have been greatly struck  
“ with the improvement which might be made in the value of  
“ the large body of statistics which the different government  
“ departments regularly collect each for itself, if some authority  
“ existed charged with the duty of keeping them all in line,  
“ ensuring the employment of uniform standards and inter-  
“ related methods and unifying the whole by the application of  
“ principles of scientific co-ordination.”

This recommendation, which is all the more striking because it has only an indirect relation to the subject of the report, is supported by Professor Ashley and Professor Bowley, members of the committee whose authority in such matters is unquestioned.

There is no need to point out in detail the directions in which the new department would prove of value. Its least recommendation is that the cost of it would be infinitesimal compared with the gain to the country which must result say from competent examination of the statistics relating to the trade of our own country now handled by the Board of Trade, and of the similar statistics obtainable from foreign countries.

In one other direction to which I have already referred its advice and assistance would be of the greatest value, even if it did not actually undertake the work, now admirably performed, within the limitations imposed upon them, by the Registrar-General and his staff. It has long been a dream of actuaries and statisticians that in the perfect state there shall be a permanent census office, constantly examining and re-arranging in different lights the results of each census. It is further urged that censuses should be taken at least quinquennially, instead of decennially as at present. The Registrar-General has carried this idea even further. He has pointed out to me that with slight alterations the present machinery of national registration could be adapted to the purpose of an annual census.\*

The social and economic value of a census which would be practically continuous can hardly be over-estimated, and having regard to the comparatively simple character of the main facts required, the Registrar-General's proposal seems eminently practicable. The question which department should carry out the work would no doubt form the subject of a fierce struggle between my friends Sir Bernard Mallet and Sir Alfred Watson, but whichever won the thing would be properly done and we should have got it, which after all is the main consideration.

To National Health Insurance I need hardly refer. We know it to be in good hands and it is fortunate that it is so, for the war must have created for the department a great many problems which will tax to the utmost the skill and judgment of its actuarial and financial advisers. In the approaching valuations of the approved societies, the disturbance of the rates of mortality and sickness both immediate and future, the possibility of increased benefit as a result of political and economic pressure, and the depreciation in pre-war investments, are all facts which must be faced.

I have mentioned State Pensions as a subject which will be one of anxiety to the state and to every thoughtful citizen for many years. In dealing with this matter it is necessary to speak plainly. For obvious reasons it has been taken up enthusiastically by a certain class of politicians, of all shades

\* The Registrar-General has developed his suggestion in a lecture to the Royal Institute of Public Health. The lecture was published in "The Journal of State Medicine" for August 1918, and has been reprinted by Messrs. John Bale, Sons & Danielson, Ltd., 83-91, Great Titchfield Street, London, W. 1.

of opinion, who vie with each other in extravagant promises to the electorate, and in applying pressure to a facile minister. To the lazy and self-seeking nothing is so easy as to be generous with the money of the state, and nothing requires more courage than to resist a demand when resistance lends itself to interested misrepresentation. Already we have seen one pliant minister succeeded by another still more obliging, while official publications contain many protests against laxity of administration by both, and warnings as to its consequences.\* The Government, since the dissolution and therefore without consulting parliament, has increased the scale of pensions by 20 per-cent until June next. "If they do these things in a green tree what shall be done in the dry?" With the example of the United States before him the ordinary taxpayer may well ask the question, while we whose special knowledge and experience tell us how insidiously the burden of pensions grows and how heavy it can become, cannot but regard the whole matter with profound anxiety. In my view it is absolutely essential, both on actuarial and political grounds, that the administration of War Pensions should be removed, as far as possible, and if possible entirely, from the influence of the government and of parliament. It is greatly to be hoped that in the new parliament there may be a body of members sufficiently independent to adopt this view and strong enough to enforce it. But in this respect a study of election addresses is decidedly discouraging.

As a result of the war and of the gathering interest in social problems the question of Superannuation Funds is likely to assume considerable importance in the near future. Existing Funds will need careful examination and the most expert management if they are to escape disaster and fulfil their useful functions. No one can foresee the exact effect of the war on the rates of mortality and retirement in these funds, but we know that some of the direct results may be (a) heavy payments of death benefits in respect of young members; (b) earlier retirement on pension as a result of disability or invalidity; (c) an increase in the future mortality of pensioners. There are indirect results to which my

\* These protests and warnings are too long to quote, but the members may be referred to (a) the report by the Government Actuary, paragraph 6 appended to Cd. 8485, 26 February 1917; (b) The 2nd Report of the Committee on National Expenditure No. 30 of 1918; (c) Report by the Government Actuary, paragraph 5, Cd. 9054, 18 April 1918; (d) Report of Committee of Public Accounts, No. 1000 of 1918, pp. xii and xiii and 227, &c.; (e) Ninth Report of Committee on National Expenditure No. 121 of 1918, paragraphs 27 and 29 (15).

attention has been called in connection with one very important fund and these I should like to bring to your notice. In the first place there is a not unnatural agitation for an increase in the pension of those annuitants who have retired before or in the early days of the war and now find that a pension based on pre-war conditions of pay and prices is wholly inadequate to present needs. The constitution of these superannuation funds is generally very democratic. The rules as to benefits are easily altered, but in my experience it is the rarest thing to find that the funds will stand any increase in benefits, while in many cases a diminution is more or less urgently needed. Again, in those funds where the pension is a function of the final salary or of the average salary of the last few years, to base the pension on the increased salaries or wages which have been given during and in consequence of the war will upset the whole foundation of the funds. Then there is the universal depreciation in securities to be faced, although there is compensation for this in the higher rate of interest which seems likely to prevail for some years.

These points cannot be elaborated here, but they and many others deserve the consideration of all those who are interested in these funds. The matter will become of special interest if the railways are nationalized because in that case it is to be presumed that the government will assume the responsibility for the funds connected with them. Those of us who remember the Report of the Departmental Committee on Railway Superannuation Funds which appeared in 1910 will remember the condition of many of them, and how several of the railway companies dealt with the position by guaranteeing the solvency of their funds. Unless the government take over this liability, which will then be a charge on the taxpayer, the amount of it must be deducted from the purchase price. In that case the value of the junior securities of the companies interested may be somewhat seriously affected.

I have now detailed the main directions in which, as I think, the services of the Institute and of its members can be further utilized. But if they are to be employed to the best advantage then it is necessary that additional training to that which the new curriculum affords should be provided. The members of the Institute will agree with me that if it is in any way possible this training should be given within our walls. I believe that it is possible and in this belief I venture to outline the following scheme.



There are five subjects which seem to me suitable to form the substance of what for convenience I will call post-graduate courses. They are :

- (a) General statistics, including graduation in its more scientific developments.
- (b) National Health Insurance and Unemployment Insurance if unemployment is found to be susceptible of insurance.
- (c) Friendly Societies, Pension Funds, and Widows' and Orphans' Funds.
- (d) Law.
- (e) Economics and finance.

I see no reason why all these subjects should not be taught within this building and all, except possibly the last, by members of our own body. If this is feasible then the objects at which I am aiming will be completely attained.

If I may explain my idea rather more fully it is this :

(1) To enlarge the boundaries of the Institute's work and training so as to cover not only those fields (principally insurance and various forms of thrift) which it has hitherto cultivated, but also some others not now within its special domain.

(2) To secure that our members are properly trained for this work or for one or two branches of it, and do not, as too often in the past, acquire merely a smattering of all branches.

(3) To provide means by which ambitious students may fit themselves for other employment than that to which in practice they have so far confined themselves.

(4) To insure that those whom the Institute has trained, should remain members of it, whatever career they may ultimately select.

I have already referred to the difficulty of finding a definition of an actuary, and the existence of that difficulty implies that the limits of our work are not, even now, very strictly marked off. There can therefore be no objection to enlarging them, even at the expense of sacrificing all existing definitions, including my own. This will at least create opportunities for further ingenuity, and I am not sure that the definition which I have criticized, "a scientific financier" would not hereafter apply with fair accuracy to the actuary who qualified under the syllabus and subsequently took honours in statistics, economics and finance. In any case I cannot conceive any man who should have a better claim to the title than such an actuary.



As I said before, there is no reason why the first four subjects should not be taught here and by our own members. Some persons unaware of the facts might object to the Institute attempting to teach law, but these I would remind that we have among us many barristers, who although they do not practice in the courts, are constantly required to examine various aspects of conveyancing and commercial law, who thus acquire a more general knowledge than many practising barristers, and are certainly capable of imparting it to others. Indeed, if our members are to be taught law we may have to do it ourselves, for I am told on enquiry that the latest regulations might possibly be construed as barring admission to an Inn of Court to actuaries, as to members of certain other professions such as chartered accountants.

As regards economics and finance, I have expressed some doubt whether we could provide lecturers from our own body. Personally I do not think that we should be able to do so entirely, although there are certain aspects of finance, including the choice and care of investments, in which the experience of some of our members is as wide as that of anyone. It includes one class of investments, reversions and life interests, which only actuaries are competent to handle, while we may not unfairly claim that our general knowledge of all the incidents attaching to mortgages is at least equal to that of any other professional body, except lawyers.

But as regards Stock Exchange securities I cannot help feeling that we are not sufficiently in touch, not only with the influences, personal and otherwise, which produce temporary variations in values, but also with those factors and tendencies which make for changes more general and permanent in character, and, on that account, are specially important. But although we cannot supply from amongst us lecturers on all the subjects included under the wide heading of this section it is quite possible that we may be able to find teachers for some of them. For the rest we might rely on the admirable course of the London School of Economics; and in taking this course students have the opportunity of becoming at the same time graduates of London University. It might be feasible, if we can guarantee enough students, for the professors of the School of Economics to lecture here, a course which most of us would prefer if it could be arranged, but we must avoid if possible the overlapping which would be inevitable if the Institute became a separate teaching centre for the same subjects as are now so well taught by the

School of Economics. I confess, however, that I do not at present see how to meet this difficulty.

I attach very great importance to the question of financial education. I am prepared to maintain that the most important function of the administration of a life office is the management of its funds, whether that management be actually vested in the board, a finance committee or the manager. The highest prizes which actuaries, speaking generally, have been able to reach have hitherto been the chief positions in life assurance offices, or in the life branches of the composite companies. Apart from character and personality the attainment of these positions is due in my view not so much to purely actuarial knowledge as to the general training provided by the Institute, and particularly the financial training, inadequate as this has been. How many actuaries who are also chief officers of their companies concern themselves with more than general supervision of the actuarial work? None, I imagine, and although my own opinion is that the chief officer of a life office should be an actuary, this claim must be justified in the view of those responsible for such appointments, and if actuaries are to retain their present almost universal control of life offices, it must be on the strength of wider qualifications than the purely actuarial. It follows, if my views are right, that the complete omission of financial questions from the syllabus imposes on the Council, as the trustees of the general interests of the members, the duty of providing training in finance outside the syllabus. In my opinion scientific finance may quite properly be included in the domain of the actuary, particularly if we assume, as I do, that economics and general statistics will in future also form part of his training.

In that case not only will the actuary retain unchallenged and indeed unchallengeable his position in the insurance world, but, having regard to the character of his training in other respects, I can see no reason why, if he likes to leave that world and possesses the personal qualities indispensable to success in any of the higher walks of life, he should not be capable of treading them with the best. But as I heard Mr. Arthur Balfour say recently: "No education in the world will make up for the original deficiency, when there is a deficiency, in the raw material to be educated. Nothing will compensate for the absence of mother wit. There is no substitute for that energy, originality, tact, insight, courage

“and enterprise which, in commerce as in every other walk of life, are sure and certain instruments of success.”

In this connection there is one further point to which I must refer, namely, the difficulty, however good our theoretical teaching may be, of affording opportunities for practical training. But this difficulty should not be insurmountable if directors and managers regard the movement with sympathy and understanding. They can, for instance, allow members of their staffs who are specializing in finance to attend finance committees or meetings of the board, inviting them to submit opinions and suggestions and to join in discussions. No harm could come from such association, while the value of the investment department as a source of profit must be increased. Individual managers might allow students in their offices access to the investment ledgers and encourage them to make enquiries and suggestions in regard to particular securities. They might also submit to them the offers which daily come before them, asking for a reasoned expression of opinion as to their merits. I believe that the managers would derive at least as much benefit as the students from this course. To assist and encourage the personal contact which is so necessary an element in successful finance, intelligent and tactful students should be allowed to call from time to time on the company's financial connections and urged to make such connections for themselves.

If these “postgraduate” courses are instituted, as I hope that they will be, then the Council or the Board of Examiners must prescribe appropriate courses of reading, and in my view it is essential to its success that the efficiency of the system should be tested by examinations. No one believes that examinations are an infallible test of knowledge, but they put to some proof both it and the training for it. It should be easy to devise a diploma which in the candidate's hands would be witness to the fact that he had gained honours in one or more of the additional subjects, while distinguishing marks or letters could be attached to his name in our annual list of members. If in financial matters we had to rely entirely on the training provided by the London School of Economics then their examinations, and the degree of the London University, would meet the needs which I have indicated.

The course of these remarks leads me to suggest for your consideration whether it is possible and if so whether it is

desirable that the Institute should itself become attached to London University. There are naturally difficulties in the way—the question of matriculation for example—but I see no reason why the influence and importance of the Institute should be in any way diminished by association with the University—rather the reverse. I do not know what are the exact relations between the London School of Economics and the University, but apparently it would not be impossible to reconcile incorporation in the University with the maintenance of the Institute's independent existence and training, while it might be arranged that the University degree, whatever form it took, should be an addition to and not a substitution for our own F.I.A. However, this is a large subject and I have not sufficient knowledge of the details involved to be competent to discuss it. Nevertheless I think that it should be considered, particularly as it seems to me that it is in the direction of co-ordination and centralization of teaching that the times are moving and that, if this is so, then we must go with them or risk losing our position and influence.

Since these words were written the movement towards the institution of degrees in Commerce by the London University inaugurated in July has advanced to the stage that the insurance community has been directly approached by representatives of the University with a request for such co-operation and financial assistance as has already been promised by the banking and other interests. The subject was recently discussed at a meeting at which the University was strongly represented and was referred for consideration to a committee to be nominated by the chairman of the British Insurance Association, Mr. Roger Owen. It is therefore premature to discuss it, but in the draft scheme of examinations for these degrees insurance is only one of five subjects to be covered by one paper. As the remaining subjects for this single paper are (a) Commercial Law, (b) Sale of Goods, (c) Carriage of Goods by Land and Sea, and (d) Elements of the Law relating to Negotiable Instruments, Partnership and Companies, and as about 20 papers have to be taken for the full degree it is obviously impossible that either the teaching of insurance or the test by examination can be sufficient for any practical purpose. On the other hand the full course for the degree is so extensive that it does not seem that students of the insurance institutes would

be able to take it concurrently with their present training, while for actuarial students it would be impossible. Nevertheless if the scheme is ultimately put into practical shape I think that we may assure its promoters that any help which the Institute can give by way of advice or assistance in its special subjects will be forthcoming.\*

This concludes the main portion of my address. The suggestions which I have made might have been thought revolutionary a few years ago, but to us who have been tried in the fires of these last four years a revolution, especially one that is purely domestic, is a small matter. For this reason I commend my proposals to your consideration with the assurance in my own mind that they will be examined and discussed on their merits, and with the sympathy which is due, not to your President or to me, but to any well meant effort to improve the world in which we move, and that particular part of it which is the scene of our own endeavours. Frankly, though the main object which I had in mind in writing this address was to suggest some ways in which the status of the Institute and the profession might be raised and their scope enlarged, yet I hope that in doing this we may increase the value and therefore the emoluments of the individual actuary. I have no patience with the cant which affects to believe that all our efforts should be purely unselfish. We have seen and heard a great deal of it recently in connection with national and international affairs, but I do not notice that certain classes of the community in pursuing their ideals have overlooked their own interests, even though their action seriously endangered the objects which they profess to have in view. Private gain is not incompatible with public good. On the contrary the better the man the more should he be worth not only to himself but also to the community. But within the limits of his opportunities he must work as hard for the community as for himself. Only thus shall he earn, and even possibly receive, the full measure of his reward.

I must ask you to bear with me for yet a few minutes while I refer to a question which, though not at present a burning one, is certainly smouldering, and therefore requires examination and careful handling. I mean the admission of women to the Institute. I am not one of those who think

\* On this subject see a letter from Mr. S. G. Warner, which appeared in the *Insurance Record* of 20 December 1918.



that because women have done nobly in the war, as they have, that therefore they should be invited to undertake all sorts of work and responsibilities for which they are fitted neither by temperament nor by training. Women's brains and physical capacity are no greater and no less now than before the war, but the war has given them the opportunity to prove their precise worth in both these respects. In my opinion they have proved it to be sufficiently high to entitle them to admission, on equal terms with men, to many spheres of activity hitherto closed to them. In the past the refusal so to admit them was due to want of knowledge which has now been supplied, or to prejudice on the part of men, which amounted to a denial of social justice. I do not see therefore why the Institute should affect to be deaf, or should bar its doors when women begin to knock at them. It is unlikely that any but women of high intelligence will seek admission here, and for intellect, whether it resides in man or woman, there is always room in every profession. For these reasons my personal opinion is that women should be admitted to the Institute on equal terms with men, and that if the members decide that this should be done, it should be done completely at the appointed time. I mention this because it has been suggested that for the present women should be admitted only to sit for the examinations. To me this looks like fencing with a question which must be faced. We may be certain that the demand for full admission will not be suppressed, even if at their first examination all the women candidates were to fail, and it is better to make a concession as a result of considered judgment than to yield to importunity.

As I have said, these are only my personal opinions. Officially I may say that while there is no reason why the question should not be discussed at this time, yet there are good and obvious reasons why it should not be decided before the return to our hall of those of our members who have been forth to fight. If, therefore, the matter is pressed I should use such influence as your President may possess and deserve, to prevent action the result of which might be regarded, in existing circumstances, as an injustice, either by absent members on the one hand, or by aspiring women on the other.



*Newton's Interpolation Formulas. Further Notes by* DUNCAN  
C. FRASER, M.A., F.I.A.

THE following references to the subject will be found in the "Commercium Epistolicum" :

*Letter from Leibnitz to Oldenburg, dated 3 February 1672/3.*—The "Commercium Epistolicum" includes ten letters from Leibnitz to Oldenburg, and references to five others, besides four letters from Leibnitz to other mathematicians.

Only one of these, the earliest of the series, discusses questions relating to Finite Differences ; and as it is entirely devoted to this subject and furnishes an interesting indication of the state of knowledge at the time, I have given a version of it in full. It will be observed that while the heading clearly states that it is addressed to Oldenburg, he is referred to in the course of the letter in the third person and by name. It seems to have the form of a memorandum rather than of a letter, and we may suppose it to have been written for private circulation, or possibly with a view to publication in the *Transactions* of the Royal Society, of which Oldenburg was Secretary.

The paragraph in which Leibnitz explains a general rule relating to the differences of any kind of powers is merely a verbal description of the formula

$$a^n - b^n = a^{n-1}(a - b) + b(a^{n-1} - b^{n-1}).$$

In his final paragraph he gives some examples of series with fractional terms, the general term of the  $r$ th series being

$$\frac{r}{(n+2)(n+3) \dots (n+r+1)}.$$

The rest of the letter is taken up with a discussion of the fundamental formula of Finite Differences,  $u_n = (1 + \Delta)^n u_0$ , for the case when  $n$  is an integer ; and with remarks on the properties of the coefficients which are employed.

It appears that Leibnitz and writers before him found no difficulty in writing down the formula for any given integral value of  $n$ , the necessary coefficients being obtained by reference to a table. Leibnitz employed a table drawn from Pascal's *Treatise on the Arithmetical Triangle* (printed in 1654, but not published until 1665) ; an account of which is given in the "History of the Theory of Probability" by Isaac

Todhunter, who mentions the application of the Arithmetical Triangle by Pascal in solving questions of Combinations and Probabilities, and in obtaining the powers [*i.e.*, integral powers] of binomial quantities. The particular point of interest to us in the present connection is that mathematicians before Newton were in the habit of using one and the same table for obtaining the integral powers of binomial quantities, and for expressing any one of a series of values in terms of the initial value and its leading differences.

In vol. xiv of the *Journal* (pp. 1 and 73) references will be found to the early use of methods of Finite Differences by Briggs and by Mouton, both of whom are mentioned in Leibnitz' letter.

*Newton's letter of 13 June 1676.*—This letter needs little comment. In it the first enunciation appears of what we now call the Binomial Theorem, a name not used by Newton; and he gives detailed examples of its application. A series is also given for finding a number from its logarithm, *i.e.*, the exponential series. There are some remarks of a vague and general character on methods of obtaining approximate series which probably refer to methods of differences. In my version I have omitted a short section on the roots of equations.

*Newton's letter of 24 October 1676.*—This letter constitutes the principal document in the controversy between Newton and Leibnitz as to priority in the discovery of the Differential Calculus. The letter is a long one; and in my version I have omitted more than half of it, retaining only those sections (fortunately non-controversial) which contain matter of interest in relation to the subject of interpolation by methods of finite differences. It opens with a clear and detailed account of the process of discovery of the Binomial Theorem; and it is interesting to note that the discovery arose out of a problem in interpolation, the question which Newton set himself to solve being to ascertain the form of the series which would represent the function  $\int (1-x^2)^m \cdot dx$  for a fractional value of  $m$ , the forms of the series for a number of integral values of  $m$  being known. This was an example of the "interpolation of series", *i.e.*, of the insertion of a new series among a number of known series. The solution of this question suggested the solution of the simpler problem of finding the general form of the series for the

expansion of  $(1-x^2)^m$ , for any value of  $m$  integral or fractional, which led at once to the Binomial Theorem; and the Binomial Theorem in turn suggested many of the familiar processes which are included in the general description of "Algebra up to the Binomial Theorem."

In a letter dated 21 June 1677 to Oldenburg, Leibnitz says, "His description of the way in which he was led to some of his very elegant theorems is singularly happy; and what he says on the interpolations of Wallis is especially pleasing, because by this argument a proof of these interpolations is obtained which (so far as I know) had previously been given by induction only."

The date of discovery of the Binomial Theorem is fixed by a note left by Newton in which he says: "In the beginning of the year 1665 I found the method of approximating series and the rule for reducing any dignity [*i.e.*, power] of a binomial into such a series."

Newton's proof depends solely on the properties of the coefficients, and is not affected in any way by the nature of the quantities, algebraical or otherwise, with which they are associated. Remembering that the mathematicians of the time were in the habit of familiarly using the same table of coefficients for obtaining the expansions of the integral powers of binomials, and for expressing any one of a series of known values in terms of the initial value and its leading differences, it will be appreciated that Newton's discovery of a general formula for the expansion of the fractional powers of binomials gave him command at the same time of the fundamental formula of interpolation  $u_n = (1 + \Delta)^n \cdot u_0$ , which he embodied more than twenty years later in Lemma V of Book III of the Principia.

That he at once proceeded to apply his theorem in this direction is suggested by the fact that in the summer of the same year he engaged in extensive calculations of logarithms. In two separate sections of the letter he gives elaborate details of his methods, and in a note dated 4 July 1699, quoted in Brewster's Life of Newton, he says: "In summer 1665, being forced from Cambridge by the plague, I computed the area of the hyperbola at Boothby, in Lincolnshire, to two and fifty figures." The MS. is still preserved in the University Library at Cambridge, and is mentioned in the "Catalogue of the Portsmouth collection of books and papers written

“by or belonging to Sir Isaac Newton” (Cambridge, 1888), under the title “Calculation of the Area of the Hyperbola”, being item No. 4 of sub-section “Early papers by Newton. (Holograph).” An examination of this MS. and of other MSS. in the same collection, for example the “Regula Differentiarum”, which is No. 5 of sub-section “Miscellaneous Mathematical subjects” might bring to light valuable information.

Newton remarks in his letter that at a later date he used other methods which gave logarithms more exactly, and the editors of the “Commercium Epistolicum” refer to the “Geometria Analytica.” This work was first published in 1736 in a translation by Colson and it contains formulas for the interpolation of logarithms derived from the properties of the logarithmic series.

The “Logarithmotechnia, sive methodus construendi logarithma nova, accurata et facilis” (4to. London), of Nicholas Mercator (not Mercator of the Maps), which is mentioned by Newton was published in September 1668. In this work Mercator took the equation to the hyperbola in the form  $y = \frac{1}{1+x}$ ; and by simple division, explaining each step of the process in great detail, he obtained the series  $1 - x + x^2 - x^3 +$ , &c., the integration of which term by term gave the logarithmic series. This was the first time that the operation of division with algebraical symbols had appeared in print and it excited extraordinary interest among mathematicians, though as the editors of the “Commercium Epistolicum” explain it was already known that the expression  $\frac{1}{1-x}$ ,  $x$  being less than unity, represented the sum of the series  $1 + x + x^2 + x^3 +$ , &c., taken to infinity. On his attention being drawn to this publication, Barrow, then Professor of Mathematics at Cambridge, made it known that a young friend of his, by name Newton, had previously arrived at general propositions of which Mercator's example was only a particular case.

Newton mentions that he wrote some papers on series in 1671. Wallis states that these papers, or some of them, were destroyed by fire. Some portions of them appear to have been included in the “Geometria Analytica.”

In a following section of the letter will be found the references, which I have already quoted, to the “Methodus Differentialis.”

In the second of the two sections on the subject of logarithms, the process of interpolation by intervals of one-tenth is repeatedly mentioned, and there can be little doubt that this is the process described in detail in the letter of 8 May 1675 to J. Smith. We may reasonably suppose that the same process is referred to in the final paragraph of my version where Newton says that he had almost decided to describe his method of inserting intermediate terms in the construction of trigonometrical and other tables.

It is a remarkable conclusion that as early as the year 1665, when he was still under the age of 23, Newton appears to have had at his command practically all the methods and facilities of computation which are now in use, with the exception of calculating machines; and when we consider that we owe the originating germ of such machines to Newton's contemporary Pascal, we may realize what a direct and vital connection there is between the ideas of the mathematicians of the 17th century and the practical work of the present-day Actuary.

#### EXTRACT FROM LETTER OF FEBRUARY 1672/3.

*Leibnitz to Oldenburg.*

Recently when I happened to meet the eminent mathematician, Pell, at the house of the famous Boyle, we began to talk about numbers, and I was reminded by our conversation that I had a method of my own of constructing the terms of a series of any kind, either increasing or decreasing continuously, by a class of differences which I call generating differences. If the differences of a given series are found, and the differences of the differences, and the differences arising from the differences of the differences, &c.; and if a series be constructed consisting of the first term, and the first difference, and the first difference of the differences, and the first difference arising from the differences of the differences, &c.. that will be the series of generating differences; so that if the continuously increasing or decreasing series be  $a, b, c, d$ , then putting  $\infty$  as the sign of the difference the generating differences will be:

$$\begin{array}{llll}
 1. & a & 2. & a \infty b \\
 & & 3. & \overline{a \infty b \infty b \infty c} \\
 & & 4. & \overline{a \infty b \infty b \infty c} \infty \overline{b \infty c \infty c \infty d} \\
 & & 4. & \overline{a \infty b \infty b \infty c} \infty \overline{b \infty c \infty c \infty d} \\
 & & 3. & \overline{a \infty b \infty b \infty c} \quad \overline{b \infty c \infty c \infty d} \\
 & 2. & a \infty b & b \infty c \quad c \infty d \\
 1. & a & b & c \quad d
 \end{array}$$



Or, in numbers, if the series be the series of cubes increasing in succession from unity, the generating differences will be 0, 1, 6, 6. I call them by this name, because the terms of the series are produced from them when multiplied in a particular way. Their use appears to be greatest when the generating differences are finite in number, but the terms of the series infinite; as in the example proposed of cube numbers:

		0	0	0		
		6	6	6	6	
	6	12	18	24	30	
1	7	19	37	61	91	
0	1	8	27	64	125	216

When the eminent Pell heard this, he replied that it had already been described by Mouton, Canon of Leyden, from the observation of the most noble Francis Reynald of Leyden, a man long celebrated in the literary world, in a book of Mouton on the apparent diameters of the Sun and Moon. From a letter of Reynald's addressed to Monconisius, and from a diary of the journeys of Monconisius, I had become acquainted with the name of Mouton, and the two works he had in view; namely, the apparent diameters of the luminaries, and a scheme for transmitting the measures of things to posterity: but I did not know that the book had actually appeared. Wherefore I hurried off at once to Oldenburg, the Secretary of the Royal Society, and borrowed the book and found that Pell had spoken truly. But nevertheless I thought that I ought to take pains that no suspicion should remain in their minds of my having wished to appropriate the credit of another man's ideas by suppressing the name of the discoverer, and I hope it will be apparent that I am not in such want of ideas as to be compelled to pilfer those of others. Moreover, I shall vindicate my honour by two arguments; first by showing my rough notes, in which not only my discovery but also the manner and the occasion of the discovery appear: and then by adding some things of the greatest importance not remarked by Reynald and Mouton, which it is not very likely should have been contrived by me in a night, and which could not reasonably be expected to be produced by a mere transcriber.

From my papers it appears that the occasion of the discovery was as follows: I was seeking a method of finding the differences of every kind of powers; just as it is known that the differences of the square numbers are the odd numbers; and I had found a general rule of this kind.

The preceding power of a given order being known, to find the power following (or the reverse) at a given distance, that is the powers of given terms; or to find the differences of the powers of a given order, whatever their distances apart. In the powers of the



next lower order, let the power of the greater term be multiplied by the difference of the terms ; and let the difference of the powers (still in the next lower order) be multiplied by the smaller term. The sum of the products will be the required difference of the powers of the given terms. I had adapted the same rule in such a way that to ascertain the powers of the terms for a higher order it was sufficient to know the powers of the given terms for any lower order. And I showed that what is observed to be the case for squares, namely, that their differences are the odd numbers, is not outside the basis of the rule proposed.

My mind being fixed on these ideas, as in the case of square numbers the differences are the odd numbers, so also I enquired what might be the differences of the cubes ; and since these appeared to be irregular I sought the differences of the differences, until I found the third differences to be all sixes. This observation produced another. For I saw that the terms and the successive differences were generated from the preceding differences in the same way as all the successive terms arise from the primary differences, which I call on that account the generating differences, namely, in this case 0, 1, 6, 6. Having come to this conclusion it remained to find by what kind of addition or multiplication, or combination of these, the successive terms could be produced from the generating differences. And thus by solution and experiment, I perceived the first term, 0, to be composed of the first generating difference, 0, taken once or by itself ; the second term, 1, to be composed of the first generating difference, 0, taken once ; and the second, 1, taken once ; the third term, 8, of the first generating difference, 0, taken once ; the second, 1, taken twice ; and the third, 6, taken once ; for

$$0 \times 1 + 1 \times 2 + 6 \times 1 = 8 ;$$

the fourth term, 27, of the first generating difference, 0, taken once ; the second, 1, taken three times ; the third, 6, taken three times ; and the fourth, 6, taken once ; for

$$0 \times 1 + 1 \times 3 + 6 \times 3 + 6 \times 1 = 27 ;$$

and further calculation proved to me that this was general. This was the occasion of my observation, far otherwise from Mouton's way of approaching it : who happened upon this convenient method of calculation along with Reynald, when he was at work on the construction of his tables. Nor should either he or Reynald have any less praise because Briggs also had in some degree turned his attention to certain methods of this kind in his logarithmic tables, as Pell observes. For me, this much remains ; that I may add some things not remarked by them so as to avoid the reputation of being a transcriber merely ; for in the commonwealth of knowledge it does not matter who made an observation ; the thing that matters is what was observed. First then I direct attention to a question

which is not noticed in Mouton's works, and yet is the head of the whole matter, namely, what are those numbers of which he gives a table to be continued to infinity, by the multiplication of which into the generating differences and by combining the products, the terms of the series may be produced. For you may see, from the very way in which the table is set out on p. 385 of his book, that it has not been sufficiently examined by him; for otherwise it is likely that the table would have been set forth in such a way that the connection and harmony of its numbers would be apparent; unless one is to say that he has been at pains to conceal it; for a part of the table is as follows:

1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

It appears from this table that the relationship of correspondence of the generating numbers is only with the number of the term generated; so that when the term is the fourth it is produced from the first difference taken once, the second difference taken three times, the third taken three times, and the fourth once; and therefore in the same transverse line (4) are placed the numbers 1, 3, 3, 1. But the author has either not observed, or if he has observed it he has concealed that he knew the correspondence of the numbers if they are arranged in columns proceeding from the top downwards in the following manner:

1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

For in this way their real and genuine nature and origin are apparent; that they are in fact the numbers, which I am accustomed to call combinatorial numbers, of which I have written at length in my dissertation on the art of combination, and which others call the numerical orders; unities in the first column; natural numbers in the second column; triangular numbers in the third column; pyramidal numbers in the fourth; triangulo-triangular in the fifth, &c., of which a whole treatise of Pascal's deals under the title of the Arithmetical Triangle; in which nevertheless I have wondered that such a conspicuous and natural property of these numbers has not been observed.

[*Note by Editors of the "Commercium Epistolicum."*—On the contrary, it has been observed. See Pascal's Arithmetical Triangle, published in Paris, in the year 1665, p. 2, where the last definition but one is this:

"The number in each cell is equal to that of the cell which precedes it in the perpendicular column, added to that of the cell which precedes it in its parallel column. So the cell F, that is the number in the cell F, is equal to the cell C plus the cell E; and similarly for the other cells."]

But there is indeed an element of fortune in discovery, which does not always offer the best things to the greatest abilities, but often gives some of them to moderate abilities.

Hence the true nature of these numbers and the construction of the table is perceived, whether concealed or not by Mouton or by Reynald; for any given term of a given column is composed of the preceding term in the same column and of that in the previous column; and it also appears, that it is not a work involving any

troublesome calculation to continue the table set forth by Mouton as he demands, since these series of numbers are now everywhere described and used in calculation.

Moreover, from the observation of Mouton for interpolating proportional means between two extremes, I drew the conclusion that it could be used for continuing the extreme numbers themselves to infinity. He found a use for the rule only when the ultimate differences vanish, or almost vanish; I, however, detected innumerable cases, included in a rule which had been overlooked, where although the differences do not vanish I can from given finite numbers multiplied in a certain way, produce the numbers to infinity of very many series.

From the same foundations, I can work out many problems in progressions either in integers or fractions. For I can add and subtract progressions, and even multiply and divide them, and that very conveniently:

$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{21}$
$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{35}$	$\frac{1}{56}$
$\frac{1}{15}$	$\frac{1}{35}$	$\frac{1}{70}$	$\frac{1}{126}$
&c.	&c.	&c.	&c.

Many other points about these numbers have been noticed by me, of which the above is one of the most important; I have a method of finding the sum of a series of fractions decreasing to infinity, of which the numerator is unity, and the denominators are the triangular or pyramidal numbers, or triangulo-triangular numbers, &c.

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LETTER OF 13 JUNE 1676.

*Newton to Oldenburg.*

(To be communicated to Leibnitz.)

Although the modesty of Leibnitz, in the extracts which you have lately sent me from his letter, attributes great credit to our countrymen for their investigations into Infinite series of which there begins to be talk; nevertheless I feel no doubt that he has not only discovered, as he claims, a method of reducing all sorts of quantities into series of this kind, but also a variety of convenient processes very like ours if not even better.

But since he wishes to know what discoveries have been made in England in the subject, and it happens that several years ago

I engaged in investigations of this kind, I have sent you some points that have occurred to me so that I might meet his wishes at least in some degree.

Fractions are reduced into infinite series by division, and radical quantities by the extraction of roots; by performing these operations on symbols in the same way as they are usually performed on decimal numbers. These are the fundamental principles of such reductions.

But the extractions of roots are greatly shortened by this theorem

$$\overline{P + P.Q.}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} A.Q. + \frac{m-n}{2n} B.Q. + \frac{m-2n}{3n} C.Q. \\ + \frac{m-3n}{4n} D.Q. + \&c.$$

where  $P + P.Q.$  signifies the quantity of which a root, or a power, or a root of a power is to be found:  $P$  is the first term:  $Q$  the remaining terms divided by the first. Also  $\frac{m}{n}$  is the numerical

index of the power of  $P + P.Q.$  whether the power be integral or fractional, positive or negative. For as mathematicians are accustomed to write  $a^2$ ,  $a^3$ , &c., for  $a.a$ ,  $a.a.a$ , &c., so for  $\sqrt{a}$ ,  $\sqrt[3]{a}$ ,  $\sqrt[4]{a}$ , &c., I write  $a^{\frac{1}{2}}$ ,  $a^{\frac{1}{3}}$ ,  $a^{\frac{1}{4}}$ , &c., and for  $\frac{1}{a}$ ,  $\frac{1}{a.a}$ ,  $\frac{1}{a.a.a}$ ,

I write  $a^{-1}$ ,  $a^{-2}$ ,  $a^{-3}$ . And in the same way for  $\frac{a.a}{\sqrt{c.a^3 + b.b.x}}$

I write  $a.a \times \overline{a^3 + b.b.x}^{-\frac{1}{3}}$  and for  $\frac{a.a.b}{\sqrt[3]{c:(a^3 + b.b.x) \times (a^3 + b.b.x)}}$

I write  $a.a.b \times \overline{a^3 + b.b.x}^{-\frac{2}{3}}$ . In the last case if  $\overline{a^3 + b.b.x}^{-\frac{2}{3}}$  be taken to stand for  $\overline{P + P.Q.}^{\frac{m}{n}}$  in the formula,  $P$  will be  $= a^3$ ;  $Q$  will be  $= \frac{b.b.x}{a^3}$ ;  $m = -2$ ,  $n = 3$ . Finally for the terms found in the

quotient in the course of the work, I employ the symbols  $A$ ,  $B$ ,  $C$ ,  $D$ , &c.; namely,  $A$  for the first term,  $P^{\frac{m}{n}}$ ;  $B$  for the second,  $\frac{m}{n} \cdot A.Q$ ; and so on in succession. In other respects the use of the formula will be plain from examples.

[The examples, each of which is expanded to several terms, are

$$(1) \quad (c^2 + x^2)^{\frac{1}{2}}$$

$$(2) \quad (c^5 + c^4x - x^5)^{\frac{1}{5}}. \quad \text{Here } P \text{ may be taken as } c^5 \text{ or as } -x^5. \\ \text{"The former is to be preferred if } x \text{ is very small; } \\ \text{the latter if } x \text{ is very great."}$$

$$(3) \quad N \times \overline{y^3 - a^2y}^{-\frac{1}{2}}$$

$$(4) \quad (d + e)^{\frac{4}{3}}$$

- (5) "In the same way simple powers also are produced."

$$\text{Ex: } (d+e)^7$$

- (6) "Moreover, Division, whether simple or repeated, is accomplished by the same rule."

$$\text{Ex: } \frac{1}{d+e} = (d+e)^{-1}$$

$$(7) \quad (d+e)^{-3}$$

$$(8) \quad N \times (d+e)^{-\frac{1}{3}}$$

$$(9) \quad N \times (d+e)^{-\frac{3}{5}}]$$

By the same rule, expansions of powers, divisions by powers or by radical quantities, and extractions of the higher roots of numbers, are also conveniently performed.

\* \* \* \* \*

It would take too long to describe how, from equations so reduced to infinite series, the areas and the lengths of curves, the volumes and the surfaces of solids, or of any segments of any figures, and their centres of gravity, may be determined; and also how all mechanical curves can be reduced to equations of infinite series of this kind, and problems concerning them resolved just as if they were geometrical curves. It may suffice to review some specimens of such problems; and in these I shall sometimes for the sake of brevity use the letters A, B, C, D, &c., to indicate the successive terms of a series taken from the beginning.

(Nine problems are discussed. The first three relate to trigonometrical functions, and the next three to the ellipse; the eighth to the quadratrix, and the ninth to the spheroid. The seventh is as follows):

- (7) if C D be an hyperbola, whose asymptotes EB, EF make the right angle BEF; and if on EB are erected perpendiculars AC, BD, meeting the hyperbola in C and D; and if EA be called  $a$ ; AC,  $b$ ; and the area CADB,  $z$ ;

$$\text{AB will be} = \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} +, \text{ \&c.}$$

Where the coefficients of the denominators arise from the multiplication into one another continuously of the terms of the A.P. 1, 2, 3, 4, 5, &c. And hence from a given logarithm the number corresponding to it can be found.

From these examples it may be seen how the bounds of Analysis are enlarged by means of infinite equations of this kind; indeed, by their aid the method extends, I had almost said, to all problems except Diophantine and similar questions. The method however



is not quite general except by means of certain further methods of forming series. For there are problems, in which one cannot arrive at infinite series by division, or by the extraction of roots, simple or affected. But there is no time to say what the procedure should be in such cases; nor to describe some other things which I had devised relating to the reduction of infinite into finite series, when the nature of the case permits. For I am writing somewhat briefly because these speculations have begun for some time past to be less interesting to me, so that indeed I have abstained from them now for almost five years.

Nevertheless I shall add one point; that after any problem is expressed in terms of an infinite equation, then various approximations for mechanical use can be found with hardly any labour; which when sought by other methods usually involve much labour and the expenditure of time.

[The letter concludes with an investigation of Huygen's approximate formula for the length of the arc of a circle:

If  $A$  be the chord of the arc,  $B$  the chord of half the arc, and  $r$  the radius, then if  $z$  be the length of the arc,

$$z = \frac{8B - A}{3} \text{ with an error less than } \frac{z^5}{7680r^4} \text{ in excess;}$$

and some further approximations are given relating to the circle, the ellipse, and the hyperbola].

LETTER OF 24 OCTOBER 1676.

*Newton to Oldenburg.*

(To be communicated to Leibnitz.)

I could hardly say with how much pleasure I have read the letters of those illustrious men, Leibnitz and Tschirnhausius.

The method by which Leibnitz obtains convergent series is very elegant indeed, and would have shown the ability of the author if he had written nothing else. But remarks, most worthy of his reputation, which occur here and there in his letter, lead us to hope for the greatest things from him. The diversity of methods by which the same results are obtained is all the more attractive because three methods had already become known to me of arriving at series of this kind; so that I hardly expected that anything new could be communicated to us.

One of my methods I described in my former letter; now I add another; that, indeed, by which I was first led to these series; for I was led to them before I discovered the division (of fractions) and the extractions of roots which I now use. And in the explanation of the Theorem which was set forth at the beginning of my former letter is to be found the foundation which Leibnitz desires from me.

At the beginning of my mathematical studies, when I had fallen in with the works of our celebrated Wallis and came to consider

the series by the interpolation of which he brings out the areas of the circle and of the hyperbola, since in the series of curves whose basis or common axis is  $x$  and whose ordinates are :

$$(1-x^2)^{\frac{0}{2}}, (1-x^2)^{\frac{1}{2}}, (1-x^2)^{\frac{2}{2}}, (1-x^2)^{\frac{3}{2}}, (1-x^2)^{\frac{4}{2}}, (1-x^2)^{\frac{5}{2}}, \&c.,$$

if the areas of the alternate curves, which are

$$x, x - \frac{1}{3}x^3, x - \frac{2}{3}x^3 + \frac{1}{5}x^5, x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7, \&c.,$$

can be interpolated we should have the areas of the intermediate curves the first of which,  $(1-x^2)^{\frac{1}{2}}$ , is the circle; I noted for these interpolations that in every case the first term was  $x$ , that the second terms  $\frac{0}{3}x^3, \frac{1}{3}x^3, \frac{2}{3}x^3, \frac{3}{3}x^3, \&c.$ , were in A.P.: and accordingly that the two first terms of the series to be interpolated must be

$$x - \frac{1}{2} \cdot \frac{x^3}{3}, x - \frac{3}{2} \frac{x^3}{3}, x - \frac{5}{2} \frac{x^3}{3}, \&c.$$

For the insertion of the remaining terms I considered that the denominators 1, 3, 5, 7, &c., were in A.P.; and so the numerical coefficients had to be investigated for the numerators only. But in the given alternate areas these were the figures which express the powers of the number 11; namely, 1; 1, 1; 1, 2, 1; 1, 3, 3, 1; 1, 4, 6, 4, 1, &c.

Then I set myself to enquire how in these groups of figures when the first two terms of a group were given the rest could be derived. And I found that assuming the second figure to be  $m$  the rest would be produced by the continuous multiplication of the terms,

$$\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times, \&c.$$

For example, let the second term  $m$  be  $=4$ ; then the third term will be  $4 \times \frac{m-1}{2}$ , that is 6; and the fourth  $6 \times \frac{m-2}{3}$ , that is 4; and the fifth,  $4 \times \frac{m-3}{4}$ , that is 1; and the sixth,  $1 \times \frac{m-4}{5}$ , that is 0; with which the series in this case terminates.

I applied this rule therefore to obtain the intermediate series. And since for the circle the second term was  $\frac{1}{2} \cdot \frac{x}{3}$ , I put  $m = \frac{1}{2}$  and

the resulting terms were  $\frac{1}{2} \times \frac{\frac{1}{2}-1}{2}$ , or  $-\frac{1}{8}$ ;  $-\frac{1}{8} \times \frac{\frac{1}{2}-2}{3}$ , or  $\frac{1}{16}$ ;  
 $\frac{1}{16} \times \frac{\frac{1}{2}-3}{4}$  or  $-\frac{5}{128}$ ; and so on to infinity, whence I found that the

area sought for the segment of a circle was

$$x - \frac{1}{2}x^3 - \frac{1}{8}x^5 - \frac{1}{16}x^7 - \frac{5}{128}x^9, \text{ \&c.}$$

By the same method also the intermediate areas of the remaining curves are obtained ; as also the area of the hyperbola and other alternate terms in this series,—

$$(1+x^2)^{\frac{0}{2}}, (1+x^2)^{\frac{1}{2}}, (1+x^2)^{\frac{2}{2}}, (1+x^2)^{\frac{3}{2}}, \text{ \&c.}$$

And the method for obtaining intermediate series in other cases is the same, whether the intervals between two terms or more are wanting.

This was my first entrance into these speculations, which would have quite passed out of my memory had I not cast my eyes on certain memoranda a few weeks ago.

But when I had obtained these results, I soon began to consider that the terms  $(1-x^2)^{\frac{0}{2}}, (1-x^2)^{\frac{2}{2}}, (1-x^2)^{\frac{4}{2}}, (1-x^2)^{\frac{6}{2}}$ , could be interpolated in the same manner as the areas generated by them : and for this nothing more was necessary than the omission of the denominators 1, 3, 5, 7, &c., in the terms expressing the areas. That is to say, the coefficients of the terms of the quantity to be interpolated,  $(1-x^2)^{\frac{1}{2}}$ , or  $(1-x^2)^{\frac{3}{2}}$ , or generally  $(1-x^2)^m$ , arise from the continuous multiplication of the terms of this series,

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$$

So, for example,

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6, \text{ \&c.}$$

$$(1-x^2)^{\frac{3}{2}} = 1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6, \text{ \&c.}$$

In this way therefore the general reduction of radical expressions into infinite series became known to me by the rule which I set forth at the beginning of my former letter, before I discovered the extraction of roots. But when I had learned this, the other could not long be concealed. For in order that I might prove these operations I multiplied  $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6, \text{ \&c.}$  into itself and the result was  $(1-x^2)$ , the remaining terms to infinity vanishing throughout the continuation of the series. And in the same way  $1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6, \text{ \&c.}$  multiplied twice into itself produced  $(1-x^2)$ .

When I was sure of the demonstration of these conclusions I was led to try on the other hand whether these series which it proved to be roots of the quantity  $(1-x^2)$  could not be extracted

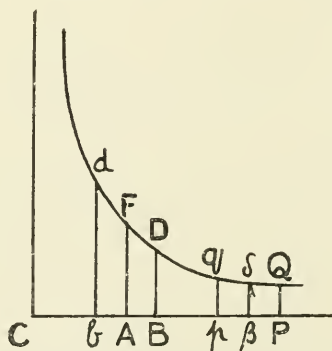
from it in the arithmetical manner; and the attempt was quite successful. The form of the operation in the case of the square root was this:

$$\begin{array}{r}
 1 - x^2(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - , \&c. \\
 \hline
 \frac{1}{0 - x^2} \\
 -x^2 + \frac{1}{4}x^4 \\
 \hline
 -\frac{1}{4}x^4 \\
 -\frac{1}{4}x^4 + \frac{1}{8}x^6 + \frac{1}{64}x^8 \\
 \hline
 -\frac{1}{8}x^6 - \frac{1}{64}x^8
 \end{array}$$

After realizing these consequences I entirely neglected the interpolation of series; and employed only these operations, as being more essentially fundamental. Nor was reduction by division concealed from me, a method very easy to use.

But I soon attacked the solution of affected equations, and obtained that also: from which, the ordinates, the segments of the areas and any other lines became known at once when the areas or arcs of the curves were given. For the regression to these results required nothing beyond the solution of the equations by which the areas or arcs were given in terms of the given lines.

At that time the increasing plague (which fell in the years 1665, 1666) compelled me to fly from this place and to turn my thoughts to other things. Nevertheless, immediately afterwards I produced a scheme, which I here subjoin, for the calculation of logarithms from the area of the hyperbola. Let  $dFD$  be a hyperbola whose centre is  $C$ , vertex  $F$ , and intercepted square  $CAFE=1$ .



[The completion of the square, and the letter E do not occur in Newton's diagram.]

In AC take AB and Ab on one side and the other,  $= \frac{1}{10}$  or 0.1 ; and having erected the perpendiculars BD, bd, with their extremities on the hyperbola, the semi-sum of the spaces, AD and Ad

$$= 0.1 + \frac{0.001}{3} + \frac{0.00001}{5} + \frac{0.0000001}{7}, \text{ \&c. ;}$$

and their semi-difference

$$= \frac{0.01}{2} + \frac{0.0001}{4} + \frac{0.000001}{6} + \frac{0.00000001}{8}, \text{ \&c.}$$

And the calculation of these terms gives the following results :

0.100, 000, 000, 000, 0	0.005, 000, 000, 000, 0
333, 333, 333, 3	25, 000, 000, 0
2, 000, 000, 0	166, 666, 6
14, 285, 7	1, 250, 0
111, 1	10, 0
9	1
0.100, 335, 347, 731, 0	0.005, 025, 167, 926, 7

The sum of these 0.105, 360, 515, 657, 7, is Ad ; and their difference 0.095, 310, 179, 804, 3, is AD. And in the same way AB and Ab being taken on one side and the other = 0.2, Ad will be found = 0.223, 143, 551, 314, 2 ; and AD = 0.182, 321, 556, 793, 9. After the hyperbolic logarithms of the four decimal numbers 0.8, 0.9, 1.1, and 1.2, have been obtained in this way ; since  $1.2 \times \frac{1.2}{0.9} = 2$  ; and 0.8 and 0.9 are less than unity, by adding their logarithms to twice the logarithm of 1.2, you get 0.693, 147, 180, 559, 7 as the hyperbolic logarithm of the number 2. Since  $\frac{2 \times 2 \times 2}{0.8} = 10$ , by adding the logarithm of 0.8 to three times the logarithm of 2 you will get 2.302, 585. 092, 993, 3 the logarithm of the number 10 ; and then by addition the logarithms of the numbers 9 and 11 are at once obtained, and thus the logarithms of all these prime numbers, 2, 3, 5, 11, are ready at hand. And further, simply by the depression of the numbers used in the decimal calculation detailed above, and by addition, the logarithms of the decimal numbers 0.98, 0.99, 1.01, 1.02, are obtained ; as also of the decimal numbers 0.998, 0.999, 1.001, 1.002 ; and thence by addition and subtraction, the logarithms of the prime numbers 7, 13, 17, 37, &c., are obtained. Which, together with those found above, when divided by the logarithm of the number 10 give the true logarithms for insertion in a table. But these I afterwards obtained more exactly.

[Footnote by Editors of "*Commercium Epistolicum*." Vide "*Geometriam Analyticam*", Cap. ix, pp. 36-48.]

I am ashamed to say to what a number of decimal places I carried these calculations being then at leisure. For, indeed, I took too much pleasure at that time in these investigations. But when the ingenious Logarithmotechnia of Nicholas Mercator appeared (whom I suppose to have been the first discoverer of his methods), I began to take less interest in them; imagining that he was acquainted with the extraction of roots as well as the division of fractions; or that others, once the method of division had been brought to light, would soon discover the rest before I was of mature age for writing.

[Footnote by Editors of "*Commercium Epistolicum*."]—Earlier mathematicians discovered this theorem, that the sum of the terms of a Geometrical Progression proceeding to infinity, is in the same ratio to the first and greatest of the terms, as this term is to the difference between the first and second terms. This is proved arithmetically by multiplying the extremes and means of the ratio. Wallis proved it by dividing the product of the means by the last term of the ratio. See the *Opus Arithmeticon* of Wallis, published in 1657, Cap. xxxiii, § 36. By using Wallis' method of division, Mercator proved and extended the quadrature of the hyperbola, previously discovered by Brounker. And Gregory proved the same geometrically. But none of these discovered the general method of finding the areas of curves. Mercator nowhere claimed this. Gregory who was admittedly a man of the greatest ability, and who had his attention drawn to the subject by the letters of Collins, at length and with difficulty found a method of this kind. Newton found it by the interpolation of series, and afterwards by the divisions and extractions of roots as being more familiar.]

At the time when that book appeared, a compendium of the method of these series was communicated to Collins by my friend Barrow, the Professor of Mathematics at Cambridge. In that compendium I had shown how the areas and lengths of all curves and the surfaces and volumes of solid bodies could be found from their given ordinates, and *vice versâ*, how the ordinates could be found if the areas, &c., were given; also I had illustrated the method there given by various series.

A regular correspondence having thereupon sprung up between us, Collins, who was devoted to the promotion of mathematical knowledge, did not cease to urge that I should make these results public. And five years ago (1671) when on the advice of my friends I was planning the publication of a treatise on the refraction of light and on colours which I then had in readiness, I began again to think of these series and wrote a treatise on them with the intention of publishing both together.

But, arising out of the catadioptric telescope, after I had written you a letter in which I briefly explained my ideas on the subject of light; something unexpected brought it about that I felt it to be of importance to me to write to you hastily about the printing of that letter. And the number of questions that immediately arose, through letters of various people crammed with objections, &c., deterred me entirely from my plan: and had the result that I accused myself of imprudence, because by grasping at the shadow I had lost my peace, a thing of real substance.

About that time James Gregory, from a single series of mine which Collins had sent him, arrived at the same method, after, as



he said in a letter to Collins, a great deal of consideration ; and he left a treatise on the subject which we hope will be published by his friends. For, from the ability he possessed he could not but add much new matter of his own, and it is of importance in the interests of mathematical knowledge that it should not be lost. Moreover, I had not completely finished my treatise when I desisted from my plan ; nor to this day has my mind returned again to the things that were left to be added. That part indeed was wanting in which I had proposed to explain the method of solving problems which could not be reduced to quadratures ; granted that I might have done something towards the foundation of it. Moreover, infinite series did not occupy much space in that treatise.

\*            \*            \*            \*            \*

Although many things remain to be investigated about methods of approximation, and about different kinds of series which may serve for that purpose ; nevertheless I should hardly hope with Tschirnhausius that simpler or more general methods of reducing quantities to the kind of series in question can be given than the divisions and the extractions of roots, which Leibnitz and myself use ; at any rate not more general, because for quadrature and rectification of curves and similar questions, no series can be given depending on these simple algebraical terms (involving only one indefinite quantity) which it is not possible to obtain by this method.

For the number of convergent series for the determination of the same quantity cannot be greater than the number of indefinite quantities from the powers of which the series are produced, and I am acquainted with methods of obtaining a series from any indefinite quantity that may be employed ; and I believe that Leibnitz also has that in his power.

For although by my method there is a free choice, for the construction of the series, of any indefinite quantity on which the question depends, and the method which he has communicated to us seems to be adapted for the choice of such indefinite quantities as can conveniently be reduced to fractions, which, by division only, produce infinite series ; nevertheless any other indefinite quantities whatever can be employed for the construction of series by means of the method used for the solution of affected equations, provided they are solved in appropriate terms, that is by constructing the series only from terms which are involved in the question.

Moreover, I do not see why it should be said that by using these divisions and resolutions, problems are solved by accident, since these operations have the same relation to this kind of algebra as the common operations of arithmetic to ordinary algebra.

But as regards simplicity of method ; I would not have fractions and radicals resolved invariably into infinite series without previous reduction ; when complicated quantities occur, all kinds of reductions are to be tried ; whether by increasing, diminishing, multiplying or dividing the indefinite quantities ; or by Leibnitz' method of transmutation ; or by any other method which may happen to

fit the case ; and then resolution into series will be suitably employed by division and extraction.

Moreover, efforts should specially be made to reduce the denominators of fractions, and quantities under the radical sign, to the fewest and least complicated terms possible ; and to such also as are most rapidly expanded into convergent series, although the roots may be neither converted into fractions nor depressed. For by the rule given at the beginning of the earlier letter, the extraction of the highest roots is as simple and easy as the extraction of the square root or division ; and series which result from division are usually the least convergent of all.

Hitherto I have spoken of series involving only one indefinite quantity. But series can also by the method investigated be constructed at pleasure from two or from more indefinite quantities. Moreover, by the aid of the same method series can be formed for all curves, of a character similar to the series given by Gregory for the circle and hyperbola ; that is series of which the final term gives the area sought. But I would not willingly undertake this calculation.

Finally, series can be derived from complex expressions by the same method. As for example, if  $\sqrt{a^2 - ax + \frac{x^3}{a}}$  be the ordinate of any curve, I put  $a^2 - ax = z^2$ , and the extraction of the square root of the binomial,  $z^2 + \frac{x^3}{a}$ , will produce  $z + \frac{x^3}{2az} - \frac{x^6}{8a^2z^3}$ , &c.

All the terms of this series can be quadrated by the theorem already described. But I attach little importance to this method because when simple series are not obtainable with sufficient ease, I have another method not yet published by which the problem is easily dealt with. It is based upon a convenient, ready and general solution of this problem, *To describe a geometrical curve which shall pass through any given points.*

Euclid has shown how to describe a circle through three given points. Also a conic section can be described through five given points and a curve of three dimensions through seven given points ; so that I have in my power a description of all the curves of that order which are determined by seven points only. These are done at once by geometrical methods, without any calculation. But the above problem is of another kind, and although it may seem to be intractable at first sight, it is nevertheless quite the contrary ; perhaps indeed it is one of the prettiest problems that I can ever hope to solve.

\* \* \* \* \*

Nor when he [*i.e.*, Leibnitz] divides this series

$$\frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} +, \&c.,$$

does he seem to have observed my general method of using letters in the place of quantities affected with their signs + and -. For

since the hyperbolic area AD, here signified by  $z$ , is positive or negative according as it lies on one or the other side of the ordinate AF; if that area given in numbers is  $l$ , and  $l$  is substituted for  $z$  in the series, the result will be either  $\frac{l}{b} + \frac{l^2}{2ab^2} + \frac{l^3}{6a^2b^3} + \frac{l^4}{24a^3b^4} +$ , &c., or  $-\frac{l}{b} + \frac{l^2}{2ab^2} - \frac{l^3}{6a^2b^3} + \frac{l^4}{24a^3b^4}$ , &c., according as  $l$  is positive or negative. This being understood, if  $a = 1 = b$ , and  $l$  stands for the hyperbolic logarithm, the number corresponding to it will be  $1 + \frac{l}{1} + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24}$ , &c., if  $l$  be positive; and  $1 - \frac{l}{1} + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24}$ , &c., if  $l$  be negative. In this way I avoid the multiplication of theorems, which otherwise would increase to an extraordinary degree. For to take an example, that one theorem which I gave above [omitted from the present version] for the quadrature of curves would be resolved into 32, if it were multiplied in accordance with the variations of sign.

Moreover, I do not yet understand what my eminent friend says about finding a number, greater than unity, from its hyperbolic logarithm by the use of the series

$$\frac{l}{1} - \frac{l^2}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} - \frac{l^4}{1 \times 2 \times 3 \times 4} +, \text{ \&c.,}$$

rather than by the use of the series

$$\frac{l}{1} + \frac{l^2}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^4}{1 \times 2 \times 3 \times 4} +$$

For if one term more be added to the latter series than to the former, the latter will give a better approximation. And certainly it is less laborious to calculate one or two figures of this additional term than to divide unity by the number extended to many decimal places, derived from the hyperbolic logarithm, in order that the required number greater than unity may then be obtained. Therefore let either series, if it be right to speak of two, be employed for its appropriate work. Nevertheless the series

$$\frac{l}{1} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^5}{1 \times 2 \times 3 \times 4 \times 5}, \text{ \&c.,}$$

depending on half the terms can be best employed, since this will give the semi-difference of two numbers; from which, and from the given rectangle either number is given. So also from the series

$$1 - \frac{l^2}{1 \times 2} + \frac{l^4}{1 \times 2 \times 3 \times 4}, \text{ \&c.,}$$

the semi-sum of the numbers is given, and thence the numbers themselves. From which arises a relation between the two series so that when one is given the other is found.

From the following simple process which depends on such series, you will readily agree that the construction of logarithms need not be attempted in any other way.

By the method previously explained the hyperbolic logarithms of the numbers 10, .98, .99, 1.01, 1.02 are investigated, which would occupy an hour or so. Then, dividing the logarithms of the four last numbers by the logarithms of the number 10, and adding the index 2, the true logarithms of the numbers 98, 99, 100, 101, 102 are obtained for entry in a table. These are to be interpolated by intervals of one-tenth (*Hi per dena intervalla interpolandi sunt*) and the logarithms of all numbers between 980 and 1,020 will be found; the numbers between 980 and 1,000 being again interpolated by intervals of one-tenth, the table will so far be constructed. Then from these are to be collected the logarithms of all the primes, less than 100, and their multiples; for which nothing but addition and subtraction are required. Thus:

$$\begin{array}{l} \sqrt[10]{\frac{9984 \times 1020}{9945}} = 2; \sqrt[4]{\frac{8 \times 9963}{984}} = 3; \quad \frac{10}{2} = 5; \quad \sqrt{\frac{98}{2}} = 7; \quad \frac{99}{9} = 11; \\ \frac{1001}{7 \times 11} = 13; \quad \frac{102}{6} = 17; \quad \frac{988}{4 \times 13} = 19; \quad \frac{9936}{16 \times 27} = 23; \quad \frac{986}{2 \times 17} = 29; \\ \frac{992}{32} = 31; \quad \frac{999}{27} = 37; \quad \frac{984}{24} = 41; \quad \frac{989}{23} = 43; \quad \frac{987}{21} = 47; \\ \frac{9911}{11 \times 17} = 53; \quad \frac{9971}{13 \times 13} = 59; \quad \frac{9882}{2 \times 81} = 61; \quad \frac{9949}{3 \times 49} = 67; \quad \frac{994}{14} = 71; \\ \frac{9928}{8 \times 17} = 73; \quad \frac{9954}{7 \times 18} = 79; \quad \frac{996}{12} = 83; \quad \frac{9968}{7 \times 16} = 89; \quad \frac{9894}{6 \times 17} = 97. \end{array}$$

And so, having found the logarithms of all numbers less than 100, it remains only to interpolate these also once and again by intervals of one-tenth.

\* \* \* \* \*

[After a section on the construction of Trigonometrical Tables the following paragraph occurs]:

What has been said about Tables of this kind can be applied to others where Geometrical considerations have no place. Moreover, it is sufficient by means of these series to calculate 30 or 20, or even fewer terms at suitable distances apart, since the intermediate terms are easily inserted by a method which I had almost decided to describe here for the use of computers. But I pass on to other matters.

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*Notes on some recent developments of Pension Problems in America.* By W. J. H. WHITTALL, F.I.A., F.A.S.

THE CARNEGIE FOUNDATION (*continued from p. 117*).

SUMMARIZING the policy of the Carnegie Foundation it is seen that it covers the following fundamental principles : (i) that contributions by the employed are not only necessary, but desirable in themselves ; (ii) that death benefits are desirable ; (iii) that the proper method of procedure is by contractual methods (*i.e.*, by what we should call the money purchase method), based on contributions provided in advance ; (iv) that the reward for past service must be separately provided for ; and (v) that disablement must be regarded as a separate problem. In all these respects the conclusions are exactly the same as those arrived at independently, after a long argument, in the Report on Hospital Officers' Pensions, recently published by King Edward's Fund.

Of the points I have enumerated the first appears to me to be the most important. The Carnegie Foundation appears to have faced this question boldly and have actually converted the Teachers as a class to the view that contributions are not only necessary, but desirable in themselves. The Joint Commission, which included representatives of the University professors, laid down the fundamental principle, as a "primary obligation of the existing social order," that the individual should provide protection for himself and for his family. To this they added that the employer must assume some part of the obligation. The Carnegie Trustees go further, and assert (i) that no system of free pensions can be devised which will not in the end affect the teacher's pay ; (ii) that a contributory system is the only one which society can permanently support, and (iii) that, according to the "experience of the world," any group in the body politic receive their best service from society if they are assisted to economic independence.

The amended rules of the Carnegie Foundation have recently reached me. Its method of reducing its own commitments for the existing staffs is to increase the age of retirement to 70, with the option of smaller allowances at earlier ages. This solution of their difficulty will of course leave the beneficiaries free to supplement their pensions on their own account by contributing to the new Association for annuities in respect of their



future service. As regards all new appointments the Foundation will guarantee an average return of  $4\frac{1}{2}$  per-cent on payments made to the Teachers' Association for deferred annuities, and definitely undertakes the provision of the disability allowances. On disability, the teacher will assign his annuity policy to the Foundation and will receive an immediate annuity of two-thirds of the amount of the ultimate benefit therein assured. This undertaking is limited to the next 20 years, after which time the Trustees think that experience will enable the disability risk to be measured actuarially and contracted for by the Teachers' Association. No suggestion is made that the Death Benefit, if any, assured in the Association should be assigned to the Foundation or that this should in any way be co-ordinated with the other assets to provide disability benefit in the manner suggested by Mr. F. L. Collins in the Report on Hospital Officers to King Edward's Fund. On the contrary, there is to be a special provision in all the Insurance Contracts of the Teachers' Association that the policy shall continue in full force without further payment of premiums in the event of complete disability before age 65. It would thus seem to be the intention, in that event, to preserve for the assured the full benefit of such insurance as he may have effected. The British Offices who make a speciality of catering for pension schemes might well consider the possibility of providing for this contingency.

The detailed "Handbook of the Teachers' Insurance and Annuity Association of America" gives effect to the scheme on the lines already indicated.

It is perhaps to be regretted that the Life Insurance Companies of the United States cannot have the opportunity of contributing towards the solution of these pension problems. Our experience here is that such companies, with their widespread interests and established businesses, are in a better position to enter into annuity contracts than any new organization can be.

As the Carnegie Foundation supplementary benefits are limited to holders of and are based on the amounts of the annuity contracts, the joint contributions will be spent mostly if not wholly on that form of contract. It may be worth while to give a few specimens of the rates quoted for annuities and insurances.



*Rates of Monthly Annuity granted by the Teachers' Association.*

Age	DEFERRED TO AGE 65				Age	IMMEDIATE	
	Per Monthly Premium of \$10		Per Single Premium of \$100			Per Single Premium of \$1,000	
	Male	Female	Male	Female		Male	Female
25	\$109.81	\$96.27	\$4.53	\$3.97	50	\$6.26	\$5.66
30	85.11	74.62	3.72	3.26	60	8.06	7.15
40	48.13	42.19	2.51	2.20	65	9.52	8.33
50	23.14	20.29	1.70	1.49	70	11.59	10.00

*Annual Premiums for \$1,000 Insurances.*

Age	TERM INSURANCE				Whole Life	Endowment Insurance at 65
	Ten-Year	Expiring at Age 60	Expiring at Age 65	Expiring at Age 70		
21	\$7.93	\$9.81	\$10.55	\$11.43	\$13.91	\$16.50
30	8.69	11.07	12.20	13.64	17.40	21.91
40	10.78	13.34	15.31	17.60	23.88	33.32
50	...	17.59	20.82	24.71	35.75	61.02

To meet the case of members who withdraw from the teaching profession, the rates of premium reserved in the contracts will be higher than the foregoing, a reduction of 10 per cent being allowed so long as the assured remains in the profession. The full premiums payable on withdrawal from educational employment would therefore be the quoted rates increased by one-ninth. It seems that a provision of this kind is quite fair if it means, as I assume, that a withdrawing member will in all cases have the option of maintaining his annuity and other contracts for his own benefit, together with the full benefit of the cash surrender values and other standard provisions of the New York Insurance Law.

On this question of the "vested rights" principle, to which I have drawn attention on another page and which seems to be essentially the proper complement and corollary of the contributory method, I cannot find that the Carnegie Foundation has laid down a definite ruling, but the Handbook of the Teachers' Association clearly contemplates that the principle of the vested right accruing from year to year will be recognized

in practice. "Teachers who have followed", it is remarked, "the discussions of pensions during recent years will understand that the contribution made by a college or university to a teacher's annuity will inevitably in the course of time be considered as part of his salary. This result must always follow in any such arrangement between two parties who have to each other the relation of employer and employee. Nevertheless, even when this stage has been reached, it is still of great advantage to the teacher to have his college advance this payment, even though it be in the nature of deferred salary. Not only is it an advance payment, but its full value can be secured to the teacher whether he lives out his term of activity and makes use of the annuity, whether he changes his occupation, or whether he dies prematurely."

It would seem that in this question of vested rights, the Trustees of the Foundation have taken up very much the position that the Committee on Hospital Officers Pensions recommended King Edward's Fund to adopt, namely, that while not going the length of laying down the principle as an actual condition of its support, it should cast its influence as far as possible on the side of establishing it.\*

#### THE STATE OF ILLINOIS.

The State of Illinois appointed in 1916 a Commission (with an Actuary, Mr. Donald F. Campbell, as its Secretary) to investigate all its pension laws, and there was found to be not only want of foresight and ignorance as to the ultimate cost of the liabilities, but a quite chaotic variety of conditions for determining the pensions themselves. In order to determine the liabilities the principal funds were thoroughly investigated, Life and Service Tables being prepared. On four of the principal valuations the total present deficiency appears to be about £11,000,000. As to the future, the Commission proposed that the pensions should be provided in advance, that the employees should contribute, and that there should be death benefits for dependants. A minimum compulsory pension is suggested at fifty-five, with a voluntary "sur-pension" proportionate to any increase of salary. The cost of existing staffs is to be met mainly by the State and charged over forty years.

\* Since writing this I learn that the Deferred Annuity Form of Contract definitely assures the whole benefit to the member, whatever proportion (even if the whole) of the premium be paid by the Institution.

The Report makes the following statement in regard to our Civil Service Pension system :

After a hundred years of experience with Civil Service pensions in England, the pensions are regarded very generally as deferred pay. . . . . It was in the law of 1909 that the principle was recognized that even with a non-contributory system the employee who withdraws from the Service should receive certain benefits. . . . . In other words, even a non-contributory system of pensions has in England become virtually contributory, because wages and salaries are held at a lower level on account of the pension prospects.

It seems clear that this statement is inaccurate in regard to withdrawal benefits. It is to be regretted that the language of our Act of 1909, presumably Sec. 1 (2), should have misled an American student on such an important point, but the practice of the Treasury is well known. Throughout our Civil Service no benefit whatever is paid on voluntary withdrawal earlier than the "retirement" recognized by the Acts. For American practice I quote the following precise and important statement in the Report :

Without exception, Pension Acts in the United States\* that do not provide for contributions from salaries of employees make no provision for any payment to employees upon separation from active service on account of resignation or dismissal ; and in those cases where such contributions are required, the payment never exceeds the amount contributed by the employee.

#### NEW YORK CITY PENSIONS COMMISSION.

The Corporation of New York City has recently made what is probably the most thorough investigation into pension liabilities ever undertaken by a public authority. A Pensions Commission was appointed, of which Mr. George W. Perkins, known to many of us as a former Vice-President of the New York Life Insurance Company, was Chairman ; and a special Actuarial staff was engaged with Mr. G. B. Buck, F.A.S., as its chief. The results are contained in several volumes packed with statistical and actuarial information. The first part of the Report was presented in February 1916, consisting of a preliminary statistical account of the nine systems covering the various city services. This was followed by an actuarial investigation and valuation of them ; and finally a new scheme was proposed which in turn required a large amount of further

\* Over 200 are cited !

actuarial work to estimate the cost of it. For the following notes I have consulted the original reports.

The New York City pension systems originated with the employees. The Government was indifferent. Thus in 1857 the Police Force procured special legislation giving them pensions. In 1871 the Fire Brigade secured pensions, and in 1894 the Teachers. Other funds followed, including the "Grady" Fund, and (in 1911) one for the Street Cleaning Department. In 1911 also the Grady Fund was extended to include all employees not covered by the other Funds. Finally, the system consisted of the foregoing five funds and four smaller funds. The five large funds include 75,359 employees out of the total of 77,310, and for brevity I will deal only with these in giving details. The statistical position was as follows in 1914 :

Fund	Founded	Employees 1914	Pensioners 1914	TOTAL RECEIPTS SINCE ESTABLISHMENT FROM		Balance Dec. 31 1914
				Contribu- tions	City Funds and Interest, &c.	
				£	£	£
Police ...	1857	10,708	4,234	917,030	5,521,689	146
Fire Dept. ...	1871	5,004	1,686	...	2,816,050	170,324
Teachers ...	1894	21,317	1,549	444,525	1,817,104	181,321
Grady ...	1906	32,856	166	...	34,261	None
Street Cleaning	1911	5,474	484	85,743	175,643	201,260
All Funds ...	...	77,310	8,232	1,469,812	10,535,843	622,379

The 8,232 pensioners consisted of 5,779 employees with an average pension of £152, and 2,453 dependants with an average of £61. The total salary list of all the funds as at June 1914, was £19,590,182.

In the Fire Department and Teachers Funds, Superannuation of half final pay is granted after 20 and 30 years' service respectively, irrespective of age. A similar scale applies to the Police after 25 years and the Street Cleaning Department after 20 years, but with minimum ages of 55 and 60 respectively. In the Grady Fund there is no superannuation provision; but on disability after 30 years' service half final pay is given. Disability is also provided for in the other four Funds on a generous scale, which is frequently half final pay after 20 years' service, and sometimes after 10 years. There are also benefits for dependants in the case of the Police, Fire and Street Cleaning Funds.

For these benefits the Police contribute 2 per-cent of salary, the Teachers 1 per-cent, and the Street Cleaners 3 per-cent. In the other two Funds there are no contributions. Many of the Funds enjoy special appropriations of excise monies and other special items of the City's Income, which therefore do not pass through the City Accounts and thus reduce the apparent ultimate cost to the City by way of direct contributions. In the above table this differentiation is disregarded, as the whole of the City's contributions are public money. As will be judged from the small balances in hand, it has been usual to pay pensions out of the funds, as in the case of our own Police Pensions, regardless of any actuarial valuations; and it is the recent increase in direct contributions demanded from the City which has led to the investigation. The Pensions Commission found that, "due to their fitful and unsystematic development, the existing pension laws present a tangled mass of conflicting provisions."

It would serve no practical purpose to follow the Commission in their painstaking exposition of the inequalities and inconsistencies which they found. In their summary they remark

- (i) that employees in different branches get varying benefits not justified by differences in their work;
- (ii) that the employees in each fund get varying benefits through the operation of crude flat pension rates and contributions;
- (iii) that about 37 per-cent of the municipal service could retire after 20 to 30 years service irrespective of physical condition;
- (iv) that on the other hand about 43 per-cent of the service had no superannuation but only disability allowance to look forward to, and there were already 3,323 employees above age 60 who could not retire even if inefficient;
- (v) that the frequent employment of pensioners at fresh salaries led to absurd results; and generally that the regulations as to disability, as to dependants and as to contributions were all inconsistent with one another and based on no general principles.

As regards the financial provisions of the existing system the Commission find

- (i) that the schemes were all launched without actuarial advice or proper financial provision for their cost;



- (ii) that the City's support was disguised by the allotment of indirect sources of revenue ;
- (iii) that the City was directly responsible for four funds, of which the two important ones, the Police and Fire Department, were likely to develop respectively a pensions expenditure of 35 per-cent and 45 per-cent of the pay roll ; and
- (iv) that five of the funds, including the important Teachers Fund, are limited to definite sources of income, and are now, or soon will be, bankrupt.

The Commission also reported that owing to inadequate departmental records a costly census and much delay had been necessitated before the actuarial staff could construct the mortality and service tables needed for calculations of cost.

Meanwhile, the Commission proceeded to arrive at certain preliminary conclusions as to the proper fundamental principles of a new system. They did not attempt to reconcile the illogical features of the existing system, which, they say, would have "collapsed under the weight of its own absurdities, but for the lack of general knowledge of the details, and the oblique methods of financing by which its cost in the past had been hidden"; but before determining on any new financial basis they made a broad survey of existing pension systems in operation, both in the United States and abroad. They found "that the development of pension measures as a result of an experience of over a hundred years is in the direction of equal division of cost between the employer and the employee, and that this tendency applies equally to systems for public employees and for industrial workers."

Part II of the Report comprising the Actuarial Investigation was published in May 1916. The Actuarial Society of America had appointed by request an honorary advisory Committee of Actuaries to assist the Commission, consisting of Messrs. William A. Hutcheson (Chairman), Robert Henderson and Henry Moir ; and the work of Mr. Buck and his actuarial staff was carried out with the help and approval of these gentlemen. A six-year period, from 1908 to 1914, was selected, for which particulars of each employee were obtained on an elaborate form of card, and the ensuing statistical analysis, the Commission say, "required more than two years' work, and is unparalleled in its scope by any similar study of a body of employees ever made anywhere in the world."



The ultimate financial result, to which the whole Investigation led up, was to show, on the basis of existing laws and past experience, a present liability for existing staffs of £43,104,083. Deducting funds in hand, £769,931. and the value of future contributions, £1,779,038, there is shown a deficit of £40,555,114. The City, the Commission remark, had blindly embarked on a vast financial programme with no thought of the cost involved, and even if the inequalities of the existing schemes had not called for reform, the financial burden demanded a revision which would relieve taxpayers in part and provide for accumulation of reserves.

Part III of the Report, to which the whole of the foregoing investigation may be said to be ancillary, is dated February 1918, and contains the draft of the proposed new scheme followed by a full account of the actuarial calculations of its cost.

It is satisfactory to find that the new scheme begins by differentiating clearly between past service and future service. The futility of attempting to combine the two is amply shown in the history of our own Municipal schemes as given by Messrs. Manly and Ackland (*see J.I.A.*, vol. xlvi, p. 327). For the recompense of past service the City will assume the whole liability in the case of all employees who elect to enter the new scheme. Such entrance, optional for the time being, is subject to payment of contributions as at their present ages.

Commencing then with all future service the scheme provides service pensions on the salary percentage method, the basis being the average salary for the final ten years of service. The employee will contribute the cost of one half of this pension on a returnable basis; the City will provide the cost of the other half, but non-returnable. The City will further contribute

- (i) the whole cost of disablement pensions, as well as pensions to widows and children, due to hazards arising in performance of duty;
- (ii) the whole cost of certain benefits on death from ordinary causes; and
- (iii) the greater part of the cost of pensions on disablement arising from ordinary causes after 10 years' service.

The scale of these proposed benefits varies in different branches of the service. Thus for service pensioners the street cleaners are to receive 1/60th for each year of service at minimum age 55, the police, firemen and labourers 1/66th at age 58, the

mechanics 1/68th at age 59, the clerks and all administrative officers 1/70th at age 60, and teachers 1/74th at age 62. Retirement is to be compulsory in all cases at 70. The past service of existing officers entering the scheme is arranged for by the City undertaking to provide the whole "service fraction", instead of one-half, in respect of every prior year's service.

On disablement arising at any time in actual performance of duty the City will pay a pension of three-fourths of the final average salary, and the officer will retain the right to his own contributions. On disablement after 10 years' service from other causes, the pension will be nine-tenths of the corresponding service pension, with a minimum of 25 per-cent of final average salary. In this case the officer's own contributions are used *pro tanto* and the City provides the balance.

On death in actual performance of duty the City will pay to widow, children or dependent parents a pension of one-half of the final average salary, in addition to the returnable contributions. On death from other causes the benefit is a lump sum of one-half of the last year's salary, in addition to the contributions. On withdrawal from any cause the officer's own contributions are returnable with interest. Finally, the service pensions may be varied (as in the case of the Carnegie scheme) to provide a capital sum at death or to introduce a second life ; and it is contemplated that all City contributions will be provided in advance, that proper reserves will be maintained on an actuarial basis, and that the management will be in a Board of Trustees on which each group of employees will be represented.

It may be thought I am lingering too long over details, but it appears to me that the great and pressing need at present in the pension problem is to settle fundamental questions of principle. The New York investigation probably represents the most thoroughgoing enquiry that has been made hitherto not only into objective facts but also into the general principles that should be adopted for future guidance. It is worth noting therefore the following general principles proposed for adoption :

- (i) After the fullest consideration of alternatives it was decided that New York (like London) should exact contributions from its employees. Employer and employee are each to contribute strictly in advance a moiety of the estimated cost of the service pension. But while the employee contributes on a returnable basis the City retains its profit on withdrawals towards

other benefits. Thus there will be fostered a "sense of mutual responsibility for the provision of benefits which will be equitable and advantageous to both parties."

- (ii) Benefits should be given if possible with equal fairness to all. Hence, the services being in complex variety, the City charges itself with the whole cost of special hazards where such exist. Thus arise the special benefits where death or disablement occur in performance of duty—benefits which, as I understand, replace the benefits under the Workmen's Compensation Acts for which ordinary employers would be liable; and the frank recognition of the employee's claim to his own contributions in addition.
- (iii) In adjusting scales of contributions and benefits the "coat was cut according to the cloth." The general aim was to provide for an entrant of 25 a service pension of one-half of the final average salary. It was then decided that the employee's contribution at that age on a savings bank basis should be as nearly as possible 4 per-cent of salary; and the retirement ages according to the various occupational experiences, with the resulting "service fractions", were thus arrived at as the conclusion of the process.
- (iv) The City frankly accepts the additional cost for existing staffs. For example, a clerk can retire at 60, his service fraction being 1/70th for each year, of which 1/140th is provided by his own contributions. In the case of an existing officer electing to enter the scheme the City provides, in addition to its own normal moiety, the clerk's 1/140th for each year of prior service.
- (v) Disablement from ordinary causes is not viewed too sympathetically, but I suspect that abuses under the existing system account in part for this. It is admitted that from the standpoint of the employee "disability is probably the danger which is viewed with the gravest concern"; but to avoid offering temptation to retire early, and seek other employment, it was decided to limit the pensions to nine-tenths of the corresponding scale for service pensions. It is, however, contemplated that special additional grants might be

made by the City in meritorious cases and that the general scale of benefit might need to be broadened later.

On the foregoing basis rates of contribution are given for at least eleven different occupational groups. I select four of the groups for quotation.

*Salary Percentages required.*

Age at Entry	EMPLOYEE'S CONTRIBUTION				CITY CONTRIBUTION			
	Police	Fire Depart.	Teachers (Women)	Street Cleaning	Police	Fire Depart.	Teachers (Women)	Street Cleaning
20	4·01	4·35	4·08	3·57	4·53	3·93	2·77	2·14
25	4·12	4·47	4·12	3·96	5·10	4·41	3·15	2·85
30	4·33	4·69	4·22	4·41	5·58	4·86	3·63	3·53
35	4·67	4·94	4·43	4·91	5·53	5·13	3·98	4·23
40	5·10	5·24	4·77	5·46	5·48	5·60	4·35	5·31

We are finally presented with the results of a complete valuation of each benefit for each Fund on the assumption that every existing employee elects to come under the new scheme. The total liability for the whole service is as follows :

Existing Pensioners and Dependants ...	...	£10,590,342
Pensions in respect of Future Service ( <i>less</i>		
Employees' half) ...	...	5,010,260
Disability : Ordinary ( <i>less</i> Employees' share) ...		4,713,281
Do. Accident ...	...	490,661
Death Benefits : Ordinary ...	...	1,292,746
Do. Accident ...	...	793,979
Cost of Past Service ...	...	9,125,208
Grand Total ...	...	32,016,477
Less Cash in Existing Funds ...	...	769,931
Deficiency, to be provided by City ...	...	£31,246,546

It remains to be seen in what way the City will provide the cost of benefits for past service in the case of existing staffs. The London County Council in 1914 consolidated all its pension liabilities then outstanding and decided to liquidate the amount (£1,181,398) by an annuity spread over 50 years.

*Note on a fundamental question underlying the Pension Problem; with a reference to the School Teachers' Superannuation Act, 1918. By W. J. H. WHITTALL, F.I.A., F.A.S.*

IT may be worth while, in view of current pension developments in America and elsewhere, to take stock briefly of the present position of the Pension Problem in this country; I mean of its broad aspects. I exclude from consideration the enormous liabilities the State has assumed for naval and military pensions. They have received insufficient attention from actuaries and statisticians, and the public has little conception of the burden it has assumed or how it is to be provided for when the existing wild orgy of expenditure shall cease.

These pensions, however, are incumbent, and will gradually decrease. Subject to the public insisting that they be provided for out of current taxation, and that no attempt be made to pay them out of borrowed money, they will settle themselves in time. For the purpose of this note I shall speak only of the civilian pensions of the future.

If we follow existing preconceptions we shall begin by differentiating between State pensions, having unlimited public money behind them, and all others. These others may be subdivided into categories,

- (i) Where the employers have ample funds available for contributions or guarantees, including local authorities with rating powers, and
- (ii) Where the beneficiaries must seek security for themselves.

Now the first fact which emerges from a review of the position in America is that a great heartsearching is taking place there as to the basic propriety of postponing State pension liabilities to be met by future generations. The growing tendency there seems to be to regard each State like any other employer, and to call on it to face its pension liabilities in advance. The questions involved in the proper determination of this apparently simple but fundamental question are so numerous and involved, and the sums concerned so great and so constantly growing, that it calls for particular and separate investigation. The Institute might well initiate this by devoting an evening specially to its discussion. A strict provision in advance for all State pensions might conceivably be the key to another pressing problem, namely, how to reduce the National Debt.



After the State pensions there is a large category of well secured benefits enjoyed by employees of municipalities or other statutory bodies with local rates to fall back on, and of various wealthy industrial, commercial and financial corporations. Of these not much is known of the financial and commercial companies' schemes, which are kept private as a rule; but a good deal is known of the municipal and similar schemes, and of the Railway Funds. The county and municipal authorities have hitherto been left to seek such special powers as they may think fit, and the consequence is an existing lack of uniformity which is quite bewildering, and into which a strong departmental Committee is now enquiring. The great Railway Funds, as we know from the report of Lord Southwark's Committee, present a chequered history of deficiencies for which the Companies have made themselves liable, and of grievances which are believed still to persist.

Of the numerous class of employees in my final category very little is known. They are either unprovided with pensions altogether, or they are dependent on such quasi-eleemosynary recognition as the financial capacity of the employer may permit of, or they are dependent on mutual funds unguaranteed by any external resources. After studying the history of the Railway Funds I doubt if many such unsupported mutual funds can be actually in existence and functioning normally to-day. One cannot, however, prove a negative. Such societies, if existent, are private institutions of which it is impossible to get complete and trustworthy *data*. I have made an effort to hear of such funds, but without success.

It has been obvious for some time past, that over all Mutual Pension Funds, whether supported by external guarantee or not, there has been hanging the shadow of the gradually growing longevity of annuitants. To this cause of anxiety must now be added, as a consequence of the war, the increase in salary scales and the depreciation of securities, of which the former is probably the more important. To investigate the concrete effects of these adverse influences on the finances of mutual funds—particularly those giving benefits based on salary percentages—is beyond the scope of this article, but there is clearly a need for this to be done by someone. In particular, the aggregate deficiency on the Railway Funds will reach very large figures if new valuations be made with a view to nationalization.

It is not a cheerful story. As soon as we get outside the



zone of free State pensions there seems to be running through each of the other broad categories an increasing and now fairly general recognition that contributions from the beneficiaries are necessary ; and it is possible to trace also a general demand for some form of death benefit. All the evidence points to this aspiration having received insufficient recognition from employers in the past. As regards the best methods of attaining the desired ends there seems to be in recent schemes a tendency to resort (perhaps one might say to revert) to what our American friends call "contractual methods", *i.e.*, the money purchase principle ; though this applies more to the category where employers are poor or fleeting. Where guarantees are available and service is more settled, there is evident reluctance to dispense with a scale of pensions bearing relation to salary, and of course for preference final salary.

From this summary of the position to-day, as to which it is difficult to dogmatize but for which a good many facts can be cited in support, I pass to an important matter belonging more to the region of opinion. In all the categories the various services, even the State services with free pensions on the most generous scales, give evidence of discontent. Why should this be so ? My own explanation is that there is a main fundamental question lying at the root of all pension questions to which insufficient attention has hitherto been given. It is involved in the contingency, to which I have already made reference (*see pp.* 116 and 235), of migration from one service to another.

The primitive pension apparently originated in the benevolent wish of an employer to provide for the old age of an employee who had long served him faithfully and could work no longer. But the benevolence of the wish vanished if the employee sought to better himself by transferring his service elsewhere. In that event the faithful servant found he must take his fate in both hands and cast himself on the world or on his new employer ; but the old one washed his hands of him and even was glad sometimes to be rid of an accruing liability.

This idea of a pension seems primitive and rudimentary to-day, and more adapted to patriarchal times than to the post-war period. It exists, however, and is still firmly entrenched. I need not pile authority for this statement. The typical pension fund almost invariably provides that on withdrawal the benefit is limited to the member's own contributions.

As evidence for the prevailing conception in services where

there are no contributions, we may turn to the Report of Lord Courtney's Commission, which carefully reconsidered and reviewed the general principles underlying Civil Service Pensions, as well as the question of adjusting their form to meet new needs. Here, we find, Lord Courtney and the majority of his Commission, while they admitted "that a deferred pension is remuneration for services as much as an immediate money payment," proceeded to lay down the general principle that the pension is, in part at least, "remuneration for continuity of service contingently payable on the continuity being maintained during "a defined period, and not accruing from year to year as an "indefeasible interest." The minority, consisting of Sir Ralph Knox and Sir Edward Brabrook, were still more conservative. They adhered to the primitive conception of a pension as the reward of long and faithful service, and objected even to the foregoing limited recognition of the "deferred pay" principle.

Now the Civil servants, on their side, are very conscious as they get older of the grip of the system in which they are held. The recent increase in prices, with no adequate increases of salary in the higher ranks, has emphasized their position of dependence. We have, further, the testimony of our own expert on pension questions, the late Mr. H. W. Manly, that he always recommended employers to provide free pensions, the chief reasons being that superannuation after faithful service should be the aim, that contributions introduced claims for other benefits, and that free pensions would pay for themselves by making the service popular and keeping salaries moderate.\* If Mr. Manly could have survived to feel the new impulses now current I doubt if he would retain those views to-day.

The aspiration to-day is not only for free pensions, but for freedom. Old thralls are being thrown off. *A priori* reasoning points rather to a universal principle that every year of service, given by high or low, should carry with it not only the living wage but also its proper quota towards the ultimate old age provision of the worker; and that such quota, once earned, shall enure as a vested right.† This principle already exists in the Federated Universities Superannuation System, and it is adopted (see p. 236) in the new Carnegie scheme for American Colleges.

\* See *J.I.A.*, vol. xxxvi, p. 258, and vol. xlviii, p. 34.

† This is what has been known as the "deferred pay" principle when that term is applied in its strict sense; but it has often been used loosely, and I think it better now to use a more exact phrase.

It is found in embryo in the National Health Insurance Act, where every contribution paid by the employer is irrevocable and final. It is, however, independent of the method of providing the contributions. Though we have seen a tendency to divide these between employer and employee, it may be applied equally if they are provided wholly by the employer.

Supposing such a principle established, let us try to visualize some of the sociological results. Many say that employees need not fear in the end the burden of their own contributions, if any. Mr. Carson Roberts has remarked "there is no such thing "as a non-contributory scheme."\* When once the pension is secure the free play of the market will determine the salary; and what more can the able official wish? Many employers, and especially the State, would have to pay higher salaries to retain the able men; but why not? There might be other effects. Less able men could be dispensed with more easily if they had pension rights. The net effect to the employer would undoubtedly be a fresher and healthier organization. For the State, above all, any principle that would promote an interchange of officials with the real world outside is to be welcomed. Business and administrative life would alike be quickened and every worker would find increased opportunities either for advancement or for leading the life for which he is suited.

Actuaries in the past have worked well at formulas for measuring the protean benefits and contingencies of existing pension funds and systems—largely self-developed. I hope the Institute may follow the impulse given by our President in his inaugural address, and now seek to be the leader and inspirer of true principles rather than the humble handmaid busy with the evaluation of self-evolved and fortuitous benefits. We can easily see what actuarial results would flow from a general recognition of the broad principle of vested pension rights. While the forms of benefit would remain manifold and the methods of providing contributions as between employer and employee be various, strict provision in advance would be necessary as well as strict provisions for apportioning individual vested interests and for maintaining actuarial solvency. These in turn would probably rule out all salary percentage plans and dictate resort to contractual methods. In other words the money purchase principle would prevail, with definite benefits assured either by ordinary life offices or by a system of "approved societies." We

\* *J.I.A.*, vol. xlv, p 373.

should have to say farewell, if by some a sad farewell, to the absorbing actuarial work based on unknown rates of salary progression, on unknown future benefits, on unknown future contributions, and on unknown rates of withdrawal, and the unknown profit thereon. We shall still be left, however, by way of some consolation, with the future of the rate of interest, the future rate of pensioners' mortality, and the future rates of disablement. There will be plenty here to occupy us for some time to come. As regards disablement in particular there is much work to be done before we can decide whether it is best provided for in combination with superannuation or separately; and if separately by what methods.

It is significant that since the last number of the *Journal* appeared, the opposing principle of free pensions for continuous service has received a fresh lease of life in this country through the passage of the Bill (*see* p. 107) for superannuating the whole of our elementary and secondary school teachers. A measure to raise the status of such an important class, and to improve the education of the country, would be matter of rejoicing if clear evidence had been presented of the soundness of its underlying principles. This Bill, however, was hastily conceived and passed into law without any such evidence being adduced in the expiring days of the late House of Commons, which had long been swimming in a sea of borrowed money, seemingly oblivious of the meaning of figures.

To save space I must refer the reader to the Act itself for the detailed provisions, which are complicated; but the broad effect is to apply the Superannuation Act of 1909 (the salary basis being altered to the average amount received in the last five years of service) to the teachers of all grant-aided institutions below university rank, and to some extent to the teachers of outside schools as well. Speaking generally, all past service is to count for pensions; and existing teachers are to receive in addition the accrued benefit to which they may be entitled in any existing fund. As in the case of the Civil Service, there is no benefit whatever on withdrawal before becoming entitled to a pension. In view of our President's plain speaking on the question of State pensions, it may not be out of place if I record briefly here the facts about the passage of the Bill.

In opening the debate on the second reading, Mr. Fisher apologized for the fact that, though he was speaking on a Monday,



the Bill had not been issued until the preceding Saturday. Most members, therefore, could not have seen it until the day of the debate. After referring to the existing inadequate provision for elementary teachers, the Minister spoke as follows :

“ There is no form of pensions whatever for teachers in secondary schools, although here and there there are local pension schemes. “ The absence of such systems led to the appointment in 1912 of a “ Departmental Committee to Report upon the best system by which “ provision can be made for the superannuation of teachers in “ secondary and technical schools and institutions, schools of art, “ colleges and schools for the training of teachers, pupil teacher “ centres, and other schools and institutions not being universities “ or university colleges aided by Grants from the Board of Education. “ This Departmental Committee reported in favour of a system of “ insurance for full-time teachers in secondary schools, resembling “ the system adopted by the Federation of Universities and supported “ by contributions from the teachers themselves and from the “ employers, to be supplemented by assistance from the State in the “ shape of superannuation and disablement allowances. The Report “ of this Committee was published, and although widely commented “ upon at the time, has never been given effect to. The War “ supervened, and the hopes which had been created by the “ publication of the Report have not yet been realized. In these “ circumstances, and in view of the great development of education “ which we expect to ensue from the recently-passed Education Act, “ the Government has come to the conclusion that it is essential at “ the earliest possible moment to bring under one State pension “ scheme all qualified teachers in aided schools of all kinds below “ those of university rank.

“ Let me say, in the first instance, that the scheme proposed in “ this Bill is no niggardly scheme. Its generosity has been widely “ and freely acknowledged by all who have studied the White Paper “ and who are interested in the welfare of the teaching profession. “ I think the House will realize that if the pension scheme for “ teachers is to err it should err on the side of generosity. Teaching “ is one of those professions which demand the preservation of a “ buoyant temper and of a fresh outlook under conditions which too “ often make for deadly monotony. Since the State has undertaken “ to make provision for the teaching profession it is for the State to “ find some means of relieving those school teachers who have “ reached that period of life when vitality is lower and when the “ spirit tending to usefulness is almost gone. Seeing that the “ salaries of teachers generally are on a modest scale, this affords “ additional reason why the pension should bear a high ratio to “ salary. The scheme, generous in its terms, is non-contributory. “ I gathered from some observations from my hon. friend the “ Member for the University of Glasgow and Aberdeen University “ (Sir H. Craik) while the Financial Resolution was being discussed, “ that he was inclined to prefer a scheme under which contributions

“should be demanded from teachers and from their employers. I do not wish in any way to disparage the value of contributory schemes. They have their place in the national system. But a contributory scheme to which the State makes a contribution is open to certain objections, the force of which is increased in direct ratio to the size and complexity of the scheme. If you have a contributory scheme it may be worked by an insurance company, and if it is worked through an insurance company you are at once confronted with the objection that public money is going in dividends to the shareholders of those companies. You are also confronted with the objection that the Minister of Education will be besieged by different insurance companies pressing on his notice the advantages which they are enabled to offer to their clients. You will be confronted with the objection that the State is subsidizing and guaranteeing a private company, and these objections become, of course, seriously aggravated when the sums involved are large.\*

“If you do not go to the insurance company, and if you compel the teachers and the employers of teachers to make their contributions to the fund, then you are open to another set of objections. Teachers know that such a fund earns a comparatively low rate of interest, and they know that the benefits to be obtained from such a fund compare unfavourably both as to amount, elasticity and variety of options with the benefits which may be obtained from an insurance company. The teacher will say with some reason that if he is to be asked to contribute his money—to make a contribution from his modest salary in order to secure provision for his old age—he should at least be allowed to take that money to the quarter in which it will earn the best rate of profit. There is another objection. If you have a fund, then that fund must be subjected to periodical valuation. I have a very great respect for the sombre science of the actuary. But the science of the actuary is not an exact science, and whenever a fund of this kind comes up for revaluation there will always be disputes as to the rate of mortality, and there will also be disputes as to the rate of interest. Again, I say, the larger the operation the more complex it becomes. We have decided that this scheme of superannuation shall be non-contributory.”

Mr. Fisher proceeded to repudiate the suggestion that the Bill would increase the bureaucratic tyranny of the Board. The suggestions that had been made in that direction were, however, limited to the effect of the Bill on schools at present independent of the Board, and its possible effect in forcing them to become

\* The companies might well consider whether and how they should contest this supposed principle, which has been invoked before (Cd. 7365). The Government to which Mr. Fisher belonged must have paid many hundreds, if not thousands, of millions to armament and equipment companies—concerns which certainly do not return the whole or greater part of their profits to their customers as do life offices.



“grant aided,” and thus to enter the pension scheme. I cannot find that the fundamental question whether free pensions would or would not in the long run be for the advantage and add to the dignity of the profession as a whole was ever discussed in the way that has been common in America, or even raised by anyone. The claims put forward for the Bill were : (i) that it would promote the unity of the teaching profession ; (ii) that it would improve the teaching ; (iii) that it would attract the army of men and women teachers who would be required for the purposes of the new Education Act.

Sir James Yoxall, a representative of the teachers, subsequently asserted in the debate that the justification for the Bill was that the Board would not otherwise get the teachers they wanted ; and I fancy that is the real reason of the Government for this hasty legislation. This view is confirmed by a statement by Mr. Fisher who, in resisting in the Committee stage of the Bill an amendment for adding to existing pensions, said :

“The broad principle of the Bill is that it is an attempt to make “the condition of the teaching profession attractive, and to bring “into the profession more teachers in the future. The Bill is “founded on broad considerations of public policy, and it has for its “purpose the securing of a certain definite public advantage. It is “not designed to give benefit to individuals unaccompanied by service “of public advantage. Consequently this Bill is not concerned with “those who have already left the service in Grant-aided schools, and “who will never return to the service, either because they have “migrated to other occupations or because they have retired from “active work.”

So far as fundamentals are concerned the only reasons for setting aside the departmental Committee's contributory scheme of 1912 for secondary teachers are contained in the unconvincing statement by Mr. Fisher which I have already quoted ; and as regards the private members, twenty in number, who took part in the second reading debate, all with one accord encouraged the expenditure and asked for an extension of it in one direction or another. One or two members had some misgivings on the subject of cost and were referred to the debate on the Financial Resolution.

I have therefore referred to this earlier debate and find that the only information as to the cost of the measure given to the House by Mr. Fisher was as follows :

“There is one question which has been asked by the Hon. “Member for Somerset (Mr. King), which is germane to the financial

“resolution which we are now discussing. I have been asked to furnish an estimate of the cost. Of course, hon. Members will realize that the cost of the Pensions Bill, or any system of pensions, cannot be very accurately estimated. There are a number of uncertain factors in any preliminary calculation, but I will, however, offer the House an estimate which will, I think, be as good as can be provided at the present juncture.

“There are at present about 100,000 teachers serving the State under the existing law, and this Bill will increase that number by about 70,000, and it will also increase the amounts of the pensions to be granted in the future, besides giving other benefits in addition. The existing Pensions Scheme now costs us £256,000 yearly, and would probably cost in ten years’ time, if left undisturbed, about £428,000. The Pensions Scheme now proposed will probably, in ten years’ time, cost about £2,000,000 per annum more than this. I ought to add that, in giving this estimate, I make no attempt to forecast the rise in salaries which may take place in ten years’ time. I have endeavoured to take account of the rise which has been going on through this year but not beyond that. Any further increase in salaries will ultimately bring with it a corresponding increase in pensions.

“I have taken account of the increase in the number of pensionable teachers which will immediately result from the inclusion within the pensions system of 70,000 teachers who are now outside the system, but I have not attempted to forecast what may be the increase in the number of teachers in this country due to the extension of our educational system. As that grows so must also the expenditure grow, both on salaries and on pensions. To carry the forecast beyond ten years would involve too many problematical results. My hon. friend asked me for an estimate twenty years hence, but that would be very difficult to give. On the whole the best indication I can give of the expected cost of the present measure is to say that at the present moment the salaries of teachers in grant-aided schools amount to something over £20,000,000 a year, and on those teachers and those salaries we expect the pension system ten years from now to cost something like £2,500,000 per annum, of which £2,000,000 will be new money, and the remainder is what the present system would cost if left unaltered. That is as far as I can go at the present moment, and I hope that indication will satisfy my hon. friend opposite.”

Our science may be sombre, and it may be inexact, but the Government Actuary could upon request have furnished the House with something much closer than this. Mr. Fisher states one interesting fact, namely, that on the basis of present establishments the salary list affected will be over £20,000,000 a year. I take this sum though it looks low for 170,000 teachers. On establishments not rapidly growing the “non-effective” vote for pensions on the Civil Service scale has usually varied between

20 and 30 per-cent of current salaries, so that an ultimate charge to the state of £5,000,000 a year will be nearer the mark than the £2,500,000 given to the House as the probable cost in ten years' time. But, as he frankly said, Mr. Fisher made no allowance for the certain growth of the total salary list owing (i) to increases of scale ; (ii) to normal increases of establishments ; and (iii) to the special increases that will be necessitated by the new Education Act. It is therefore clear that the future expenditure under this legislation must amount to a very large sum.

The Committee stage of the Bill need be referred to only so far as general principles were touched upon. The principal debate centred in a somewhat general desire to extend the Bill to outside secondary schools which, though not grant-aided, were admittedly efficient. The Minister himself proposed an amendment giving the Board power to make rules which would include many of these cases. Sir Henry Craik, the only vocal member who really weighed the fundamental principle of the Bill, again spoke as follows :

“ I warn him (Mr. Fisher) that, however small this Bill, and  
“ however little attention is given to its various stages, he is taking  
“ a very new and a very long step forward towards making teachers  
“ in public schools into Civil servants, and I object to that on the part  
“ of the teaching profession. I think that sooner or later this will  
“ work evil in the teaching profession. Why are you to take one class  
“ of the community more than another, unless they are to be considered  
“ as Civil servants in the service of the State, and give them pensions  
“ without any contribution ? There is no other class who hitherto  
“ have been pensioned on a non-contributory system. This matter  
“ does not deal only with England. It affects also Scottish teachers  
“ very largely. Hitherto it has been the case that Scottish and  
“ English teachers were on an equal footing. . . . . Now you are  
“ digging a gulf between the two sets of teachers. A teacher cannot  
“ change from England to Scotland without dislocating entirely the  
“ scheme of pensions for which he stands. In Scotland we have, so  
“ far, only contributory pensions. I am quite aware that teachers  
“ are not a very reticent class in Scotland if they have any grievance,  
“ and they have not, so far as I am aware, asked that the contributory  
“ system should be entirely abolished. I have always insisted on  
“ liberal aid being given, but that it should be started on a principle of  
“ small contributions, and I do not see why you have suddenly started  
“ an entirely new scheme for England, which is to leave Scotland out  
“ altogether, and which is to exclude from the category for which  
“ provision is made one class of schools which do not come within  
“ the hard-and-fast official regulations. That is the centre of my  
“ doubts about your Bill, and it is not removed by the Amendment

“ of the Right Hon. Gentleman, though it is the only Amendment  
“ which has been introduced to alleviate this hardship. I do not  
“ think that it is sufficient to do away with very many of the  
“ objections which can be urged against this Bill.”

Opposed as he was to the principle of the Bill, Sir Henry Craik, as I think a little unfortunately, said that if passed at all it should be extended to all schools. The battle raged, in fact, round the fear of the bureaucratic tyranny of the Board. He was followed by Sir William J. Collins, who was a member of the recent Committee at King Edward's Fund on Hospital Pensions, and who spoke as follows, after welcoming the Bill :

“ As has been pointed out, in the matter of principle this Bill  
“ has made a most important change in the whole method of retire-  
“ ment legislation, and I think that that might be given as a reason  
“ for not pressing the Right Hon. Gentleman to go much further  
“ than he has done in the case of these non-Grant-aided schools.  
“ After all in 1912 a committee was appointed to consider the  
“ pensions of secondary school teachers, and the Right Hon. Gentleman  
“ has told us that he threw over the proposal of a federated  
“ university scheme on the contributory principle, and, under this  
“ Bill, pensions are being given out of moneys provided by Parliament  
“ to persons who are not in the employment of the State without  
“ any contribution either from the beneficiaries, or from those who  
“ employ them in the shape of local authorities or management.  
“ These, indeed, may be described as revolutionary principles, and  
“ very generous principles, in the matter of pensions, but I think one  
“ can hardly press the Right Hon. Gentleman to go much further  
“ when we are dealing with these private schools which are still less,  
“ as it were, connected with any State organization. If the teachers  
“ in them are to enjoy pension rates on the Civil Service scale without  
“ any contribution whatever, if that principle be conceded, I can  
“ hardly see where the development on these lines will stop. You  
“ must remember that the schemes of the police, the Poor Law, the  
“ Asylums officers, the Metropolitan borough councils are contributory  
“ schemes, and obviously in the case of the municipal employé's and  
“ the borough and county councils, about whom a committee is  
“ sitting, the principle in this Bill will be cited as a model for future  
“ pension schemes. I confess that to press the Right Hon. Gentleman  
“ further in the matter of these private schools than he has already  
“ gone or to widen the limitations he has prescribed in this  
“ Amendment is scarcely fair. . . . I do not think in the Committee  
“ stage or now we have heard any estimate of the final cost out of  
“ the public revenue of pensioning these teachers in private schools.  
“ For these reasons I do not think the Right Hon. Gentleman should  
“ be pressed to go further. He has made generous provision for  
“ the private teacher and I hope this Bill will satisfactorily pass  
“ into law.”



Mr. Dillon not only supported the Bill, but pressed for its universal application as opposed to leaving any discretionary power with the Board. As an old parliamentarian he may well be listened to on this point :

“ All experience in this matter goes to show that when you introduce measures of this character which set up a privileged class the results of the operation of these measures go far in excess of what their originators dreamed of. They set up a tendency which grows and becomes irresistible, and although it is only recently that the Minister of Education has got control of a Government Department he must know the inevitable law, like one of the great laws of science, that all the Government Departments are consumed with a desire to eat up everything that comes their way. I am not blaming them. It is of their essence and nature and Government Departments universally—I never knew an exception—imagine that their methods are the best, and their whole desire and tendency is to gather everything under the machine, run it all in accordance with the machine and get it under their control.”

After similar appeals for extension, Mr. Fisher brought the discussion back to a sense of proportion by remarking that, while the original Bill would apply to 170,000 elementary and secondary teachers, there were left outside only 2,600 teachers in 105 schools who were on the Board's efficient list, and that his object was to try and provide for some of those. If the whole were included the annual cost in 10 years' time would be about £90,000. His proposal was then agreed to. I think nothing else of main interest occurred in Committee apart from the general endeavour to secure extensions—especially in the case of administrative officers of education authorities ; and I would have made my quotations still shorter, but for the side-lights thrown on important questions of principle.

What is the bearing of the new Act on the general principle of vested rights, with which in this note I am more particularly concerned ? I cannot see that it carries any weight at all in theory, whatever it may do in practice. If it is to the temporary and pecuniary advantage of the State to bribe or induce entrants to a profession by the promise of free pensions, and thus secure them in a grip for life, then surely it must be to the advantage of the profession to cry “ *timeo Danaos* ” and refuse the proffered gift. As I have said, the only vocal member of Parliament who faced fundamental principles was Sir Henry Craik, and he on the second reading remarked : “ I should be false to the whole

“work of my life if I did anything which would delay and  
“hinder a real benefit to the teacher. As I have urged over and  
“over again, be liberal in your salaries and pensions ; but I am  
“not quite sure that this benefit, accompanied as it is by a  
“transformation of the whole profession into what will come  
“very nearly to a Civil Service system, will be in the long run  
“for the real advantage of that profession or for the advantage  
“of education.”

To the onlooker it would seem that prior to this legislation the teaching profession was in a strong position. The State was faced with the apparently impossible task of securing the teachers it required on the basis of existing inducements. By combination the teachers could have raised the whole level of remuneration of their profession ; while the superannuation system recommended by the Departmental Committee of 1912, or something similar, would have secured to them the vested pension rights, and thus at the same time their liberty. As Mr. Fisher happily remarked “the casuistry of life is infinite”, and one must not dogmatize without full knowledge ; but as the case stands we must refuse to accept this new Act, based as it is on opportunist reasoning, as settling the question of principle whether pension rights should or should not vest from year to year. Meanwhile it seems to me that the University staffs in this country, who do not come within the scope of the new Act, but who have already succeeded in establishing the principle of vested rights, would be wise to regard the preservation of that principle as fundamental in any steps they may take to improve their own position.

Whatever may be our opinions on the merits of this question of principle it is clearly matter for regret that Parliament should have failed signally to realize the importance of discussing it and dealing with it advisedly. The need for some clear thinking on it still exists, and it is thus that the points involved present themselves in summary to my mind :

- (i) If men and women are to do their best work must they not be free ?
- (ii) If they are to be free must not pension rights enure and vest from year to year ?
- (iii) If the answer to these questions is in the affirmative economic enquiry should follow into a remaining question, namely : On whom, whether employers or employees or both, does the cost of pensions really fall ?



In other words, we need to determine clearly whether, as is commonly supposed, pensions can only be paid for in the long run by the labour of the employees. If that be so and if the cost really falls on the latter, we shall then be on firm ground in asserting that they cannot sacrifice freedom and accept free pensions, where employers hold the power of the purse, without economic detriment.

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*The American-Canadian Mortality Investigation, 1900-1915.\**

[Communicated by MR. ARTHUR HUNTER, A.I.A., F.F.A., Chairman of the Committee on Mortality Investigation.]

SYNOPSIS OF VOL. I OF THE REPORT.

THERE are two tables in use in the United States for the calculation of premiums and of reserves, namely, the "Actuaries' Combined or Seventeen Offices", and the "American." The former, published in 1843, has been little used in recent years except by companies which assumed 4 per-cent interest. The principal reason for the "Actuaries' " Table remaining in use so long is that it is the basis for valuation in States in which 4 per-cent interest is permitted. The "American," with interest at 3 per-cent or  $3\frac{1}{2}$  per-cent, is the reserve standard in most of the States and is the basis of premiums in the principal American companies. The table was completed in 1860 and was based largely on the experience of the Mutual Life Insurance Company. The rates of mortality are considered as "ultimate," but the American Table at the younger ages is known to be distinctly higher than the ultimate experience. As the reserves based on that table are considered sufficient and as the premiums based thereon result in a satisfactory margin of safety, there has not been a disposition to make the radical changes which a new table might necessitate. These might involve not only a change in the premium rates and in the reserves, but also in the provisions of the laws of the State of New York with regard to initial commissions to agents, the measurement of the company's first year's and total expense allowances and the rule for determining the maximum new business permitted.

The National Convention of Insurance Commissioners in

\* Published by the Actuarial Society of America.

August 1911 requested the Actuarial Society to "turn its attention to the construction of new mortality tables covering the general experience of the companies among normal lives." It was then pointed out by the Society that the new table would probably show slightly higher reserves and that the premiums charged by the insurance companies would not necessarily be affected by such a new table because a larger loading might be required. A larger percentage of loading would doubtless be necessary in the case of the companies issuing non-participating policies with a loading of from 0 to 5 per-cent on the net American  $3\frac{1}{2}$  per-cent premiums. These companies expect to pay part of their expenses out of the savings in mortality which would largely be eliminated under the new table.

The Actuarial Society expressed its intention of preparing a new mortality table after the Medico-Actuarial Mortality Investigation had been completed. In 1914, before the publication of the last volume of the report on the M.A.M.I., the Insurance Commissioners again asked the Society "to construct a mortality table which is in keeping with the ascertained experience of American companies." In 1915 the first meeting to make the plans for the investigation took place. The Insurance Commissioners appointed a committee of Insurance Department Actuaries to co-operate with the committee of the Actuarial Society, while the latter invited the American Institute of Actuaries to appoint a committee of co-operation. The work has been conducted by the Actuarial Society and the other two bodies have given their advice and assistance. While the main purpose of the investigation was to obtain "select" and "ultimate" tables based on the experience of men resident in the United States (American men), advantage was taken of the opportunity to obtain corresponding tables for men resident in Canada (Canadian men), and to investigate the mortality among women. These results are given in Volume I, while in Volume II will appear the report on other investigations such as the mortality by plan of insurance among American men, by groups of States in the United States and by provinces in Canada.

As an investigation based on recent data was desired, it was decided to take the experience during the years 1900 to 1915 inclusive. This included the issues of 1900 to 1914 carried to the anniversary of the policies in 1915, and also the issues prior to 1900, if continued in force until after the anniversary of

the policies in 1900. Thus, a policy issued in 1895 was observed from its anniversary in 1900 to the anniversary in 1915, or its previous termination.

The investigation is based on amounts insured, but policies issued at the same age on any individual life for more than \$100,000 were treated as \$100,000.

If all the data of the large companies had been taken, they would have had a preponderating influence in the investigation, and accordingly a rule was adopted whereby the larger the company, the smaller would be the percentage of its data to be contributed.

For the benefit of students the report gives the reasons of the committee for adopting the several rules such as those with regard to policies (a) dated back, (b) changed in amount and in plan, (c) lapsed and reinstated. The reasons are also given for omitting policies re-insured in bulk, or issued on the group insurance plan and those issued incontestable from date.

The mean duration method of determining the duration of the insurance was used.

Probably fewer refinements in technical treatment were made than in any similar investigation, but it is believed that the methods used give as accurate results as are necessary for the purposes of the tables.

A study of the methods used in collecting and in recording the data shows that the companies in general gave their share of the data on "policy sheets", and that "perforated" cards were prepared by the Bureau established by the Actuarial Society to carry on the work. Both the perforating and the sorting machines are developments of those used in the M.A.M.I. The tabulating machine, in addition to making summations of different columns, printed a record of each card, giving such items as year of issue, age at entry, duration, sex, mode of termination, plan of insurance, habitat at date of application for insurance, and amount of insurance.

#### EXTENT OF THE STATISTICS.

Fifty-nine American and Canadian companies, representing 95 per-cent of the regular insurance in the United States and Canada, contributed to the investigation.

From the following table, the extent of the data may be judged :

	American Men	American Women
Total exposures ...	\$26,003,321,900	\$787,819,500
Dead ...	436,345,200	10,878,500
Average exposure...	6.88 years	6.18 years
	Canadian Men	Canadian Women
Total exposures ...	\$2,743,665,100	\$69,144,100
Dead ...	29,400,800	672,500
Average exposure...	6.05 years	6.37 years

## AMERICAN MEN.

A number of tests showed that the effect of medical selection apparently did not extend beyond the fifth policy year. At the younger ages the mortality was much lower than in the American table, the ratio to the American table, for example, at attained age 30 was 51 per-cent. It was accordingly to be expected that the mortality during the early policy years would not show as great an effect of selection as was generally assumed.

In the following table the ultimate  $q$  of the new experience of American Men—to be known as the A.M.—is compared for decennial ages with the  $q$  of four standard tables :

*Comparison of Rates of Mortality with A.M.<sup>(5)</sup>*

Attained Age	Ratio to O.M. <sup>(5)</sup>	Ratio to American	Ratio to M.A. ultimate	Ratio to U.S. Males (1910)
25	63%	53%	90%	77%
35	57	55	96	58
45	66	69	103	61
55	84	95	112	82
65	96	103	106	95
75	99	96	98	97

The foregoing percentages show (1) that there has been a marked improvement in the mortality among insured lives at the younger ages, but little, if any, at the older ages ; (2) that there is not a wide difference on the whole between the Medico-Actuarial Mortality Table (ultimate) and the American Men ultimate table ; (3) that there is a much lower death rate (ultimate) in the insurance companies than in the population of the United States except at the older ages when there is little difference.

A feature of the ultimate curve of mortality is that neither the Gompertz nor Makeham method is suitable prior to age 50, and it is necessary, on account of the flatness of the curve at the younger ages, to apply a supplementary curve. In the report are given at quinquennial ages the results of graduations by Spencer's 21 Term Formula, King's Osculatory Interpolation, Larus' 23 Term Formula, G. F. Hardy's Friendly Society Formula, a Makeham graduation with a supplementary curve, and a Gompertz graduation with a supplementary curve. The method finally adopted was based upon Makeham's second modification of Gompertz's Law with a supplementary curve.

The same method was applied to the graduation of the first five insurance years with modified values of the constants other than  $c$  for each duration. A short table is now given of the graduated rates of mortality at decennial ages at entry :

*American Men.*

Age at Entry	$q(x)$	$q(x)+1$	$q(x)+2$	$q(x)+3$	$q(x)+4$	$q(x)+5$
15	·00247	·00324	·00341	·00355	·00372	·00392
25	·00293	·00384	·00402	·00412	·00427	·00446
35	·00316	·00429	·00457	·00480	·00523	·00584
45	·00482	·00690	·00782	·00866	·00992	·01158
55	·00994	·01447	·01712	·01951	·02271	·02668
65	·02252	·03267	·03931	·04517	·05263	·06147

In the M.A. Mortality Table, based on policies, the effect of selection was not apparent for more than five years, and the same is true of the new table. In the latter the ratio of actual to expected deaths by the ultimate table for ages at entry 15 to 34, is 100·1 per-cent for the sixth to the tenth insurance years and 98·7 per-cent for the eleventh to the fifteenth insurance years. There is no evidence that the ultimate rates should have been prepared by including more policy years at the young ages or fewer years at the older ages at entry.

As the last death in the American Experience Table is assumed to be at age 96, the custom of the companies is to pay the face amount of the insurance at that age. The history of these cases was traced by the companies to the date of death, or the policy anniversary in 1915.

The net premiums under the American Men Mortality Table (ultimate) are naturally lower than under the American Experience Table. At age 20 this amounts to 18 per-cent, and



at 60 to 1 per-cent on the Ordinary Life plan. Comparing the new table with the  $O^{M(5)}$ , the corresponding percentages are 17 per-cent and 2 per-cent. On the 20-year Endowment Insurance plan at age 20 the A.M.<sup>(5)</sup> is  $5\frac{1}{2}$  per-cent lower than the American and 4 per-cent lower than the  $O^{M(5)}$  while at age 60 there is practically no difference in the premiums under the American and A.M.<sup>(5)</sup>, but the  $O^{M(5)}$  is 2 per-cent higher than the A.M.<sup>(5)</sup>

The difference between the mean reserves on the A.M.<sup>(5)</sup> and the American varies with age at entry, duration of policy and plan of insurance, although they are generally higher under the former than under the latter on the Ordinary Life plan. A valuation of the entire business of an old established company showed 2 per-cent higher reserve on the A.M.<sup>(5)</sup> than on the American basis.

It is rather rash to express an opinion with regard to the chance of the new table taking the place of the American for the calculation of premiums and reserves. The chance is less than it was before the influenza epidemic. The deaths during the last three months of 1918 were in most companies approximately the same in amount as during the first nine months of the year—in other words the death rate for October, November and December 1918, was three times the normal. In these months the deaths in the population from influenza and epidemic pneumonia was about 4.5 per 1,000 inhabitants.

#### CANADIAN MEN.

As may be seen from the following comparison of the rates of mortality for the sixth and succeeding insurance years combined the mortality among Canadian men is much lower than that shown by the  $O^{M(5)}$ , the standard for valuation in Canada :

*Comparison of Canadian Mortality Table C.M.<sup>(5)</sup>*

Attained Age	Ratio to $O^{M(5)}$	Ratio to A.M. <sup>(5)</sup>
25	64%	102%
35	53	93
45	58	88
55	71	84
65	83	87
75	93	94



Except at the young ages the ultimate mortality among Canadian men is lower than among American men and the same result was found to apply generally to the first five policy years.

The method of graduation was the same as that applied to the American Men Table, but there were greater percentages of deviation in the expected deaths as the material for Canadian men was only 10 per-cent of that for American men. In the following appears the graduated rates of mortality for decennial ages at entry :

*Canadian Men.*

Age at Entry	$q[x]$	$q[x]+1$	$q[x]+2$	$q[x]+3$	$q[x]+4$	$q[x]+5$
15	·00195	·00307	·00342	·00370	·00393	·00412
25	·00254	·00365	·00394	·00411	·00423	·00428
35	·00280	·00380	·00414	·00453	·00495	·00533
45	·00469	·00582	·00659	·00761	·00877	·00987
55	·01011	·01186	·01386	·01657	·01964	·02258
65	·02425	·02783	·03303	·03996	·04773	·05507

The net premiums on the C.M.<sup>(5)</sup> are lower than on the O<sup>M</sup>(<sup>5</sup>). The mean reserves are sometimes higher and sometimes lower on the former than on the latter, depending on age at entry, duration of policy and plan of insurance.

### MORTALITY AMONG WOMEN.

It was shown in the Medico-Actuarial Mortality Investigation that the mortality among women depended upon their conjugal condition at the date of application for insurance. The lowest ratio of actual to expected deaths was 81 per-cent of the Medico-Actuarial Table (M.A.) among spinsters and the highest 126 per-cent among married women whose husbands were the beneficiaries. When the data were divided by plan of insurance the relative mortality among the four classes of women differed only slightly from the experience for all plans combined. The present investigation does not make a division according to either conjugal condition or plan of insurance, and its chief value therefore is in enabling us to compare the experience by amounts insured among women with that among men.

The ratio of actual to expected deaths by the A.M. Table is 110·8 per-cent for American women for the first to the fifth insurance years, 91·5 per-cent for the sixth and succeeding years, and 94·4 per-cent for the entire experience. The detailed

tables show that the mortality among American women was not as good as among American men in the early insurance years, but was distinctly better in the sixth and succeeding years at attained ages over 40. On the whole the companies have experienced a better mortality on women than on men.

Measuring the mortality on Canadian women by the select and ultimate table for Canadian men (C.M.) the ratio of actual to expected deaths is 117.9 per-cent for the first five insurance years, 93.3 per-cent for the sixth and succeeding years and 97.9 per-cent for the entire experience. The relative mortality was slightly lower than among American women, but this may be due to a different distribution by conjugal condition, or to a different distribution of the insurance between very healthful and less healthful habitat, or to accidental fluctuations. (The amount of material on Canadian women is small, the actual deaths amounting to \$672,500).

The statistics show that the experience on Canadian women is slightly more favourable than among Canadian men, but less favourable in the earlier insurance years and at the younger attained ages.

There was no difficulty in choosing a name for the experience on Canadians. As there has not been any table prepared in Canada, based on either the population or insured lives, it was obviously proper to use the word "Canadian" as a designation. When it came to selecting a name for the American experience considerable difficulty was met. The name "American" had been pre-empted by the table constructed in 1860 by Sheppard Homans and D. Parks Fackler, although it did not merit such a comprehensive title. The phrase "United States Life Tables" could not be used because the Government had selected that designation for the experience based on the population of 1910 and the deaths for the three years 1909, 1910 and 1911. There is an Actuaries' Table and an American Offices Table. While not very satisfactory on account of the limitations thus imposed the title chosen was American Men Mortality Table. It is therefore written A.M. and not A<sup>M</sup>.

The instructions and rules were sent to the companies in June 1916, and the data recorded on policy sheets were received by the Bureau at the end of that year or the beginning of 1917. With a force of less than twenty women the work of perforating over 2,000,000 cards, of sorting and tabulating the data was completed with the help of modern electrical machinery in less

than a year. The calculation of the expected deaths was done by a force of clerks of one of the insurance companies, working after their regular office hours. If the war had not intervened, making it difficult to obtain the services of trained clerks, delaying the printing for many months on account of lack of proper paper and expert type-setters, the volume containing the investigations here described would have been available in less than two years after the companies were asked to collect their statistics.

*Two recent Legal Decisions affecting Titles to Reversions.*

By A. H. WITHERS, *Barrister-at-Law.*

*The decision in Hill v. Peters* [1918] 2 Ch. 273.

IT has long been settled law that if A, being entitled to a life or reversionary interest in personal estate held by a trustee, assigns it first to X and secondly to Y, then X and Y are entitled to priority as between themselves in the order of time in which the trustee receives notice of their respective assignments. This is known as the rule of *Dearle v. Hall*. The exceptions to this rule prevent Y obtaining priority over X (1) where Y has not given value for his assignment or is a trustee in bankruptcy of A, &c., and (2) where Y at the time of taking his assignment has notice (actual or constructive) of the assignment to X.

The recent decision of Mr. Justice Eve in *Hill v. Peters* says there is yet another exception, namely, where the assignment to X takes the form of a declaration of trust by A in favour of X.

In *Hill v. Peters* the material facts were shortly as follows : In September 1897 a reversion was mortgaged to two solicitors, who promptly gave notice to the trustees of their mortgage, and in the following month declared by deed (of which notice was not given to the trustees) that they held the mortgage in trust for the defendant Peters and another. In 1907 the two solicitors, in fraud of their beneficiaries and without disclosing the trust, purported to join in mortgaging the reversion to Mrs. Gwynn (whose executor was the plaintiff Hill) and notice of her mortgage was given to the trustees in 1917. It was held by the learned judge that (1) the rule of *Dearle v. Hall* did not require notice to be given of the declaration of trust, and (2) the assignment to Mrs. Gwynn being in fraud of the beneficiaries passed nothing, and (3) therefore the beneficiaries under

the prior declaration of trust (of which notice was *not* given to the trustees) had priority over the mortgage to Mrs. Gwynn, of which notice was given to the trustees.

It is impossible to tell from the report in the Law Reports what exactly was the transaction of 1907. Presumably the solicitors did not in 1907 receive payment of their mortgage debt; because if they had been paid off, the rights of their beneficiaries would probably have been destroyed by Section 20 of the Trustee Act, 1893, which provides that "the receipt in writing of any trustee for any money securities or other personal property or effects payable, transferable, or deliverable to him under any trust or power shall be a sufficient discharge for the same, and shall effectually exonerate the person paying, transferring, or delivering the same from seeing to the application or being answerable for any loss or misapplication thereof."

The learned judge decided definitely that the rule in *Dearle v. Hall* does not apply to a declaration of trust so as to require the beneficiary under the declaration to give notice of his interest to the trustee holding the fund. It is certainly curious that the point seems never to have been considered previously, and there is a good deal of force in the observation that where an equitable interest is lawfully left in the name of a trustee it is unnecessary, on the point of priority, to give notice to the holder of the fund that other persons claim *through* the trustee: such a notice merely supports the right of the trustee to receive the equitable interest left in his name. But on the other hand Courts of Equity usually regard the substance rather than the form of a transaction, and it seems absurd, and opening the door wide for fraud, to say that the rule of *Dearle v. Hall* applies where the document takes the form of an assignment and not where it takes the form of a declaration of trust by a vendor or borrower in favour of a purchaser or mortgagee.

The arguments and judgment in the case suggest that the point would have arisen just the same had there been no formal declaration of trust, *e.g.*, if the solicitors had been trustees of a fund and rightly or wrongly invested it upon the mortgage of 1897 and had executed no declaration of trust. It is difficult to see how *Dearle v. Hall* could apply to such case. Of what assignment could notice be given to the trustees?

The rule in *Dearle v. Hall* having been brushed aside, the way was clear for the application of an exceedingly inconvenient rule, which undoubtedly applies to land and shares, namely,

that if a purchaser or mortgagee of land or shares in a company obtains only an *equitable* estate or interest in the land or shares, then he takes subject to all prior equitable estates or interests in the land or shares, *e.g.*, if the vendor or mortgagor has declared himself a trustee for others and subsequently *makes the sale or mortgage in fraud of his beneficiaries* then, in the absence of special circumstances, the beneficiaries are not prejudiced by the sale or mortgage. The second part of the decision in *Hill v. Peters* is that this rule applies to choses in action, as well as to land and shares, in cases where the rule in *Dearle v. Hall* does not apply.

So far as the present writer is aware, this is quite a new point and has not previously been put forward. It has always been considered that one is safe in dealing with the ostensible owner or mortgagee, regardless of any secret trust that such owner or mortgagee may have created or been subject to. It is now decided that this is quite wrong.

The decision is one of the utmost importance to purchasers and mortgagees of reversions; more important indeed than any case that has been decided for a good many years past.

The practical result of the decision, if correct, is that if an ostensible owner or mortgagee (past or present), is, unknown to the purchaser or mortgagee, a trustee for others and acts in breach of trust, then the purchaser or mortgagee takes subject to the rights of the undisclosed beneficiaries. It is absolutely impossible to prove the negative, or to obtain satisfactory evidence on the point or gain protection by enquiry of or notice to the trustees of the fund.

If *Hill v. Peters* is correct, its inconveniences may be reduced in some cases by the section above quoted from the Trustee Act, 1893, *e.g.*, if in *Hill v. Peters* the two solicitors had been paid off and had given a receipt for their mortgage, their beneficiaries would have lost their security. That section, however, is of somewhat limited extent, *e.g.*, it does not seem to apply where the ostensible owner or mortgagee is only one of several trustees, or where a trustee is selling without any trust or power of sale, or where a trustee gives a receipt or release without receiving payment in full.

*The decision In re Pain*, 1919, 1 Ch. 38.

In this case Mr. Justice Younger decided that a purchaser or mortgagee of a reversion is usually not affected by the reversioner,



after the trustee has received notice of the assignment, commencing and prosecuting litigation against the trustee and being ordered to pay the costs of the proceedings. But if the assignee stands by and allows costs to be incurred by the reversioner *which the assignee could have prevented*, then the costs payable by the reversioner to the trustee are charged on the reversion in priority to the claims of the assignee. Compared with *Hill v. Peters*, this decision is of trifling importance; but it is a decision that has to be borne in mind. When a purchaser or mortgagee is made a party to an action affecting the trust, he has to consider whether he can and ought to stop the action. If he cannot stop the action, then he cannot be blamed or made in effect to bear the costs; but if he can but does not prevent the costs being incurred, he allows the proceedings to continue at some risk to himself. This point will trouble mortgagees more often than purchasers.

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## LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law*.

Policy issued to  
cover funeral  
expenses.  
Conflict of Claim.

A FURTHER case on the construction to be placed upon Section 36 (2) of the Assurance Companies Act, 1909, is that of *Hatley v. Liverpool Victoria Friendly Society*, L.T.R. 118, 687.

This case, which was an appeal from a decision of the City of London Court, dealt with the question as to who was entitled to the proceeds of an industrial policy of life assurance effected by a father on the life of his son. The decision of the Court was in accord with the decision of the Court of Appeal in the case of *Da Costa v. Prudential Assurance Company* (J.I.A., vol. li, p. 51).

The facts are as follows:

In 1891, George Lake effected a policy of insurance in the Liverpool Victoria Friendly Society on the life of his son, William Oliver Lake, aged two years. He retained the policy and continued to pay the premiums until his death in March 1915. There was evidence that he gave the policy to his daughter, Mrs. Jenner, and that she paid the premiums after his death. William Oliver Lake died in June 1916, and in June 1917, letters of administration to his estate were taken out by the plaintiff who claimed the policy moneys.



The trustees of the defendant Society made an award in favour of Mrs. Jenner, whereupon the plaintiff sued them in the County Court. The action was dismissed and upon appeal by the plaintiff the Court upheld the decision of the County Court Judge.

In the course of his judgment, Ivory, J., said: "All the  
" authorities which have been cited to us I think result in this,  
" that the question is whether there is admissible evidence to  
" show that the father at the time he made the purchase of the  
" policy in respect of his child intended that purchase to be for  
" his own benefit or for the benefit of the child. If there is  
" admissible evidence to show that he intended it to be for his  
" own benefit, then the presumption that it was for the child's  
" advancement is rebutted. Now, in the present case I think  
" there was ample evidence to rebut the *primâ facie* presumption  
" that this policy was intended for the benefit of the child.  
" Looking at the terms of the policy itself and at all the  
" circumstances of the case, it is clear, I think, that the father  
" took it out for his own benefit, in order to cover the possible  
" expenses of the child's funeral. That view of this case is  
" confirmed by a reference to the Assurance Companies Act,  
" 1909, s. 36, the second paragraph of which, in my opinion,  
" applies to this policy: 'No policy effected before the passing  
" ' of this Act with a collecting society or industrial assurance  
" ' company shall be deemed to be void by reason only that the  
" ' person effecting the policy had not, at the time the policy  
" ' was effected, an insurable interest in the life of the person  
" ' assured.' It is to be observed that prior to this Act there  
" had been a question, at all events, raised as to whether the  
" fact of a father being liable for the funeral expenses of his  
" child gave him an insurable interest in the life of the child.  
" Then the section a little lower down says: 'If the policy was  
" ' effected by or on account of a person who had at the time a  
" ' *bonâ fide* expectation that he would incur expenses in  
" ' connection with the death or funeral of the assured, and if  
" ' the sum assured is not unreasonable for the purpose of  
" ' covering those expenses. . . . such policy shall enure for  
" ' the benefit of the person for whose benefit it was effected or  
" ' his assigns.' I think that applies to the present case, and  
" that therefore shows, in conjunction with the rest of the  
" evidence, that this policy, which was one taken out for the  
" benefit of the father, did enure for his benefit, and therefore

“ formed part of his estate. Under those circumstances the  
 “ plaintiff had no claim or title to it. On that short ground I  
 “ think the judgment of the learned County Court Judge must  
 “ be upheld and the appeal dismissed.”

Shearman, J., concurred.

Company subject  
to British and  
Colonial Income  
Tax.

What is proper  
deduction from  
dividend on  
Preference  
Shares?

A case of some interest to life assurance officials who have to deal with questions of income tax is that of *Rover v. South African Breweries, Limited*, T.L.R. 34, 478. This was a special case stated by consent to decide whether a company which has paid British income tax and also Colonial income tax, and has obtained from the British authorities repayment of the Colonial income tax, is entitled to deduct from a shareholder's dividend the total amount paid on account of British income tax or such amount less the amount of Colonial tax recovered.

It was decided by Astbury, J., that the company could only deduct from the dividend due to the preference shareholders the amount of British tax less the Colonial tax recovered.

The facts are as follows :

The plaintiff was the holder of 500 5 per-cent cumulative preference shares and 1,300 ordinary shares in the South African Breweries, Limited. The directors of the company paid 5 per-cent on the preference shares and a dividend on the ordinary shares.

The company paid income tax on its profits in South Africa at a rate of not less than 1s. 6d. in the £, and was assessed to British income tax at the rate of 5s. in the £ on its assessable profits. Relief was granted under Section 43 of the Finance Act, 1916, to the extent of 1s. 6d. in the £, subject to proof of payment of the tax in South Africa.

The company deducted income tax at the rate of 5s. in the £ from its preference share dividends and 3s. 6d. in the £ on the ordinary share dividends.

In the course of his judgment Astbury, J., said :

“ The Colonial Tax is, *qua* the shareholder in this country,  
 “ not a duty charged within the meaning of Section 54, but is an  
 “ outgoing of the company, as any other administrative expense,  
 “ in the Colony, which has to be satisfied before any profits can  
 “ be ascertained and distributed in this country by way of  
 “ dividend. The real question is whether the company was  
 “ charged and paid 5s. or 3s. 6d. in the £ for British income tax  
 “ for the year 1917, within the meaning of Section 54. If the

“ former it was right in making the 5s. deduction from the  
“ plaintiff’s dividend ; if the latter it still owes to him the unpaid  
“ portion of the dividend that he became in law entitled to when  
“ the declaration of the dividend was made. The company  
“ contends that under Section 43 the person who pays both  
“ Colonial and British tax is the only person entitled to recover  
“ the 1s. 6d. in the £, and to keep it It is said that the share-  
“ holder has not paid the Colonial tax, and therefore does not  
“ bring himself within the benefit of the section. The company  
“ also says that if the plaintiff succeeds he and those in his  
“ position will become a privileged class of British taxpayer,  
“ escaping with the payment of only 3s. 6d. in the £. It is also  
“ urged that as between the company and the plaintiff the  
“ former did in effect pay 5s. in the £ for British tax, and that  
“ it was only on proof of this that the right to repayment arose  
“ under Section 43, although the matter was, as between the  
“ company and the Crown, dealt with in account as I have stated.  
“ I think that the company, having paid both the taxes in the  
“ first instance, is the person entitled to the repayment of the  
“ 1s. 6d. ; but the question remains whether when it has obtained  
“ it, it can reasonably and in truth be said to have paid or to  
“ have become liable to pay 5s. in the £ for British tax. The  
“ whole object of Section 43 was that the payment of income  
“ tax was to be reduced as therein provided, and it is difficult to  
“ see how a company which has paid 5s. on the terms that it  
“ should immediately be repaid 1s. 6d. cash out of it can be said,  
“ after such repayment, to have expended more than 3s. 6d. If  
“ this is so it follows that the only deduction authorized by  
“ Section 54 of the Income Tax Act, 1842, is 3s. 6d., and that the  
“ balance, 1s. 6d., of the dividend is still unpaid or payable.”

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Appointment  
of Royal  
Commission on  
Income Tax.

In view of the complexity of the income tax laws, a complexity which does not seem to have been decreased by the passing of the codifying Act in 1918, it is interesting to record that the Government have appointed a Royal Commission to enquire into the whole subject. Actuaries will have been gratified to learn that the President of the Institute, Mr. Geoffrey Marks, is to serve on this Commission.

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## On Certain Inequalities and Methods of Approximation.

By J. F. STEFFENSEN, D.Phil.

IN a previous paper\* I have proved an inequality of very general nature, admitting applications to certain actuarial problems. The subject has been taken up by Mr. B. Meidell,† whose starting-point was the well-known inequalities of Dr. J. L. W. V. Jensen.‡ In the present paper I propose to give an account of these investigations, commencing with my own (with a few unessential simplifications), and generalizing to a certain extent Mr. Meidell's and Dr. Jensen's.

Let us assume that two functions  $f(t)$  and  $\phi(t)$  are integrable, that is, admit an integral from  $a$  to  $b$ ; that  $f(t)$  never increases, and that  $0 \leq \phi(t) \leq 1$ . Then, putting for abbreviation

$$\lambda = \int_a^b \phi(t) dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we can prove that

$$\int_{b-\lambda}^b f(t) dt \leq \int_a^b f(t) \phi(t) dt \leq \int_a^{a+\lambda} f(t) dt \quad . \quad . \quad . \quad (2)$$

If  $\phi(t)=1$  or  $\phi(t)=0$  or  $f(t)=\text{const.}$  for all  $t$ , the two limits in (2) coincide.

Before proceeding to the rigorous proof we may observe that the inequalities (2) are almost obvious. For instance, dividing them by  $\lambda$ , the expression

$$\frac{\int_a^b f(t) \phi(t) dt}{\int_a^b \phi(t) dt}$$

may be regarded as a *weighted mean* of  $f(t)$ , the weights being  $\phi(t)dt$ , and the total of weights  $\lambda$ . But in the mean

$$\frac{1}{\lambda} \int_a^{a+\lambda} f(t) dt$$

\* "On certain inequalities between mean values and their application to actuarial problems." Skandinavisk Aktuarietidskrift, 1918, pp. 82-97.

† "Note sur quelques inégalités et formules d'approximation." Skandinavisk Aktuarietidskrift, 1918, p. 180.

‡ "Sur les fonctions convexes et les inégalités entre les valeurs moyennes." Acta Mathematica, vol. xxx, p. 175 (1905-6). See also an earlier paper (quoted by Dr. Jensen) by O. Hölder in Göttingen Nachr., 1889, p. 38; a Paper by L. Galvani in Rendiconti del Circolo Matematico di Palermo, vol. xli (1916), p. 103; and Mr. Lidstone's Note in J.I.A., vol. xlv, p. 490.

the same total of weights has been concentrated round the distance from  $a$  to  $a + \lambda \leq b$ , that is, round the larger values of  $f(t)$ , and each of these values is weighted with the maximum weight.

The arithmetical proof is easy. Remembering the assumed properties of  $f(t)$  and  $\phi(t)$ , the second inequality (2) may be derived as follows :

$$\begin{aligned}
 & \int_a^{a+\lambda} f(t) dt - \int_a^b f(t) \phi(t) dt \\
 &= \int_a^{a+\lambda} [1 - \phi(t)] f(t) dt - \int_{a+\lambda}^b f(t) \phi(t) dt \\
 &\geq f(a + \lambda) \int_a^{a+\lambda} [1 - \phi(t)] dt - \int_{a+\lambda}^b f(t) \phi(t) dt \\
 &= f(a + \lambda) \left[ \lambda - \int_a^{a+\lambda} \phi(t) dt \right] - \int_{a+\lambda}^b f(t) \phi(t) dt \\
 &= f(a + \lambda) \int_{a+\lambda}^b \phi(t) dt - \int_{a+\lambda}^b f(t) \phi(t) dt \\
 &= \int_{a+\lambda}^b \phi(t) [f(a + \lambda) - f(t)] dt \geq 0.
 \end{aligned}$$

The first inequality (2) may be derived in a precisely similar way, but is more quickly obtained by putting, in the inequality just proved,

$$\begin{aligned}
 \phi(t) &= 1 - \phi_1(t) \\
 \lambda &= \int_a^b [1 - \phi_1(t)] dt \\
 &= b - a - \int_a^b \phi_1(t) dt \\
 &= b - a - \lambda_1
 \end{aligned}$$

Inequalities for *sums* (instead of *integrals*) may be derived by the same method. But it will repay the trouble to notice once for all, how the transition from integrals to sums may be effectuated. For this purpose we want a definition of a sum, when the limits of summation are not integers. If  $x = \alpha - \theta_1$ ,

$y = \beta + \theta_2$ , where  $\alpha$  and  $\beta$  are integers,  $0 \leq \theta_m < 1$  and  $y \geq x - 1$ , we propose the convenient notation

$$\sum_x^y u(n) = \theta_1 u(\alpha - 1) + \sum_\alpha^\beta u(n) + \theta_2 u(\beta + 1) \quad . \quad . \quad . \quad (3)$$

it being understood that  $\sum_\alpha^{\alpha-1} u(n) = 0$ .

As the sum depends only on *integral* values of the argument  $n$ , we may, for the purpose of demonstrating the inequalities, put

$$u(n + \theta) = u(n) \quad (0 \leq \theta < 1).$$

With this convention, (3) may be written

$$\sum_x^y u(n) = \int_x^{y+1} u(t) dt \quad (y \geq x - 1) \quad . \quad . \quad . \quad (4)$$

We have, for instance,

$$\sum_x^y 1 = y - x + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\sum_x^y u(n) = \sum_x^t u(n) + \sum_{t+1}^y u(n) \quad (x - 1 \leq t \leq y) \quad . \quad . \quad . \quad (6)$$

From (2) we may, by (4), immediately conclude, for  $y \geq x - 1$ ,

$$\sum_{y-s+1}^y f(n) \leq \sum_x^y f(n) \phi(n) \leq \sum_x^{x+s-1} f(n) \quad . \quad . \quad . \quad . \quad (7)$$

where 
$$s = \sum_x^y \phi(n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

provided that  $f(n)$  never increases, and that  $0 \leq \phi(n) \leq 1$ .

If  $\phi(n) = 1$  or  $\phi(n) = 0$  or  $f(n) = \text{const.}$  for all  $n$ , the two limits in (7) coincide.

It is evident that we may, in (2) and (7), allow the range of integration or summation to increase indefinitely, provided the expressions do not become meaningless.

We have assumed  $0 \leq \phi(t) \leq 1$  for the sake of simplicity; but if, instead of this we have quite generally  $l \leq \psi(t) \leq L$ , where  $l$  and  $L$  are any two different, positive or negative quantities, we need only apply the theorem to

$$\phi(t) = \frac{\psi(t) - l}{L - l}$$



in order to find limits for  $\int_a^b f(t)\psi(t)dt$  and for  $\sum_x^y f(n)\psi(n)$ . We leave the details to the reader.

The inequalities we have established are useful in mathematical analysis, their affinity to such theorems as Abel's Lemma\* and the Theorem of Mean Value† being obvious. Here we are, however, only concerned with their application to actuarial problems. The possibility of this application is due to the fact, that one of the limits in (2) and (7) is, in certain cases, so close to the value of the integral or sum considered, that it may pass as a first rough approximation which becomes a very fair one on applying a simple correction.

Putting, for instance, in (2)

$$f(t)=v^t, \phi(t)=t p_x, a=0, b=\infty, \lambda=\bar{e}_x,$$

we find

$$\int_0^\infty v^t p_x dt < \int_0^{\bar{e}_x} v^t dt$$

$$\text{or} \qquad \bar{a}_x < \frac{1-v^{\bar{e}_x}}{\delta} \dots \dots \dots (9)$$

that is the well known result that a life-annuity is smaller than an annuity-certain payable for the expectation of life.

Similarly, we find from (7), putting  $x=1, y=\infty, f(n)=v^n, \phi(n)=n p_x, s=e_x,$

$$a_x < \sum_1^{e_x} v^n \dots \dots \dots (10)$$

If  $e_x=k+\theta$ , where  $k$  is an integer, and  $0 \leq \theta < 1$ , (10) may be written

$$a_x < \frac{1-v^k}{i} + \theta v^{k+1} \dots \dots \dots (10A)$$

It was, in fact, this particular inequality, familiar to actuaries,‡ which suggested to me the general formulas (2) and (7).

\* Bromwich: An Introduction to the Theory of Infinite Series. London, 1908. P. 54 and p. 426.

† We may evidently write (2) in the form

$$\int_a^b f(t)\phi(t)dt = \int_\theta^{\theta+\lambda} f(t)dt$$

where  $a \leq \theta \leq b-\lambda$ .

‡ *Text-Book*, Part II, 1st Edit., p. 112.

The popular but erroneous notion that a life-annuity may be calculated as an annuity-certain payable for a duration equal to the expectation of life, yet contains an element of truth,\* which may be turned to good account. The duration  $m$  of an annuity-certain, equal in value to a life-annuity, may be calculated from the equation

$$a_x = \frac{1-v^m}{i}, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

whence

$$m = -\frac{\log(1-ia)}{\log(1+i)}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we expand  $m$  in powers of  $i$ , remembering, that  $a$  is a function of  $i$ , and neglecting in the result powers above the first, we find the approximate formula

$$m = e_x - i\epsilon_x \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\epsilon_x = \frac{1}{l_x} \sum_1^{\infty} t l_{x+t} - \frac{1}{2} e_x (e_x + 1) \quad . \quad . \quad . \quad . \quad (14)$$

which may more conveniently be written

$$\epsilon_x = \frac{1}{l_x} \sum^2 l_{x+1} + \frac{1}{8} - \frac{1}{2} \bar{e}_x^2 \quad . \quad . \quad . \quad . \quad (14A)$$

the summation being commenced from the bottom, or in *Text-Book* notation

$$\epsilon_x = \frac{Y_x}{l_x} - \frac{1}{8} - \frac{1}{2} \bar{e}_x^2 \quad . \quad . \quad . \quad . \quad (14B)$$

It may be worth noticing, that the function  $\epsilon_x$  thus introduced has an independent meaning in actuarial science. In fact, it may be easily proved that  $2\epsilon_x$  is the *mean square deviation in the estimated curtate duration of an individual life*. Supposing the life, aged  $x$  at entry, fails after  $t$  years (leaving out any fraction of a year), then the deviation from the estimated duration is  $(t - e_x)$ . Squaring this, multiplying by the probability, that the life will fail between ages  $x+t$  and  $x+t+1$ , and summing for all values of  $t$ , we find the required mean square deviation, or

\* The reason for this is, that we shall have  $\bar{a}_x = \bar{a}_{\bar{e}_x}$  provided that either  $\delta=0$ , or else  $\mu_x=0$  for all values of  $x$ .

$$\begin{aligned}
& \sum_0^{\infty} (t - e_x)^2 ({}_t p_x - {}_{t+1} p_x) \\
&= \sum_0^{\infty} (t - e_x)^2 {}_t p_x - \sum_1^{\infty} (t - 1 - e_x)^2 {}_t p_x \\
&= e_x^2 + \sum_1^{\infty} [(t - e_x)^2 - (t - 1 - e_x)^2] {}_t p_x \\
&= e_x^2 + \sum_1^{\infty} (2t - 2e_x - 1) {}_t p_x \\
&= e_x^2 + 2 \sum_1^{\infty} t {}_t p_x - 2e_x^2 - e_x = 2e_x.
\end{aligned}$$

It is seen from (13), that  $e_x$  is really a first approximation to the true duration  $m$ . We proceed to show by a numerical investigation, that the further approximation obtained by deducting  $i e_x$  from  $e_x$  produces very fair approximate values of  $a_x$ .

The experience employed for this purpose was the  $H^M$  experience (*Text-Book* graduation), and the table below gives the difference between the true and approximate values by (13) of  $a_x$  at various rates of interest. The following table of Practical Standards of Comparison, reproduces the values of  $\frac{1}{2}(a_x - a_{x+1})$ , or approx. the variation in the annuity-value,

*Differences between True and Approximate Values of  $a_x$ .*

$x$	3%	4%	5%	6%
20	-.10	-.11	-.11	-.10
30	-.04	-.04	-.05	-.04
40	+.01	+.01	+.01	+.02
50	.03	.04	.06	.08
60	.02	.04	.06	.08
70	.02	.03	.04	.06
80	.01	.02	.02	.04

*Practical Standards of Comparison.*

$x$	3%	4%	5%	6%
20	.10	.07	.05	.04
30	.12	.09	.06	.05
40	.15	.12	.09	.07
50	.18	.15	.12	.10
60	.19	.16	.14	.12
70	.17	.15	.14	.12
80	.12	.12	.11	.10

when the age varies one half year. This is a deviation familiar to the actuary who is accustomed to calculate the "age nearest birthday", a method which is legitimate when no systematic deviation from the mean is to be feared.

Our table of differences between original and approximate values shows that these differences, when the youngest ages and highest rates of interest are left out, are small, not only in comparison with the standards, but also absolutely speaking, so that the slight systematic deviation traceable is unimportant from a practical point of view.

The best standard for measuring these deviations is, however, the *standard deviation* (or "mean error") in the value of  $a_x$  as deduced from a given experience.\* The problem of calculating this standard deviation has not until comparatively recently† attracted the attention of actuaries, but as it is of importance also on other occasions, *e.g.*, when the *weights* of the ungraduated  $a_x$  are required in a direct graduation of this function, we shall occupy ourselves with the matter here. It may be remarked beforehand, that the result at which we shall arrive is practically the same as the first formula on p. 102, of G. F. Hardy's book; but the following deduction is not only easier but also safer from the theoretical point of view, making less use of approximations.

The *mean square deviation* (or the square of the mean error) is the square of the standard deviation and is denoted by  $\mu_2$ . Let there be given  $n$  different and independent observations

$$\sigma_1, \sigma_2, \dots \sigma_n$$

and an analytical function of these

$$f(\sigma_1, \sigma_2, \dots \sigma_n).$$

Put  $\sigma_r = y_r + \eta_r$ ,  $y_r$  being the "true" value of  $\sigma_r$ , and  $\eta_r$  the error of observation. Provided these errors are sufficiently small,  $f$  may be developed in powers of them,

\* This standard deviation, has, of course, nothing to do with the standard deviation examined by Bremiker and others, in the estimated value of an annuity on a single life. The latter concerns the *application*, the former the *origin* of the table.

† G. F. Hardy: The Theory of the Construction of Tables of Mortality, &c., p. 99.

neglecting powers above the first. The coefficients of the errors of observation are, then

$$\frac{\delta f}{\delta y_1}, \frac{\delta f}{\delta y_2}, \dots \frac{\delta f}{\delta y_n}.$$

The mean square deviation in  $f$  is, therefore\*

$$\mu_2(f) = \sum_1^n \left( \frac{\delta f}{\delta y_r} \right)^2 \mu_2(\sigma_r) \quad . \quad . \quad . \quad . \quad (15)$$

In order to apply this well-known general theorem to the life-annuity, the latter must be expressed as a function of the independent  $q_x$ , that is

$$\left. \begin{aligned} a_x &= \sum_1^{\infty} v^t {}_t p_x \\ &= \sum_1^{\infty} v^t (1 - q_x)(1 - q_{x+1}) \dots (1 - q_{x+t-1}) \end{aligned} \right\} \quad . \quad (16)$$

Now

$$\begin{aligned} \frac{\delta a_x}{\delta q_{x+r}} &= - \frac{1}{1 - q_{x+r}} \sum_{r+1}^{\infty} v^t {}_t p_x \\ &= - \frac{1}{p_{x+r}} v^r {}_r p_x a_{x+r} \\ &= - \frac{N_{x+r}}{p_{x+r} D_x} \end{aligned}$$

whence, by (15),

$$\mu_2(a_x) = \sum_{r=0}^{\infty} \frac{N_{x+r}^2}{p_{x+r}^2 D_x^2} \mu_2(q_{x+r}).$$

But, denoting as usual the Exposed to Risk by  $E_x$ ,

$$\mu_2(q_x) = \frac{p_x q_x}{E_x} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

therefore

$$\mu_2(a_x) = \frac{1}{D_x^2} \sum_x \frac{q_t N_t^2}{p_t E_t} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This formula gives the mean square deviation in the ungraduated value of a life-annuity, and the standard deviation  $\sqrt{\mu_2(a_x)}$  is, therefore, the most correct measure for the permissible deviation from the observations.

It should be remembered, that of the functions employed

\* Thiele: "Theory of Observations," p 39.

in (18),  $D_x$ ,  $p_x$ ,  $q_x$ , and  $N_x$  are assumed to be true (or at least graduated) values, while  $E_x$  depends on the observations alone.

The calculation of  $\sqrt{\mu_2}$  according to (18) is not a very laborious task. A slide-rule may be used for the purpose, the approximation obtained by this instrument being quite sufficient.

We give below specimens of the standard deviations in  $a_x$  according to various tables.

*Standard deviations in  $a_x$ .*

$x$	$H^M$				$O^M$ 3%	$O^{M(5)}$ 3%	$D^{M(5)}$ 3½%
	2%	4%	5%	6%			
20	·059	·048	·040	·034	·023	·065	·268
30	37	29	23	19	15	19	·063
40	38	30	25	20	15	16	58
50	44	36	31	26	16	16	69
60	52	45	40	35	18	18	82
70	67	61	55	51	20	20	99
80	·105	99	93	88	28	27	·146

They were all calculated by (18), except the figures for  $O^{M(5)}$  which are G. F. Hardy's original approximations. It is seen that this test is somewhat more severe than our "Practical Standards of Comparison"; yet it does not alter the conclusion at which we had previously arrived, that the values of  $a_x$  as calculated by (11) and (13) are sufficiently accurate for most practical purposes. The standard deviations according to  $D^{M(5)}$  are particularly illustrative of the liberties which may be taken with figures derived from a comparatively small experience, the total of exposed to risk according to  $D^{M(5)}$  being 282,118.

It appears from (18), and has already been pointed out by G. F. Hardy, that if the Exposed for all ages are multiplied by the same constant factor  $k$ , then the standard deviation in  $a_x$  is divided by  $\sqrt{k}$ . The total exposed according to  $O^M$  is 7,659,454 and according to  $H^M$  1,199,093. If the relative distribution is proportional at all ages, the standard deviation according to  $O^M$  should, therefore, be about four-tenths, of the standard deviation according to  $H^M$ , which is seen to agree very fairly except at the highest ages.

If monetary tables at a particular rate of interest  $i$  have



already been prepared, these may with advantage be taken as starting-point for the calculation of annuity-values at a different rate of interest, say  $i'$ . Putting  $i' = i + h$  and

$$\begin{aligned} a'_x &= \sum_1^{\infty} (1+i')^{-t} p_x \\ &= \sum_1^{\infty} (1+hv)^{-t} v^t p_x \end{aligned}$$

we find by (7), if  $h > 0$ ,

$$a'_x < \sum_1^{a_x} (1+hv)^{-t}. \quad . \quad . \quad . \quad . \quad (19)$$

It is, therefore, natural to put, in analogy with (11)

$$a'_x = \frac{1 - (1+h)^{-n}}{h} \quad . \quad . \quad . \quad . \quad (20)$$

where it may be expected that  $a_x$  will be a first rough approximation to  $n$ . We find, in fact, developing  $n$  in powers of  $h$  and neglecting in the result powers above the first,

$$n = a_x - h\alpha_x \quad . \quad . \quad . \quad . \quad (21)$$

where

$$\alpha_x = \frac{v}{D_x} \sum^2 D_{x+1} + \frac{1}{8} - \frac{1}{2} \left( a_x + \frac{1}{2} \right)^2$$

or

$$\alpha_x = \frac{vS_x}{D_x} + \frac{1}{8} - \frac{1}{2} \left( a_x + \frac{1}{2} \right)^2. \quad . \quad . \quad . \quad . \quad (22)$$

While, in (19), we must suppose  $h > 0$ , (20) is evidently valid for positive and negative  $h$ . If  $h$  is positive, the right-hand side of (20) is  $a_{\overline{n}|}$  at the rate of interest  $h$ . If  $h$  is negative and  $= -k$ , the right-hand side of (20) is easily seen to be  $s_{\overline{n+1}|} - 1$  at the rate of interest  $\frac{k}{1-k}$ , which may nearly always be replaced by  $k$ , as in the examples below.

For  $i=0$  (20) and (21) evidently reduce to (11) and (13).

We give below for  $i=3\frac{1}{2}$  per-cent,  $i'=3$  per-cent and 4 per-cent, the differences  $\Delta$  between the true and approximate value of  $a_x$ ; further the corresponding figures when  $a'_x$  is calculated by the obvious formula

$$a'_x = a_x - h \frac{vS_x}{D_x} \quad . \quad . \quad . \quad . \quad (23)$$

obtained by expanding  $a'_x$  in powers of  $h$ , neglecting powers above the first.

*Differences between True and Approximate Values of  $a_x$ .*

$x$	$\Delta$ by (20)		$\Delta$ by (23)	
	3%	4%	3%	4%
20	·02	·01	·12	·11
30	·02	·01	·09	·08
40	·01	·01	·06	·05
50	·01	·00	·03	·03
60	·00	·00	·01	·01
70	·00	·00	·01	·00
80	·00	·00	·00	·00

It should be noted that the numerical results for fractional durations were obtained by *first difference* interpolation in an interest-table (Spitzer), which is sufficient when the life-annuity is only required to two decimals. On the other hand, the value of the method in practice depends to a large extent on it being possible to content oneself with first-difference interpolation. We have, therefore, in (13) and (21) not gone beyond the first power of  $i$ , and  $h$ , though it is possible to continue these expansions.

The results by (13) and (21) seem so satisfactory, that we do not hesitate to recommend making a table of  $\epsilon_x$  or  $\alpha_x$  to three or four significant figures a regular feature of every publication of mortality-tables. The cases where isolated values of life-annuities are required may not be very frequent; still, they are sometimes wanted in transactions with life-annuities, for comparison with other tables, or for examining the effect of a change in the mortality, and the want may even increase with the facility for satisfying it.

We now proceed to show, utilizing an idea due to Mr. Meidell, how Dr. Jensen's inequalities for *convex functions* may be derived from (7).

As criterion for convex functions\* we use this, that the *second divided difference*† of the function shall always‡ be  $\geq 0$ ;

\* Compare Galvani, *l.c.* Dr. Jensen uses as criterion

$$\psi(x) + \psi(y) \geq 2\psi\left(\frac{x+y}{2}\right)$$

and proves his theorem by generalizing this inequality.

† With regard to this notion (which goes back to Newton), see a Note in this *Journal* by L. Oppermann, vol. xv, p. 145.

‡ That is: for all arguments for which the function is assumed to exist.

or, if  $\psi(x)$  be the function,  $x_1, x_2, x_3$ , any three different arguments, that

$$\frac{\frac{\psi(x_1) - \psi(x_2)}{x_1 - x_2} - \frac{\psi(x_2) - \psi(x_3)}{x_2 - x_3}}{x_1 - x_3} \geq 0 \quad . \quad . \quad . \quad (24)$$

It follows, in particular, from (24), that if  $\psi(x)$  possesses a second differential coefficient, then  $\psi''(x) \geq 0$ . This is often used as a criterion for convex functions, but is insufficient in many cases of practical importance, for instance, when dealing with an ungraduated table, or with a table graduated by mechanical or graphic methods.

For *concave* functions the sign  $\geq$  in (24) must be replaced by  $\leq$ . We need only consider one of the cases, convexity or concavity; for if  $\psi(x)$  is convex, then  $-\psi(x)$  is concave, and *vice versa*. The case where the second divided difference vanishes may be reckoned either to the convex or to the concave functions, or the expression *linear functions* may be applied.

If  $\psi(n)$  is a convex function of an integral, positive or negative argument  $n$ , we have, by (24),  $\Delta^2\psi(n) \geq 0$ . Summing from  $n$  to  $\beta \geq n$

$$-\Delta\psi(n) = \sum_n^{\beta} \Delta^2\psi(n) - \Delta\psi(\beta+1) \quad . \quad . \quad . \quad (25)$$

and, as  $\Delta^2\psi(n) \geq 0$ , the right-hand side of (25) is a never increasing function of  $n$  which may be called  $f(n)$ . Summing now again from  $n$  to  $\beta$  we find

$$\psi(n) = \sum_n^{\beta} f(n) + \psi(\beta+1) \quad . \quad . \quad . \quad (26)$$

We see from this, that a convex function of an integral argument may always be written in the form  $\sum_n^{\beta} f(n) + \text{const.}$  where  $f(n)$  is a never increasing function. Conversely, any such expression evidently represents a convex function. Similarly, the expression  $\sum_a^n f(n) + \text{const.}$  may be taken to represent all concave functions of an integral argument.

Now let  $e(n)$  be any such function that

$$0 \leq \sum_n^{\beta} e(r) \leq \sum_a^{\beta} e(r) \quad (\alpha \leq n \leq \beta) \quad . \quad . \quad . \quad (27)$$

Assuming  $\sum_a^\beta e(r) > 0$ , we put, for abbreviation,

$$\rho = \frac{1}{\sum_a^\beta e(r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and choose for  $\phi(n)$  the function

$$\phi(n) = \rho \sum_n^\beta e(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

which, owing to (27), satisfies the condition  $0 \leq \phi(n) \leq 1$ .

It is easily verified, that\*

$$\sum_a^\beta f(n) \sum_n^\beta e(r) = \sum_a^\beta e(n) \sum_a^n f(r) \quad . \quad . \quad . \quad . \quad (30)$$

whence, in particular, for  $f(n) = 1$

$$\sum_a^\beta \sum_n^\beta e(r) = \sum_a^\beta (n - \alpha + 1) e(n) \quad . \quad . \quad . \quad . \quad (31)$$

From the inequality proved above

$$\sum_a^\beta f(n) \phi(n) \leq \sum_a^{\alpha+s-1} f(n), \quad s = \sum_a^\beta \phi(n),$$

we therefore obtain by (29), (30) and (31)

$$\rho \sum_a^\beta f(n) \sum_n^\beta e(r) = \rho \sum_a^\beta e(n) \sum_a^n f(r) \leq \sum_a^{\alpha+s-1} f(n)$$

where

$$s = \rho \sum_a^\beta \sum_n^\beta e(r) = \rho \sum_a^\beta (n - \alpha + 1) e(n) = \rho \sum_a^\beta n e(n) - (\alpha - 1)$$

or, if we put  $\sum_a^\xi f(n) = -\psi(\xi)$ ,

$$\psi \left[ \frac{\sum_a^\beta n e(n)}{\sum_a^\beta e(n)} \right] \leq \frac{\sum_a^\beta e(n) \psi(n)}{\sum_a^\beta e(n)} \quad . \quad . \quad . \quad . \quad (32)$$

The function  $\psi(\xi)$  is a convex function, but not the most general convex function, for it vanishes for  $\xi = \alpha - 1$ , and is, according to our definition (3), linear in every interval

\* Formula (30) is independent of the assumed properties of  $f(n)$  and  $e(n)$ . This formula is, in fact, the analogon for *sums* of Dirichlet's formula for *integrals*.

$n < \xi < n+1$ . But both of these restrictions may easily be removed. For it is clear, that if (32) is satisfied by a certain  $\psi$ , then it is also satisfied, if we replace  $\psi$  by  $\psi + \text{const.}$  Secondly, a convex function which is not linear in the intervals  $n < \xi < n+1$  cannot be larger than the function which assumes the same values for integral values of  $\xi$  but is linear in every interval of this kind.\* The inequality (32) is, therefore, *a fortiori* satisfied if  $\psi$  signifies the most general convex function.

(32) is a particular case of Dr. Jensen's theorem, but is easily generalized into the complete theorem. The function  $e(n)$  was arbitrary, only subject to satisfying (27); it may, therefore, be chosen so that  $e(n)=0$  except at the points  $n_1, n_2, \dots n_\omega$ , which have been taken in increasing order. Putting  $e(n_r)=c_r$ , this is evidently an arbitrary coefficient, only subject to satisfying the inequality (27) which becomes

$$0 \leq \sum_r c_r \leq \sum_1 c_n \quad (1 \leq r \leq \omega) \quad . \quad . \quad . \quad (33)$$

and (32) becomes

$$\psi \left[ \frac{\sum_1 c_r n_r}{\sum_1 c_r} \right] \leq \frac{\sum_1 c_r \psi(n_r)}{\sum_1 c_r} \quad . \quad . \quad . \quad (34)$$

In this inequality  $n_r$  is still subject to being an integer (positive or negative); but this restriction also may be removed. It is, in fact, clear, that if  $\psi(\xi)$  is a convex function,

\* The characteristic geometrical quality of convexity, easily deducible from (24), is that the straight line connecting any two points of the curve leaves the part of the curve situated between these two points, below it. This may be proved as follows: Writing  $y_n$  for  $\psi(x_n)$ , and, assuming  $x_1 < x_2 < x_3$ , (24) may be written

$$\frac{y_2 - y_1}{x_2 - x_1} \leq \frac{y_3 - y_2}{x_3 - x_2}$$

whence

$$y_2 - y_1 \leq (x_2 - x_1) \frac{y_3 - y_1}{x_3 - x_1}.$$

Now, the equation to the straight line between  $(x_1, y_1)$  and  $(x_3, y_3)$  is

$$y - y_1 = (x - x_1) \frac{y_3 - y_1}{x_3 - x_1};$$

putting, here,  $x = x_2$  we see at once that  $y \geq y_2$ .

*Vice versa*, (24) follows from the geometrical characteristic. This characteristic quality is also at once proved by the criterion  $\psi''(x) \geq 0$  which is, therefore, equivalent to (24), whenever  $\psi''(x)$  exists.

and  $k > 0$ , then  $\psi\left(\frac{\xi}{k}\right)$  is also convex, as follows at once from the geometrical consideration. We may, therefore, in (34) replace  $n_r$  by  $\frac{n_r}{k}$ . If the quantities  $k_1, n_1, n_2, \dots, n_\omega$  are supposed to increase indefinitely in a suitable way, the quantities  $\frac{n_r}{k}$  may be made to approach indefinitely to any previously given quantities  $x_r$ . If, therefore,  $\psi(x)$  is a continuous, convex function, and  $x_r$  never decreases, and if the quantities  $c_r$  satisfy the inequality (33) where  $\sum_1^\omega c_r > 0$  then

$$\psi\left[\frac{\sum_1^\omega c_r x_r}{\sum_1^\omega c_r}\right] \leq \frac{\sum_1^\omega c_r \psi(x_r)}{\sum_1^\omega c_r} \quad \dots \quad (35)$$

This is Dr. Jensen's inequality between mean values, or rather a generalization thereof, for Dr. Jensen only proved it on the assumption that all the coefficients  $c_r$  are positive, which they need not necessarily be according to (33), although (33) is certainly satisfied if these coefficients are positive. In that case the  $x_r$  may be taken in an arbitrary order.

It should be noted, as pointed out to me by Mr. N. P. Bertelsen, that the argument employed for  $\psi$  on the left-hand side of (35), or  $\frac{\sum c_r x_r}{\sum c_r}$ , is not smaller than the smallest or larger than the largest of the quantities  $x_r$ . This is easily proved by going back to the inequalities (27). Summing these from  $n = \alpha + 1$  to  $n = \beta$  we find by (31)

$$0 \leq \sum_{\alpha+1}^{\beta} (n - \alpha) e(n) \leq (\beta - \alpha) \sum_{\alpha}^{\beta} e(n)$$

$$\text{or} \quad \alpha \sum_{\alpha}^{\beta} e(n) \leq \sum_{\alpha}^{\beta} n e(n) \leq \beta \sum_{\alpha}^{\beta} e(n)$$

$$\text{or, if } \sum_{\alpha}^{\beta} e(n) > 0,$$

$$\alpha \leq \frac{\sum_{\alpha}^{\beta} n e(n)}{\sum_{\alpha}^{\beta} e(n)} \leq \beta$$

whence the result follows by proceeding as above.



From (35) the corresponding formula with *integrals* instead of *sums* is derived by putting  $x_r = g\left(\frac{r}{\omega}\right)$ ,  $c_r = k\left(\frac{r}{\omega}\right) \cdot \frac{1}{\omega}$  where  $g(t)$  and  $k(t)$  are integrable functions, and letting  $\omega$  increase indefinitely. By the definition of an integral as a limit of a sum, we then obtain

$$\psi \left[ \frac{\int_0^1 k(t)g(t)dt}{\int_0^1 k(t)dt} \right] \leq \frac{\int_0^1 k(t)\psi[g(t)]dt}{\int_0^1 k(t)dt} \quad . \quad . \quad . \quad (36)$$

where  $g(t)$  never decreases, while the function  $k(t)$ , according to (33), must satisfy the inequality

$$0 \leq \int_{\theta}^1 k(t)dt \leq \int_0^1 k(t)dt \quad (0 \leq \theta \leq 1) \quad . \quad . \quad . \quad (37)$$

It is clear, that the limits 0 and 1 in (36) and (37) may be replaced by  $a$  and  $b$  (put  $t = \frac{x-a}{b-a}$  in the integrals).

The inequality (36) was proved by Dr. Jensen on the assumption that  $k(t)$  is positive which, according to (37), is a sufficient although not a necessary condition.\* In this case  $g(t)$  need not be never decreasing.

The inequalities (35) and (36) are of a very general nature, comprising, as particular cases, a great many well-known inequalities, as pointed out by Dr. Jensen. For instance,  $\log t$  being concave for  $t > 0$ , we find† from (35), putting  $\psi(t) = \log t$ ,  $x_r = b_r > 0$ ,

$$\log \frac{\sum c_r b_r}{\sum c_r} \geq \frac{\sum c_r \log b_r}{\sum c_r} \quad . \quad . \quad . \quad (38)$$

or

$$\frac{c_1 b_1 + c_2 b_2 + \dots + c_{\omega} b_{\omega}}{c_1 + c_2 + \dots + c_{\omega}} \geq (b_1^{c_1} b_2^{c_2} \dots b_{\omega}^{c_{\omega}})^{\frac{1}{c_1 + c_2 + \dots + c_{\omega}}} \quad . \quad . \quad (39)$$

If this well-known formula is applied to any other than

\* The formula derived by Mr. Meidell is the particular case obtained by putting  $g(t) = t$  in (36); he assumes, like Dr. Jensen, that  $k(t)$  is positive, and introduces certain restrictions, necessary for the manipulation of the integrals. These restrictions are avoided by operating, as we have done in establishing (32), on *sums* instead of *integrals*. The process, by which (32) was generalized into (35), would, in the case of Mr. Meidell's formula, correspond to substituting another variable in the integral.

† Putting, in (38),  $b_r = k^r$  we have (apart from the notation) inequality (A) in Mr. Lidstone's paper in *J.I.A.*, vol. xlv, p. 487.

positive  $c$ 's, (33) must, of course, be satisfied, and  $b_r$  must never decrease.

Putting  $c_m = S_m$ ,  $b_m = r^{n_m}$ , the inequality expresses that the *equated time* for a number of sums due at different times, as found by the usual approximate method (*Text-Book* I, pp. 24-27), *favours the debtor*.\*

Further,  $\psi(t) = t \log t$  is convex for  $t > 0$ . Putting  $x_r = b_r > 0$  and assuming *either* that  $c_r > 0$ , *or else* that (33) is satisfied and  $b_r$  never decreases, we obtain from (35)

$$\left( \frac{c_1 b_1 + c_2 b_2 + \dots + c_\omega b_\omega}{c_1 + c_2 + \dots + c_\omega} \right)^{c_1 b_1 + c_2 b_2 + \dots + c_\omega b_\omega} \leq b_1^{c_1} b_2^{c_2} \dots b_\omega^{c_\omega} \quad (40)$$

a formula which has been applied in this *Journal* (vol. xl, p. 121) by Mr. S. E. Macnaghten. The logarithmic form of (40) gives, if  $b_r = k^r$ , Mr. Lidstone's inequality (B) in *J.I.A.*, vol. xlv, p. 487.

If  $\psi(t) = t^2$  which is a convex function, we find from (35), putting  $c_r = \frac{b_r}{x_r}$ ,

$$\frac{\sum b_r}{\sum_{x_r} b_r} \leq \frac{\sum b_r x_r}{\sum b_r} \dots \dots \dots (41)$$

or the well-known result, that the harmonical mean is not greater than the arithmetical mean. This certainly holds if  $b_r$  and  $x_r$  are positive quantities, but also provided only their quotient  $c_r = \frac{b_r}{x_r}$  satisfies (33) and  $x_r$  never decreases.

If  $\psi(t) = t^p$  ( $p > 1$ ), then the function is convex for  $t > 0$ ; putting  $c_r = \frac{1}{\omega}$  we obtain, if  $x_r > 0$

$$\left( \frac{x_1 + x_2 + \dots + x_\omega}{\omega} \right)^p \leq \frac{x_1^p + x_2^p + \dots + x_\omega^p}{\omega} \dots \dots (42)$$

also a well-known result.†

Many other interesting examples have been given by Dr. Jensen; we content ourselves with referring to his memoir, but quote from Mr. Meidell's paper the following application to the approximate calculation of life-annuities.

\* For  $c_r = 1$  (39) shows that the arithmetical mean of positive quantities is never smaller than the geometrical mean.

† See Mr. Lidstone's paper "On the Equation of Payments, &c.," *J.I.A.*, vol. xlv, p. 484. The more general inequality on the same page is obtained by putting, in (35),  $\psi(t) = t^m$ ,  $x_r = r$ ,  $c_r = S_r$ .

Considering, that the two sides of (35) approach to each other, if  $\psi$  approaches to a linear function, we may obtain an approximate formula by choosing for  $\psi$  a convex function with comparatively slight curvature.

Applying this principle to the life-annuity  $a'_x$  at rate of interest  $i' + i + h$ , written in the form used above

$$a'_x = \sum_1^{\infty} (1 + hv)^{-t} v^t {}_t p_x \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Mr. Meidell writes

$$\psi(t) = (1 + hv)^{-t}, \quad c_t = v^t {}_t p_x, \quad x_t = t,$$

$\psi(t)$  being evidently convex, and obtains from (35)

$$a'_x > a_x (1 + hv)^{-\frac{\sum_1^{\infty} t D_{x+t}}{\sum_1^{\infty} D_{x+t}}} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

whence the approximate formula

$$a'_x = a_x (1 + hv)^{-\frac{S_x}{N_x}} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

producing a slightly too small value for  $a'_x$ .

Putting, in (45),  $i = 0$ , and replacing thereafter  $h$  by  $i$ , we find

$$a_x = e_x v^{\frac{\sum N'_x}{N_x}} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

While (46) does not, without a corrective term, produce results accurate enough for practical use, the results by (45) are very good, as appears from the following table, quoted from Mr. Meidell's paper,\* where  $a'_x$  at 3 per-cent and 4 per-cent was calculated by (45), taking  $a_x$  at  $3\frac{1}{2}$  per-cent, while  $\Delta$  denotes the difference between the true and approximate values.

$a'_x$  by (45). H<sup>M</sup>  
Text-book.

$x$	3%	$\Delta$	4%	$\Delta$
20	22.02	.04	18.62	.04
30	19.86	.04	17.13	.03
40	17.16	.02	15.12	.02
50	13.87	.01	12.51	.01
60	10.22	.00	9.44	.01
70	6.66	.00	6.29	.00
80	3.70	.00	3.57	.00

\* One or two minor errors in the last figure have been corrected.

Mr. Meidell gives other examples, but those in the table suffice to prove the practical utility of Dr. Jensen's inequality which has not yet, amongst actuaries, received the popularity it deserves.

From (36) we may, as pointed out by Mr. Meidell, derive an inequality, resembling (2) in form, but being in certain respects more, in others less general than (2). We prefer, for similar reasons as above, to operate on the sums instead of the integrals, and proceed to show how an inequality of the type (7) may be derived from (35) by Mr. Meidell's method. At the same time we shall obtain a more general result than Mr. Meidell's.

Putting, in (35),  $x_r = r$ ,  $\sigma = \sum_1^{\omega} c_r > 0$ ,  $s = \sum_1^{\omega} r c_r$ ,  $c_r = \frac{\phi(r)}{r}$ ,  $\psi(\xi) = -\sum_1^{\xi} f(n)$ , we have

$$\psi\left(\frac{s}{\sigma}\right) \leq \frac{1}{\sigma} \sum_1^{\omega} c_r \psi(r)$$

or, as  $f(n)$  never increases and, consequently,  $\psi(r) \leq -rf(r)$ ,

$$\psi\left(\frac{s}{\sigma}\right) \leq -\frac{1}{\sigma} \sum_1^{\omega} f(r) \phi(r)$$

or

$$\sum_1^{\omega} f(r) \phi(r) \leq \sigma \sum_1^{\frac{s}{\sigma}} f(r).$$

A lower limit for  $\sum_1^{\omega} f(r) \phi(r)$  may be obtained in a similar way, or simply by putting

$$r = \omega + 1 - n, \quad \phi(r) = \phi_1(n), \quad f(r) = -f_1(n)$$

and dropping the suffixes. The result is the inequality

$$\tau \sum_{\omega+1-\frac{s}{\tau}}^{\omega} f(n) \leq \sum_1^{\omega} f(n) \phi(n) \leq \sigma \sum_1^{\frac{s}{\sigma}} f(n) \quad . \quad . \quad . \quad (47)$$

where

$$s = \sum_1^{\omega} \phi(n), \quad \sigma = \sum_1^{\omega} \frac{\phi(n)}{n}, \quad \tau = \sum_1^{\omega} \frac{\phi(n)}{\omega+1-n} \quad . \quad . \quad . \quad (48)$$

it is assumed that  $f(n)$  never increases, that  $\sigma > 0$ ,  $\tau > 0$ ,  $s > 0$ , and that the quantities  $\frac{\phi(n)}{n}$  and  $\frac{\phi(n)}{\omega+1-n}$  respectively

satisfy the condition that the sum from 1 to  $r$  is not smaller than 0 or greater than  $\sigma$  and  $\tau$  respectively.

In particular, (47) is always valid, if  $f(n)$  never increases, and  $\phi(n)$  is never negative.

If  $\sigma = \tau = 1$ —a case which is evidently possible\*—(47) reduces to (7), and the previous considerations have, therefore, procured a new and different test for the validity of (7). If  $\sigma < 1$ ,  $\tau < 1$ , the limits in (47) are generally more narrow and certainly not wider than the limits in (7); if  $\sigma > 1$ ,  $\tau > 1$ , the reverse holds. For, putting in (7)  $\phi = k$ , we have for any not increasing function

$$\sum_{y+1-k(y-x+1)}^y f(n) \leq k \sum_x^y f(n) \leq \sum_x^{x-1+k(y-x+1)} f(n) \quad (0 \leq k \leq 1) \quad . \quad . \quad . \quad (49)$$

If, in this formula, we put  $k = \sigma < 1$ ,  $x = 1$ ,  $y = \frac{s}{\sigma}$  we have

$$\sigma \sum_1^{\frac{s}{\sigma}} f(n) \leq \sum_1^s f(n),$$

while for  $k = \tau < 1$ ,  $x = \omega + 1 - \frac{s}{\tau}$ ,  $y = \omega$ , we find

$$\sum_{\omega-s+1}^{\omega} f(n) \leq \tau \sum_{\omega+1-\frac{s}{\tau}}^{\omega} f(n).$$

If, on the other hand, we put  $k = \frac{1}{\sigma} < 1$ ,  $x = 1$ ,  $y = s$  resp.  $k = \frac{1}{\tau} < 1$ ,  $x = \omega - s + 1$ ,  $y = \omega$ , we find the two preceding inequalities with the sign  $\leq$  changed into  $\geq$ .

From (47) we may derive a corresponding inequality for integrals, simply by putting  $\phi(n) = \phi_1\left(\frac{n}{\omega}\right)$ ,  $f(n) = f_1\left(\frac{n}{\omega}\right) \cdot \frac{1}{\omega}$ , dropping the indices and allowing  $\omega$  to increase indefinitely. Writing

$$\lambda = \int_0^1 \phi(t) dt, \quad \mu = \int_0^1 \frac{\phi(t)}{t} dt, \quad \nu = \int_0^1 \frac{\phi(t)}{1-t} dt \quad . \quad . \quad (50)$$

the result is

$$\nu \int_{1-\frac{\lambda}{\nu}}^1 f(t) dt \leq \int_0^1 f(t) \phi(t) dt \leq \mu \int_0^{\frac{\lambda}{\mu}} f(t) dt \quad . \quad . \quad . \quad (51)$$

\* Example:  $\omega = 2$ ,  $\phi(1) = \frac{2}{3}$ ,  $\phi(2) = \frac{2}{3}$ .

it is here assumed, that  $f(t)$  is a never-increasing, integrable function, that the integrals (50) exist, that  $\lambda > 0$ ,  $\mu > 0$ ,  $\nu > 0$ , and that  $\phi(t)$  satisfies the condition

$$\left. \begin{aligned} 0 &\leq \int_t^1 \frac{\phi(t)}{t} dt \leq \mu \\ 0 &\leq \int_t^1 \frac{\phi(t)}{1-t} dt \leq \nu \end{aligned} \right\} (0 \leq t \leq 1) \quad . \quad . \quad . \quad (52)$$

The condition (52) is clearly satisfied, if  $\phi(t)$  is never negative. The limits 0 and 1 in (50)–(52) may evidently be replaced by  $a$  and  $b$ .

Comparison between (51) and (2) leads to similar results as the comparison made above between (47) and (7); we need not go into details.

Comparison between (51) and (47) shows, that it is preferable to operate on sums instead of integrals; for, in (51), we assumed the existence of the integrals (50) which puts certain inconvenient restrictions on the  $\phi(t)$ , for instance, that it must vanish at the limits of integration.

The result proved by Mr. Meidell was (51), but on the assumption that  $\phi(t)$  is positive which, according to (52), is no absolute condition.

Returning to Dr. Jensen's inequalities, it is clear that the particular case obtained by putting  $x_r = r$  in (35), or  $g(t) = t$  in (36), is the most important for the actuarial applications. The formulas are for this purpose conveniently written

$$\left. \begin{aligned} \sum_1^{\omega} c_r \psi(r) &\geq \psi(\xi) \sum_1^{\omega} c_r, \\ \xi &= \frac{\sum_1^{\omega} r c_r}{\sum_1^{\omega} c_r} \end{aligned} \right\} . \quad . \quad . \quad . \quad (53)$$

which is identical with (32), and

$$\left. \begin{aligned} \int_a^b k(t) \psi(t) dt &\geq \psi(\eta) \int_a^b k(t) dt \\ \eta &= \frac{\int_a^b t k(t) dt}{\int_a^b k(t) dt} \end{aligned} \right\} . \quad . \quad . \quad . \quad (54)$$



We repeat that a *sufficient* criterion for the applicability of these inequalities is, that  $c_r$  and  $k(t)$ , respectively, are positive, and  $\Delta^2\psi(r)$  and  $\psi''(t)$  respectively  $\geq 0$ .

Many of the functions employed in actuarial calculations satisfy these conditions, at least when Makeham's hypothesis, holds. Amongst the functions which are always or usually convex for adult ages we quote  $\mu_x$ ,  $q_x$ ,  $P_x$ ,  $\frac{1}{a_x}$ ,  $N_x$ ,  $S_x$ ,  $R_x$ , while  $l_x$ ,  $d_x$ ,  $D_x$ ,  $C_x$ ,  $M_x$  should be treated with more caution.

In order to use the limits, produced by (53) and (54), with advantage,  $\xi$  and  $\eta$  must be easier to calculate than the expression for which we seek a lower limit. If the curvature of  $\psi$  is only slight, we may, as in Mr. Meidell's example, obtain an approximate formula by replacing the sign  $\geq$  by  $=$ . But it may be added that in this case the condition, that  $\psi$  should be convex, becomes superfluous, and that the equations then express the "First Theorem of Mean Value" with the important addition that  $\xi$  and  $\eta$  are known approximately. It may further be added that if we only want an approximate formula and not an inequality, the whole of the previous argument may be spared and replaced by the simple consideration, that the obvious *identity*

$$\sum c_r(a+br) = (a+b\xi)\sum c_r \quad . \quad . \quad . \quad (55)$$

becomes an *approximation*, if the linear function  $(a+bx)$  is replaced by any function with a slight curvature. In this form the theorem was already given some years ago by Mr. Lidstone\* who further showed how the approximation  $\xi$  may sometimes be improved. If we put

$$\sum c_r F(r) = F(\zeta)\sum c_r \quad . \quad . \quad . \quad (56)$$

and, for abbreviation,

$$\xi = \frac{\sum r c_r}{\sum c_r}, \quad \xi' = \frac{\sum r^2 c_r}{\sum c_r} \quad . \quad . \quad . \quad (57)$$

then Mr. Lidstone finds approximately

$$\zeta = \xi + (\xi' - \xi^2) \frac{F''(\xi)}{2F'(\xi)} \quad . \quad . \quad . \quad (58)$$

Other formulas of a similar nature may be devised. We confine ourselves to one that produces exact results if applied

\* *J.I.A.*, vol. xlv, p. 483.

to a polynomial of the second degree. Let  $F(t)$  be such a polynomial; then

$$F_1(t) \equiv \frac{F(t) - F(0)}{t}$$

will be of the first degree, and (55) may be applied to  $\sum c'_r F_1(r)$  where  $c'_r = rc_r$ . We find

$$\sum c'_r F_1(r) = F_1(\xi_1) \sum c'_r, \quad \xi_1 = \frac{\sum rc'_r}{\sum c'_r}$$

or, introducing  $F(t)$  and  $c_r$ ,

$$\sum c_r F(r) = \left[ F(0) + \xi \cdot \frac{F(\xi_1) - F(0)}{\xi_1} \right] \sum c_r \quad . \quad . \quad . \quad (59)$$

where

$$\xi = \frac{\sum rc_r}{\sum c_r}, \quad \xi_1 = \frac{\sum r^2 c_r}{\sum rc_r} \quad . \quad . \quad . \quad . \quad (60)$$

This, being exact for a polynomial of the second degree, may be used as an approximation, if  $F(t)$  can approximately be represented by a polynomial of the second degree.

On the other hand there are cases where (58) produces better results than (59); it seems, therefore, that further investigations are desirable as to the circumstances under which these methods of approximation are efficient.

Dr. Jensen's theorem may be expressed very simply in the language of probabilities. Assume, that the coefficients  $c_r$  are all positive, and put  $p_r = \frac{c_r}{\sum_1^{\omega} c_r}$  whence  $\sum_1^{\omega} p_r = 1$ . Let

$x_1, x_2, \dots, x_{\omega}$  be all the possible values of  $x$ , and let  $p_r$  be the probability of the occurrence of  $x_r$ . Then, by definition,  $\sum_1^{\omega} p_r x_r$  is the "mathematical expectation of  $x$ ", and Dr. Jensen's theorem may be expressed thus:

*A continuous convex function of a mathematical expectation is not greater than the mathematical expectation of the function.*

If the mathematical expectation is denoted by the symbol  $E$ , then the inequality (35) may be neatly written thus

$$\psi(Ex) \leq E\psi(x) \quad . \quad . \quad . \quad . \quad (61)$$

Taking for instance,  $\psi(t) = t^2$ , we find  $(Ex)^2 \leq E(x^2)$ ; as a

matter of fact  $E(x^2) - (Ex)^2$  is the square of the mean error and, therefore, positive.

We shall not continue this subject further; but doubtless (61) must have many applications in the theory of probabilities where it is often more important to be able to indicate a limit than to find a very precise one.\*

\* Compare a paper by Mr. Meidell on the theory of limitation of risk in *Transactions of the Seventh Actuarial Congress*, vol. i, p. 88.

## ACTUARIAL NOTE.

### *Claim Acceleration Reserve for Endowment Assurances.*

1. If death claims are, on the average, payable  $\left(\frac{1}{2} - k\right)$  of a year after the moment of death, the claim acceleration reserve will be  $ki \cdot {}_tV_{xn}^1$  for a unit policy effected at age  $x$ , payable at  $x+n$ , and valued on the assumption that the  $(t+1)$ th annual premium is due and unpaid. This assumption will simplify the formulæ and is sufficiently accurate when a relatively small adjustment is in question.

2. Dropping the factor  $ki$  (which in any given case is a constant that can be applied to totals), and representing the sum assured by  $S$  and bonus additions by  $B$ , we have for the value of the adjustment:

$$(S+B)A_{x+t:n-t}^1 - S \frac{A_{xn}^1}{a_{xn}} a_{x+t:n-t}$$

or, writing  $A_0$  for  $A_{xn}^1$  and  $A_t$  for  $A_{x+t:n-t}^1$

$$\begin{aligned} & (S+B)A_t - S \frac{A_0}{a_{xn}} a_{x+t:n-t} \\ &= SA_0 - S \frac{A_0}{a_{xn}} a_{x+t:n-t} + BA_0 + (S+B)[A_t - A_0] \\ &= A_0[S {}_tV_{xn} + B] + (S+B)[A_t - A_0] \end{aligned}$$

3. The following table gives specimen values of  $A$  on the basis  $OM\ 5$ , 3 per-cent. It will be seen from this table

(i) That for any given value of  $n$ ,  $\Delta_x A$  increases approxi-

mately in G.P. This might have been anticipated, since it is known that  $\Delta_x a_{x\overline{n}|}$  and  $\Delta_x (\Delta_n a_{x\overline{n}|})$  behave in this way, and  $A = 1 - d(1 + a_{x\overline{n-1}|}) - \Delta_n a_{x\overline{n-1}|}$ , so that  $\Delta_x A = -\Delta_x [a_{x\overline{n-1}|}] + \Delta_n \cdot a_{x\overline{n-1}|}$ .

- (ii) That for any given value of the maturity-age,  $M = x + n$ , the values of  $A$  change but slowly in the most important part of the table, but they naturally fall off as the assurance nears maturity.

$O^{M(5)}$  3 per-cent Values of  $A_{x\overline{n}|}^1$ .

$x$	$(x + n)$					
	40	45	50	55	60	65
20	·104	·127	·149	·174	·200	·226
25	·089	·116	·143	·172	·203	·236
30	·069	·100	·133	·168	·205	·245
35	·040	·079	·118	·160	·205	·253
40	...	·047	·096	·147	·201	·259
45	...	...	·059	·122	·188	·259
50	...	...	...	·077	·159	·246
55	...	...	...	...	·104	·215
60	...	...	...	...	...	·145

4. From (i) it follows that a mean value of  $A_0$  for any given value of  $n-t$  (the unexpired term) can be found by means of the average maturity-age derived by the Z method. Over the whole E.A. business, the average value of  $M$  is usually rather under 60, so that if we take age 60 we should be fairly accurate with a small error on the side of stringency. A glance at the table shows that ·2 should be a fair average value of  $A_0$ , and that  $[A_t - A_0]$  is generally negative but not important except at the smallest unexpired terms. Neglecting  $[A_t - A_0]$ , which process is on the side of stringency, we have the result that the adjustment is

$$< \cdot 2ki[S_t V_{x\overline{n}|} + B]^*$$

If  $k = \frac{4}{12}$ , the claim acceleration-reserve on a 3 per-cent

\* If the adjustment were taken to be  $A_0 ki[S_t V_{x\overline{n}|} + \text{value of Bonus}]$ , the omitted terms would be  $S(A_t - A_0) + B(A_t - A_0 a_{x+t\overline{n-t}|})$ . The first term is generally positive, but the second may in many cases be negative, and it is not easy to show that on the whole the adjustment so calculated would be on the safe side.

basis is therefore less than  $\cdot 002 \times$  [the total endowment assurance reserve — value of Bonus + actual amount of Bonus]. This result amply confirms a remark made by Mr. J. M. Warden, F.F.A. (T.F.A., vol. vii, p. 348) to the effect that his experience had satisfied him that an adjustment of  $\frac{1}{2}$  per-cent on the reserve “is larger than is necessary when the average period is two months.” Similarly, Mr. H. W. Brown said (*loc. cit.* p. 346): “I think a reserve of  $\cdot 2$  of 1 per-cent of the liability in respect of bonus additions and a reserve of  $\cdot 15$  per-cent of the remaining liability—that is the liability for the sum assured—will be found to give a sufficiently stringent reserve for early payment of claims under the endowment assurance class.”

5. The whole adjustment found by the above or any more closely accurate process is so small that probably most actuaries will think it desirable to increase it in practice. A very good way to do this is to throw out altogether the adjustment in the value of the premiums and to take the claim acceleration reserve as  $A_0ki$  [total value of sums assured and bonus].

6. It will be observed from the approximation that the average value of  $A_0$  (on which alone depends the adjustment in the value of the premiums) should be a fairly close one. The principal error in the final result arises from the neglect of the term  $[S+B][A_t - A_0]$  which affects the sum assured and bonus only. Hence, if it is desired to apply the adjustment separately to the value of the sum assured and bonus and to the value of the pure premiums, the former adjustment will be  $A_0ki$  [total amount of sums assured and bonus] which will be considerably in excess; and we can also find a close approximation to the adjustment in the value of the premiums, namely,

$$-A_0ki \sum Sa_{x+t|n-t|} / a_{x:n|} = -A_0ki \sum Sa_{x+t|n-t|} (P_{jn|} + d) =$$

$-A_0ki$  [ $\sum$  value of pure premiums +  $\sum$  sums assured —  $\sum$  value of sum assured]. This assumes that the values of the sums assured are known independently of the values of the bonus additions. If, however, the sums assured and bonuses are valued together the values of the sums assured alone can be found with sufficient accuracy for the present purpose by valuing the bonus additions in broad groups and deducting the value so found from the value of the sums assured and bonuses combined.

7. Similarly, the correction to the net premium *income* is

$$-A_0ki\Sigma\frac{S}{a_{xn}} = -A_0ki\Sigma S(P_{xn} + d)$$

which is easily calculated since  $\Sigma SP_{xn}$  and  $\Sigma S$  are known.

8. In the above Notes the  $O^{M(5)}$  Table has been used for convenience, but the results on the  $O^M$  Table would be similar. The case of a combined  $O^M$  and  $O^{M(5)}$  Valuation would be more complicated, but it does not appear probable that the results could be substantially different, and there seems little objection to using a single table in calculating the value of a small adjustment, especially as the factor  $A_0ki$  found from the single table is actually applied to the larger reserves resulting from the combined tables.

9. It is not claimed that the method here outlined does more than give an easily-applied method of calculating the adjustment on the safe side: but the adjustment is itself so small that such a method is all that is usually required.

G. J. L.

## REVIEWS.

*Standard Mortality Ratios incident to Variations in Height and Weight among Men. Report of the Joint Committee of the Actuarial Society of America and the Association of Life Insurance Medical Directors.*

[New York, 1918.]

THIS Report—which is apparently intended to be followed by similar reports dealing with other factors affecting the eligibility of lives for assurance—is the outcome of a suggestion that the work of the Medico-Actuarial Investigation should be supplemented by the definite recommendation, for general use, of “standards by which to measure the value of lives for insurance.” It presents, in a series of tables, a revised and analyzed statement of the results already published by the Medico-Actuarial Investigation Committee in regard to the influence of build and excessive abdominal girth on mortality.

No useful purpose would be served by discussing the methods employed in the construction of the tables or the meaning of the results. Nor, indeed, is the material available for such a discussion, for after referring to three methods of dealing with the preponderance of the experience of early policy years and stating that certain further statistics had been furnished by the Mutual of New York the Report proceeds: “With this additional information



“and with such further data as could be obtained from the experience of other individual companies, the Committee prepared tables of percentages in practicable form for use in the selection of risks which seemed to it to be fairly representative of past experience after giving due weight to the results obtained under the three plans referred to above.” It is sufficient, in view of the form that the results take, that we have here the authoritative recommendation of a Committee of experts who have obviously given careful consideration to the various points involved. It is immaterial that other methods might have produced somewhat different results, because a mortality ratio is in any case a blend of more or less unknown constitution.

The important point about the Report is that it seems to mark the definite adoption of the mortality ratio by American actuaries as a practical representation of the mortality of a class of lives. This appears to us rather a confession of failure from the research point of view. The mortality ratio tells us in effect that, according to a particular experience, the actual deaths among a group of lives having certain defined characteristics were a specified percentage of the expected according to a given standard. That is to say, it gives

the value of  $\sum_0^n q'_{[x]+n} E_{[x]+n} / \sum_0^n q_{[x]+n} E_{[x]+n}$ —obviously a most complicated fact to translate into an extra premium. It tells us nothing as to the incidence of the mortality, nothing as to the relative values of  $q'$  and  $q$ . We fully recognize the impossibility of having a separate mortality table for, *e.g.*, 30-lb. overweights of medium height and aged 30-34 at date of assurance; but between this and the mortality ratio 1.18 there are numerous possibilities. It seems to us that it would be more illuminating, even if it should involve a broader grouping, to have (say) the values of  $\sum q'_{[x]+n} E_{[x]+n} / \sum E_{[x]+n} - \sum q_{[x]+n} E_{[x]+n} / \sum E_{[x]+n}$  for the first 5 years of assurance and for subsequent durations. The mortality ratio may, perhaps, owe its official adoption to the vogue of the numerical method of rating, by which one arrives at the percentage by which the “total value of a risk” exceeds or falls short of the normal by adding together the several percentages by which its mortality ratios exceed or fall short of the standard in such respects as build, family history, pulse rate, occupation, &c. We doubt, however, whether this is sufficient to commend it. Without insisting unduly on the fact that, as Mr. Macaulay in particular has shown (T.A.S.A., vol. xii, p. 290), the method leads to absurd results in extreme cases—for this might be true of a method which nevertheless had a scientific basis in average cases—we cannot find that so far any valid argument for it has been adduced, except perhaps the argument from experience. It has been suggested that it is a more methodical substitute for the old-fashioned plan of mentally balancing *pros* and *cons*, and that by its means the errors “due to the exercise of personal judgment on a combination of factors” are greatly reduced. It would seem, however, that even with the numerical method the impossibility of dividing a risk into absolutely independent factors

must involve liability to error. It is obvious, for example, that it would be fallacious to obtain the value of a lightweight with a tubercular history by adding together the separate mortality ratios for light weight and personal history of tuberculosis.

A second point suggested by the Report is, that the adaptation of an Experience to the purpose for which it is intended to be used appears to be increasingly regarded in America as part of the regular procedure of a mortality investigation. The principle is as old as the employment of "amounts" as the basis of a mortality table, but it is now being freely applied to the data included in an investigation. The most remarkable example of its exploitation is afforded by the recently-compiled American and Canadian Experience, in which the data have been disciplined by the restriction of the amount observed on any one life to \$100,000 and by the limitation of the contributions of certain large companies. It is a principle which affords scope for much ingenuity. In the Report under notice, for example, one of the methods of dealing with the preponderance of early policy years' experience was to reduce the actual and expected deaths of any policy year "in proportion to the number of years of issue entering therein." But even after this adjustment the several durations must have entered into the aggregate with varying weights, owing to the varying amounts of new business in different years. It is a question whether it would not be more satisfactory, in forming aggregate tables, to combine the various durations in definite and known proportions.

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*Simple Interest Tables* (£ s. d.). By WILLIAM SCHOOLING, C.B.E.,  
F.R.A.S.

[Pp. xvi + 172. London: Sir Isaac Pitman & Sons, Ltd. 21s. net.]

OF the 170 pages of tables contained in this work, 50 are taken up by a table of  $\log 240in/365$  (for  $i=1$  to 6 per-cent by eighths,  $6\frac{1}{4}$  to 7 per-cent by fourths, 8, 9 and 10 per-cent and  $n=1$  to 369), and 100 more by a table of  $\log S$  for  $S=1$  to 23999. It will be apparent that the effect of entering the first table with a specified rate of interest and a specified number of days, of entering the second with a specified number of pounds, of adding the results, and finally of entering the second table inversely with the sum thus obtained, will be to give (to the nearest penny and within the limits of the table) the simple interest (in pence), for the specified rate and number of days, on the specified amount. The second table is arranged with £S and the equivalent in £ s. d. of S pence as alternative arguments—or, rather, as argument and result—on the left and right respectively of the log, so that the computer can take out the antilog directly in £ s. d. (whence the title of the tables) instead of having to convert the pence; and there are supplementary tables for dealing with amounts outside the range of the tables. Moreover, with the object of rendering the tables capable of being used by a computer who

does not understand logarithms, the tabulated values—5-figure logs with characteristics—are printed without decimal points, and with 10 added when (as in parts of the first table) they would be negative. Mr. Schooling's idea, as set forth in the preface, is that we can use logs without understanding their nature just as we can switch on electric light without knowing how it is produced, or turn on water without understanding a pump. In the particular case under consideration this may be true as regards calculations falling within the range of the tables, but our feeling is that if a clerk who knew nothing about logarithms were asked to calculate by these tables interest on an odd amount (not an integral number of pounds), or interest amounting to more than £99 19s. 11d., anything might happen. And anyone with a knowledge of logarithms would, we think, find it more convenient to use a table of  $\log in/365$ —i.e., a table similar to Mr. Schooling's first table, but without the pence factor, in conjunction with an ordinary log table.

Any tables for the purpose of calculating simple interest for broken periods naturally challenge comparison with John Laurie's well-known work. Since the publication, in 1831, of the 1st edition (of which "Every genuine Copy" bore the author's signature), this work has become such an established institution that five years ago it reached—with comparatively little material change—a 42nd edition. It seems almost useless to contend with such a work. There is obviously something in its monumental simplicity that appeals to the human mind—something that makes it almost futile to suggest that Sellar's Tables (on similar general lines, but with a logically reduced range of amounts and with the results given to the nearest tenths of a penny) are far more serviceable. As compared with either Laurie or Sellar, Mr. Schooling gives, of course, a far more extensive range of rates. But most of these rates are very seldom wanted, and for the occasional calculation at an odd rate there are devices familiar to all computers of simple interest, or, at the worst, there is the despised multiplication. And there is, to our mind, the fatal objection to Mr. Schooling's process that it requires three openings of the tables as against Laurie's or Sellar's one.

Mr. Schooling's fertility in ideas on the subject of the calculation of simple interest is not, however, exhausted by the £ s. d. tables. In the preface to these tables he states that he has in preparation tables on an alternative plan, and a specimen is given on pp. 4-7 of the present work. This is simply a table of  $S in/365$  for  $S = 100, 200 \dots 900$ ,  $i = .01$  and  $n = 1 \dots 185$  with a few additional values—the results being given to 6 decimal places. It pushes Sellar's logical reduction of amounts to the limit rendered possible by the use of decimals, and it requires only one opening. Although involving in most cases more simple addition than Laurie, a table of this kind is a simple and reasonable instrument which will commend itself to every computer who is used to working in decimals. Whether it will compete successfully with John Laurie is another matter.

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## CORRESPONDENCE.

THE RELATION BETWEEN THE RATES OF MORTALITY, OR  
DECREMENT, FROM PARTICULAR CAUSES, AND THE  
RATES FROM ALL CAUSES, WITH SOME REMARKS UPON  
THE PREVALENT MISUSE OF THE TERM "RATE."

*To the Editors of the Journal of the Institute of Actuaries.*

DEAR SIRS,—With reference to Mr. D. S. Savory's letter *J.I.A.*, vol. li, p. 65, regarding the mortality due to the war, I beg to submit the following remarks dealing with the subject somewhat generally.

The distinction between a probability and a rate is one that deserves to be more clearly emphasized. The term "rate of mortality", as generally employed, might be defined as meaning the proportion dying in a year out of a number exposed to the risk of death for the duration of a year or until death within the year, the words "per annum" being understood. This particular definition has most conveniently led to the function  $q_x$  in an ordinary mortality table becoming invested with a dual capacity; for  $q_x$  not only represents the annual rate of mortality as above defined, but also the probability of dying within a year, at age  $x$ .

When, however, we come to split  $q_x$  into its component parts, the distinction is a real one. For we can subdivide  $q_x$ , the total probability of dying in a year, into the partial, mutually exclusive, and additive probabilities of dying in a year from cancer, consumption, &c., whose sum equals exactly the total probability of dying in a year: but we cannot so split up the rate of mortality  $q_x$  into partial rates of mortality, for the rates of mortality from different diseases are proportions of dissimilar things and are therefore not additive, *e.g.*, the rate of mortality from consumption is the proportion dying from consumption of those exposed for a year to that particular risk, while the rate of mortality from cancer is a proportion of another thing.

Coming now to the particular problem (which is somewhat similar to that dealt with by Dr. Sprague in *J.I.A.*, vol. xxi, p. 406, *see also* Ackland, vol. xxxiii, p. 194): Given the values  $q_{ng}$ , the rate of mortality from normal and war causes combined, and  $q_n$ , the rate of mortality due to normal causes, it is desired to find an expression for the rate of mortality due to the war.

Let  $d_n$  and  $d_g$  be the number dying during a war year from normal and war causes respectively, out of  $l$  persons alive at the commencement of the year; then  $\frac{d_n}{l}$  and  $\frac{d_g}{l}$  will be the probabilities of dying from the respective causes mentioned. Let these probabilities be designated  $q'_n$  and  $q'_g$ , as distinguished from  $q_n$  and  $q_g$ , the corresponding rates of mortality.





Equations D and F represent respectively the probability of dying and the rate of mortality, from war causes, as expressed in terms of the known quantities  $q_{ng}$  and  $q_n$ . Formula F agrees with that got by Prof. Cantelli except for the third term in the denominator, his result being  $\frac{q_{ng} - q_n}{1 - q_n}$ . The Professor's reasoning as quoted by

Mr. Savory evidently omits certain considerations.

With regard to the statement credited to the Professor that his result follows from Karup's theorem

$$p_{ng} = p_n \times F_g$$

the probabilities  $p_n$  and  $p_g$  in this particular instance are not altogether subjects for multiplication as they do not relate to events that are quite independent. The true relation is

$$p_{ng} = (1 - q_{ng}) = 1 - (q'_n + q'_g)$$

Formulas D and E would furnish the means of expressing  $p_{ng}$  in terms of  $p_n$  and  $p_g$ .

As regards the method adopted by Prof. Hersch, this would have been quite correct had he been dealing with probabilities instead of rates, for  $q'_g = q_{ng} - q'_n$ , but as indicated above, it is not accurate to add or subtract rates.

Among our writers there is, so far as terminology is concerned, a want of precision in discriminating between a rate and a probability. In a pension fund for instance, we require the *probabilities of exit* by death, withdrawal, &c. Out of a number who attain a given age we require to know simply the proportion that will go off the fund by each of the modes of exit within a year, and the proportion remaining on the fund at the end of the year; it is a simple splitting up of the given number, the results representing the mutually exclusive probabilities in respect of each of the possible ways in which the event can happen, and adding to unity. These proportions or probabilities are commonly and loosely described as rates of mortality, withdrawal, &c., but they are clearly not properly so described. Another term used with an appearance of greater precision is "the rate of mortality while on the active list", but this is also inexact. The rate of mortality while on the active list is in fact correctly represented by the following expression :

$$\left. \begin{array}{l} \text{Rate of mortality while} \\ \text{on the active list} \end{array} \right\} = \frac{q'_x}{1 - \frac{1}{2}(wq'_x + rq'_x)}$$

where  $q'$ ,  $wq'$  and  $rq'$  represent the probabilities of exit by death, withdrawal, and retirement. For if not, what is the correct designation of this expression?

The cause of the confusion is evident. Insurance is effected



against a probability, never against a rate; and when dealing with ordinary life insurance, actuaries could hardly help seeking for the probability of dying within a year, or the simple proportion of those dying in a year out of a number alive at the commencement. It was expedient to define the abstract function known as the rate of mortality so that it would coincide with this, rather than to define it in any other way, such for instance as that proposed by Dr. Farr; and therefore when we are dealing with a single force such as death or withdrawal, the probability and the rate have the same value. In short, actuaries obtained the probability and called it the rate. The term "rate", however, which a nice discrimination would have restricted to the abstract idea, was allowed to impose itself everywhere, even in respect of probabilities with quite a different value, to the extent that the idea of a probability with its resultant simplicity appears to have been banished to a precarious footing on the margin of consciousness.

It is curious to note how even some of the most eminent of our writers have paid homage to the usurper. The late Mr. Manly, for instance, obtained what were apparently true rates of mortality and withdrawal ( $q_x$  and  $wq_x$ ), and used them as probabilities of exit, admittedly as an approximate measure, but without an appropriate change in designation, following in this respect the prevalent custom (*J.I.A.*, vol. xxxvi, pp. 211, 260, 261). The retention of the incorrect terminology is responsible for half the difficulties alluded to on p. 4 of vol. xlii, besides rendering the reasoning obscure, and leaving its mark on the formulas there given. On the other hand, in vol. xxxix, p. 133, Mr. George King lays down very clearly the correct procedure for deriving the probabilities of exit required in a pension fund, but his ultimate retention of the incorrect term "rate" has led to his explanation being unnecessarily cumbered by the fiction (doing violence to the facts) that "in getting out the rate of mortality we must therefore treat the withdrawals and retirements as at risk for the whole year", &c.: a relative, apparently, of the older fiction that in getting out the rate of mortality simpliciter, deaths were to be treated as at risk for the whole year.

In Insurance functions of all kinds it is always *the probability* that we require, and the expulsion of the term "rate" herefrom at once removes the accompanying obscurities and rationalizations, and clears the channel of thought.

A great deal would be gained if a special symbol were used, where appropriate, to designate a probability as distinct from a rate (as in this communication for example), and if at the same time the concrete functions employed in financial computations, particularly those used in respect of pension funds, were given their proper designations. This need not in any way interfere with the undoubted convenience gained from the use of the term "rate" in the purely general sense. We could very easily speak of the general characteristics of the rates of mortality, &c., for example, and at the same time take care to describe our algebraic and tabular functions as probabilities or proportions.

I should add that the above examples are not quoted by way of criticism of the eminent writers named but merely as illustrations.

I am, Dear Sirs,

Yours faithfully,

A. T. TRAVERSI.

Wellington,

New Zealand.

10 August 1918.

P.S.—Though in the problem dealt with by him Dr. Sprague very clearly differentiated between a rate and a probability, nevertheless his awkward expression “the annual marriage rate among bachelors who do not die in the year” (*J.I.A.*, vol. xxi, pp. 413 and 415) involves some confusion regarding the definition of a rate, and it is surprising that his terminology in this instance should apparently have remained unchallenged. If “rate” is defined as in the second paragraph of this letter (*mutatis mutandis*), it is clear that the function in question could be described with the most rigid accuracy and with greater simplicity as *the annual marriage rate among bachelors*, notwithstanding Dr. Sprague’s deliberate rejection of the latter expression in favour of his own.

Dr. Sprague’s phrase is in fact a distinct misdescription of the function, and a realization of this fact would be of considerable help to students. If we were really in pursuit of the marriage rate among bachelors who do not die in the year we should require to exclude altogether from the figures the bachelors who die in the year. True, it might at first sight appear that there should be no difference between the annual marriage rate among bachelors and that among bachelors who do not die in the year, seeing that the rate among bachelors who die in the year is nil: but as indicated above, we cannot add or subtract rates.

Similar remarks apply to the phrase “the annual death rate among bachelors who do not marry in the year.”

A. T. T.

## THE NATIONALITY OF TETENS.

*To the Editors of the Journal of the Institute of Actuaries.*

DEAR SIRS,—May I be permitted, as a Corresponding Member of the Institute, to call attention to an excusable error concerning one of my countrymen, which I happened to notice in that reliable standard work, the Institute *Text-Book*. I find that the first inventor of commutation-columns (J. N. Tetens) is called a German professor,

although the place where he was born, namely, Kiel, was Danish at that time and belonged to Denmark for the next 127 years. I need not go into details, as these have already been given by F. Hendriks in the first volume of this *Journal* (p. 2); only I may add, that the error has probably arisen from the fact that Tetens, although professor at a Danish University, published scientific papers in German, in order to obtain a wider circulation for them.

Yours very truly,

J. F. STEFFENSEN.

*Ehlersvej 8,*

*Hellerup, Denmark.*

28 March 1919.

## THE INSTITUTE OF ACTUARIES.

At the Council Meeting held on 10 December 1918, the following Address to the King was ordered to be signed and sealed:

TO THE KING'S MOST EXCELLENT MAJESTY.

MOST GRACIOUS SOVEREIGN,

We, your Majesty's dutiful subjects, the President, Council and Members of the Institute of Actuaries, respectfully beg leave to present to your Majesty our loyal congratulations on the success achieved by your Majesty's Naval, Military and Air Forces in the present War, on the signing of the Armistice, and on the prospect of a Peace which will establish more firmly than ever the authority of the Empire in the Counsels of the World, and the position of your Majesty and the Royal House in the affections of the people.

We trust that our special knowledge and experience may be found of use in the work of reconstruction, in which we are prepared to assist your Majesty's Government to the utmost of our power.

We desire to tender to your Majesty the assurance of our heartfelt loyalty, and to offer our humble and grateful thanks for all that your Majesty and our gracious Queen have done during the War, by personal effort and example, and by never-failing sympathy with all classes, to maintain the courage and resolution of the people, and to promote that spirit of unity and self-sacrifice which has done so much to secure for your Majesty's Forces the success which has at length rewarded their heroic efforts and devotion to duty.

We fervently pray that your Majesty may long rule in peace and happiness over a prosperous and contented Empire.

GIVEN under the Common Seal of the Institute at  
Staple Inn Hall, this Tenth day of December, in the  
year of our Lord, One thousand nine hundred and  
eighteen.



GEOFFREY MARKS, *President.*

A. LEVINE,  
W. PALIN ELDERTON, } *Honorary Secretaries.*

The following resolution was unanimously passed at the Ordinary General Meeting of the Institute held on 16 December 1918 :

The Members of the Institute of Actuaries in General Meeting assembled, resolve :

That fraternal greetings and congratulations on the conclusion of the War be offered to the Actuarial Societies in France, Belgium, Italy, and the United States of America. They feel that the friendly relations which have always existed between the Members of the Actuarial profession in the British Empire and in the Allied countries have been rendered even more cordial by the common sacrifices made in the great struggle for liberty and justice now so happily ended. They have every confidence that the spirit of comradeship brought forth in the recent struggle will endure and will prove of lasting value in the work to be undertaken by the Members of the Profession in the coming days of Reconstruction.

## Obituary.

FREDERICK LIGHTON BRISTOW, Probationer of the Institute,  
Lance Corporal, 7th London Regiment.

*Killed in Action 20 March 1916.*

JOHN WILLIAM EWART ALEXANDER, Probationer of the Institute,  
2nd Lieutenant, 10th Battalion, Norfolk Regiment.

*Died on Service 14 April 1916.*

VINCENT JOHN AUSTIN, Probationer of the Institute, Private,  
5th City of London Regiment.

*Killed in Action 1 July 1916.*

RAYMOND COLE, Probationer of the Institute, Private, 16th  
Middlesex Regiment.

*Killed in Action 1 July 1916.*

THOMAS MIDDLETON, B.A., Student of the Institute, Captain,  
97th Brigade, Highland Light Infantry.

*Killed in Action 1 July 1916.*

CHARLES SIDNEY SHILSON, Probationer of the Institute, Private,  
5th City of London Regiment.

*Killed in Action 1 July 1916.*

MALCOLM HOWARD GRIGG, Probationer of the Institute, 2nd  
Lieutenant, 26th Battalion, Manchester Regiment.

*Killed in Action 9 July 1916.*

HARRY WALTER BROWN, Probationer of the Institute, Private,  
Royal Naval Division.

*Died on Service 23 January 1917.*

THOMAS GRAHAM CUNLIFFE, Probationer of the Institute, Private,  
20th Battalion, Royal Fusiliers.

*Killed in Action — May 1917.*

JOHN CROMWELL HURLEY, Associate of the Institute, Lance  
Corporal, 22nd Battalion, Australian Imperial Force.

*Killed in Action 3 May 1917.*

RALPH GEORGE GALE, Probationer of the Institute, 2nd Lieutenant,  
8th Battalion, South Staffordshire Regiment.

*Killed in Action 12 October 1917.*

WILFRED BRADLEY, Probationer of the Institute, Private,  
Honourable Artillery Company.

*Died of Wounds 6 November 1917.*

CYRIL PERCY MADDOX, Probationer of the Institute, Private,  
16th County of London Regiment.

*Killed in Action 20 November 1917.*

JAMES HENRY ORR, Probationer of the Institute, Captain, 210  
Siege Battery, Royal Garrison Artillery.

*Killed in Action 30 November 1917.*

GEORGE FREDERICK THOMAS ASCOTT, Probationer of the Institute,  
Private, London Rifle Brigade.

*Killed in Action 25 March 1918.*

FREDERICK DEFRIES, Associate of the Institute, Captain, 5th  
Battalion, Middlesex Regiment.

*Killed in Action 6 April 1918.*

HENRY JACKSON MILLS, Student of the Institute, Lieutenant,  
19th Battalion, Machine Gun Corps.

*Killed in Action 30 May 1918.*

ALFRED RICHARD HARRISS, Probationer of the Institute, Private,  
1/15th London Regiment.

*Died of Wounds 13 September 1918.*

CHARLES GEORGE GIFKINS, Probationer of the Institute, Gunner,  
139 Heavy Battery, Royal Garrison Artillery.

*Killed in Action 15 October 1918.*

EDWARD ALBERT NEWLAND, Student of the Institute, 2nd  
Lieutenant, 24th Battalion, Royal Fusiliers.

*Killed in Action 23 October 1918.*

DONALD ARTHUR ROBERTS, Probationer of the Institute, Flight  
Lieutenant, Royal Air Force (attached H.M.S. "Iron Duke").

*Died on Service 30 January 1919.*



# JOURNAL

OF THE

## INSTITUTE OF ACTUARIES.

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*Group Insurance.* By P. H. McCORMACK, A.I.A., of the  
*Provident Mutual Life Assurance Association.*

[Read before the Institute, 28 April 1919.]

THE President, in his recent address, referred to the rapid growth of group insurance in the United States, and suggested that the system was well worth the study of members of the actuarial profession in this country. The following notes may possibly be useful as an introduction to the subject. Most of the facts have been collected from contributions to American journals, a list of which is given at the end of the paper.

It has often been realized that all the lives in certain categories might be insured in bulk, selection against the insurance company being thereby avoided without the necessity for medical examination. An attempt to put this principle into practical operation has been made by means of a system devised in America under which the employees of a firm may be insured by a single policy without medical examination. This system is known as group insurance. Although there are various modifications of the plan, a group policy is generally renewable from year to year on a term basis, the cost of the premiums being borne by the employer. The sums payable at death in individual cases are usually commensurate with annual wages. The only qualifications necessary for insurance under the scheme

are active service and membership of an eligible group; and it is claimed that the scope is so wide as to include the unfit and those beyond the usual age limits. Incidentally, group insurance also covers workers who, through poverty or indifference, fail to make any personal provision for their dependents.

Although a policy bearing some resemblance to the present system was issued nearly twenty years ago, it was not until July 1912 that the first group insurance contract of note was effected in the modern form. Since that date the scheme has attained great popularity in the United States, having been adopted by many of the most important employers of labour in the country, including railroads, shipbuilding companies, and banks. As mentioned by the President in his address, one firm alone has insured its employees numbering half a million on the group system. It is reasonable, therefore, to suppose that the method offers some advantages over ordinary or industrial life insurance. Ordinary life insurance, subject to annual, half yearly, or quarterly premiums, is of course beyond the means of the majority of wage-earners, while industrial insurance, on which the premiums are collected weekly at a relatively high cost, is often only sufficient to meet the funeral and other expenses which are incurred immediately on death. One of the objects of group insurance is to meet this difficulty by enabling many grades of wage-earners, who could not otherwise be adequately insured, to obtain the benefit of substantial life insurance protection, at least during the period of their employment.

#### THE PRINCIPLES OF GROUP INSURANCE.

The distinctive feature of group insurance is that the unit of insurance is not the single life but a group of lives. As the group is the unit of measurement, the individual members need not all be standard lives, and no medical examination is required; for it is only necessary to consider the eligibility of the group as a whole. If, however, a group is to be eligible for insurance, it obviously must not contain an undue proportion of damaged lives. In order to protect the insurance company against adverse selection in this respect, it is essential that the group should exist for some purpose other than that of insurance; and, moreover, membership of the group should in itself be to some extent an index of good health and activity. These conditions

may be fulfilled by a group consisting of the employees of one firm, provided all or substantially all are included. The insurance may sometimes be restricted to certain departments, or to male employees only, and casual labourers are generally excluded. With a view to the prevention of adverse selection on the part of an entire group containing an undue proportion of impaired lives, a minimum limit of 100 lives is usually fixed in America for group insurance without medical examination. If the number of lives is less than 100, the group is not necessarily ineligible for insurance, but a modified form of medical examination would generally be required, the degree of stringency of the examination depending on the size of the group. In such cases it might of course be necessary to exclude impaired lives.

Another important safeguard against the exercise of adverse selection by the members of a group is that there should be no individual option as to the amount of insurance. That is to say, the sum insured must either be a flat amount or follow a definite formula.

Group insurance has sometimes been criticized as unsound because there is no medical examination as a general rule. But when life insurance business was started, medical examination was not required, and schemes of insurance without examination have been offered by English companies in recent years. Medical examination is only a means to an end ; but it is not necessarily the only means. Its object is to maintain the standard of mortality and prevent selection against the office. As is well known, the effect of medical selection for all practical purposes wears off after a few years, the select lives merging into the general body, which may be regarded as a sample population. The group insurance system under adequate safeguards automatically gives sample populations, in which there is no adverse selection. Such groups are therefore eligible for insurance, provided there is no undue occupational hazard. But it is very important that no opportunity for adverse selection against the insurance company should be given ; and for this reason the system should not be applied to clubs or societies connected with a particular trade or employment where the membership is voluntary. An interesting example of what seems to be a misapplication of the group insurance principle may be mentioned. The Federal Insurance Department, while apparently taking no objection to group insurance, has expressed the

opinion that the system is not at present legal in Canada. Certain Canadian municipalities, however, during the war have insured the lives of soldiers recruited for over-seas service with American Companies on a group basis, the premiums being paid by the municipalities. Bodies of soldiers, although good lives from a medical point of view, are not suitable for group insurance owing to the catastrophe hazard which exists in any particular regiment on active service. In other words, the risks are not sufficiently decentralized to allow of the operation of the law of average. In considering the eligibility of a group for insurance, occupational hazard is an important factor. The wages-staff of a brewery company, for example, would hardly form a suitable risk for group insurance, except, possibly, in the case of an insurance company doing a very large business and having special experience in dealing with hazardous risks. Again, two firms might carry on business of a similar character, but it does not follow that both would be accepted for group insurance on the same basis, as the general conditions of employment in the two cases might be altogether different. American companies, before accepting a proposal for group insurance and determining the scale of premiums to be charged, usually make a general inspection of the premises, similar to that made in connection with workmen's compensation insurance. Careful attention is paid to such matters as sanitation, lighting and heating, fire and accident prevention devices, class of workers employed, and conditions of labour, as well as to any special risks incidental to the employment. Companies which transact workmen's compensation business have the advantage of already possessing the necessary organization for making these inspections.

The group method may be adopted in connection with any kind of insurance, whether term, whole life, endowment, or annuity, and policies on all these plans have been issued; but group insurance, as it exists in America at the present time, is practically restricted to a modified form of term insurance, under which the employer pays the whole of the premium. As a general rule, each employee is insured, so long as he remains in the service of his employer, for one, two, or three years' salary, with say a minimum of \$500 and a maximum of \$3,000, which may in some cases be increased, subject to medical examination. Occasionally the insurance is for a flat amount, which may or may not increase with length of service. Sometimes past

service is also taken into account. A group insurance policy does not necessarily cover employees who remain in the firm's service after having passed the general age for retirement; but the system is very elastic, and members who are above the usual ages for ordinary life insurance are often included. New employees are admitted to the scheme, though there is usually a probationary period of three to twelve months' active and continuous employment, which is considered sufficient to protect the insurance company against errors of judgment on the part of the employer in engaging unfit workers. This regulation also excludes casual employees who only remain with the firm for a short time. Unless precautions are taken, there is a risk of bad lives exercising adverse selection by obtaining employment with a firm which has adopted the group insurance system. Some companies, therefore, reserve the right to require medical examination of new entrants to existing groups, but may be prepared to waive this right if they are satisfied that the methods adopted by the firm in selecting new employees are sufficiently stringent.

As the insurance is on a yearly renewable term basis, the premium in individual cases naturally increases from year to year. But this is not necessarily a disadvantage, as the employer has only to consider the total premium payable, which in an average group remains fairly constant, owing to the fact that the increased cost of insuring continuing members of the staff is more or less counterbalanced by the lower cost of the young lives who replace retiring members. In a progressive business the average premium may decrease for some years.

The payment of premiums by the employer avoids the main danger of assessmentism, which arises from the free choice possessed by individual members as to whether they will remain in the society or not and the assumption that a sufficient influx of new lives will be maintained. When the supply fails, the premium charges rapidly increase, with the result that the good lives withdraw and the company becomes insolvent. Under the group system, however, this opportunity for individual selection does not exist, as the insurance must either continue as a whole or lapse as a whole. But the financial position of the employer is a factor which may affect both the insurance company and the employees; for in the event of the firm tending towards insolvency the first lives to leave its service would in the ordinary course be the young and healthy. The loss of the young lives



would not necessarily have any serious effect on the insurance company, as the policy would lapse unless the firm were able to meet the increased premium charge. If, however, owing to the withdrawal of healthy members, the firm were left with an excessive proportion of impaired lives, the group would no longer be eligible for insurance, although there might be no direct selection against the office. But as it is the practice of most American companies to guarantee the renewal of group policies for a term of years varying from five to twenty on the original premium basis, it is possible that the insurance company might be compelled to renew the policy, in which case it would probably incur a loss on the particular group until the expiry of the guaranteed term or until the firm in financial difficulties could no longer pay the premium. As, however, the unit of insurance under this system is the group, and not the individual life, any loss on one group would be averaged with the profit and loss on others ; for it is not contemplated that every group must necessarily show a profit. Moreover, insolvency, or a tendency thereto, would be unlikely to occur in more than a small proportion of the insured firms, and the special circumstances mentioned would therefore seldom arise. But this possibility should serve as a warning against undue latitude in granting the option of renewal for a long term, especially in the case of companies whose group insurance business is on a small scale. The term for which rates are guaranteed has become a competitive feature with some American companies, but this is evidently an unfortunate development.

#### POLICY CONDITIONS.

Group insurance contracts are issued in the form of a single "blanket" policy, granted to the employer for the benefit of his staff and covering all members included in the scheme. The insurance is usually restricted at the outset to members who are in active employment. The employees are not named individually in the policy, but the necessary particulars of the insured and their beneficiaries, with the amount of insurance in each case, are recorded in a subsidiary schedule, card index, or book. Each employee receives a certificate, supplied by the insurance company, stating the terms and conditions of the scheme, the sum insured and other individual details being filled in by the employer.

Under the renewable term plan the policy is usually effected



for twelve months, but may be renewed from year to year, like a workmen's compensation policy. As already mentioned, an option is often given to the employer to continue the insurance for a term of years on the same premium scale as that adopted at the outset and scheduled in the policy. Occasionally the contract is binding on both sides for a limited period.

Employees remain insured so long as they continue in the active service of the firm; but absence for a limited time, whether due to sickness or not, does not invalidate the insurance. Sometimes, however, this privilege is conditional on payment by the employer of the whole or part salary during the period of absence. An employee leaving the service during the currency of a group policy is usually given the option within thirty-one days of effecting a whole life or endowment insurance without medical examination, for an amount not exceeding that for which he was insured under the group policy, at the ordinary tabular premium corresponding to his attained age. In a subsequent section of the paper this option is discussed from the point of view of the insurance company. The provision is compulsory in some states for all group insurance policies, the main object being, no doubt, to safeguard the employee against losing the benefit of his insurance when it is most needed, as he would do on a strictly term basis in the event of prolonged ill-health. In connection with state laws relating to group insurance, it is interesting to note that in New York group policies covering at least 100 lives are excluded from the limitation on new business which was imposed some years ago.

If the sum insured varies according to remuneration, the amount is increased, subject to a maximum, whenever an advance in wages takes place. Alterations in the amounts of insurance and the inclusion under the scheme of eligible new employees may be automatic or may take effect upon notice by the employer to the company. Group insurance policies are free from restrictions regarding occupation and travel, but war risks and those resulting from riots or civil commotion would probably be excluded. There is no suicide clause, since moral hazard is assumed to be eliminated by the methods of group selection. In the event of understatement of age, the claim is usually settled in full, and the premium payment adjusted accordingly.

The business is generally conducted on a non-profit basis; but some companies, charging higher premium rates, grant a

dividend or refund of a proportion of the profits each year. For this purpose the insured groups are arranged in a few classes according to the degree of occupational hazard involved, and, the profits for each class having been ascertained, separate rates of dividend are declared. The method is of course a little more complicated than the ordinary system, but it possesses the following advantages :

1. It avoids any necessity for refinement in premium rating of various risks ;
2. It provides a premium margin to meet over-average mortality ;
3. It prevents dissatisfaction which might arise under the non-profit system if the premium payments largely exceeded the claims ;
4. It minimizes the risk of loss through the guarantee of the original premium scale for a term of years.

#### PREMIUM BASIS.

The basis adopted for group insurance premiums in the early days of the business was the American Experience Table with interest at  $3\frac{1}{2}$  per-cent. This method, however, gives excessively high premiums at low ages, and the rates now charged for young lives are often smaller than the net premiums, although at the older ages increasing loadings are necessary. Since the American  $3\frac{1}{2}$  per-cent forms the legal standard of valuation, companies charging lower rates than the net premiums according to this table are required in some states to maintain a "deficiency reserve" equal to the commuted value of the amount by which the premium charged falls short of the statutory net premium. The following specimen rates of premium charged by one American company for five classes of group insurance

#### *Rates per 1,000.*

Age	American $3\frac{1}{2}$ % Net Rates	(1)(5) $q_x \times 1,000$	English Life No. 5 (Males) $q_x \times 1,000$	Insurance Com- missioners' Table (Males) $q_x \times 1,000$	(1)	(2)	(3)	(4)	(5)
25	7.79	6.89	6.36	4.15	6.01	6.26	6.76	7.51	8.51
35	8.65	8.37	10.19	6.44	6.41	6.66	7.16	7.91	8.91
45	10.79	12.00	15.63	11.20	9.17	9.42	9.92	10.67	11.67
55	17.94	20.83	26.02	21.87	18.87	19.12	19.62	20.37	21.37
65	38.77	42.21	50.57	46.48	45.88	46.13	46.63	47.38	48.38

risks are taken from a paper read by Mr. H. Pierson Hammond at a meeting of the National Convention of Insurance Commissioners at St. Paul in August 1917. The American Experience  $3\frac{1}{2}$  per-cent net one year term rates and the values of  $q_x$  according to the  $O^{M(5)}$  Table and the English Life No. 5 and National Insurance Commissioners' Tables (Males) are added for comparison.

As indicated by the above scales, a limited degree of extra risk arising from occupational hazard may sometimes be covered by uniform additions to the premiums; but considerable refinement in rating, such as occurs in workmen's compensation business, is not usually attempted in group insurance. Occasionally, however, an eligible risk may be offered for which special rates may have to be devised. Although the individual members may not all be exposed to the same occupational hazard, any extra premium required is applied to the whole group as the unit of insurance. Since the premiums are paid by the employer, no hardship is caused by this method.

When a proposal for group insurance is received and the premium basis agreed upon, the insurance company must be supplied with a census of employees, including details of the remuneration at each age, in order that the exact amount of initial premium may be calculated. As, however, the preparation of this statement may involve some delay, the risk is often accepted on payment by the employer of a provisional premium of from one to one-and-a-half per-cent of the total annual wages. An adjustment will probably be necessary when full information is available, but in an average group the required premium usually lies within the limits mentioned. Further adjustments in the premium, consequent upon staff changes, frequently take effect at monthly intervals, although the actual cash settlement may only be made once a year. But sometimes the premiums are paid monthly, in which case the rates may be true monthly premiums.

Since the sum insured is usually a function of the annual wages, an alternative method of rating, analogous to that adopted in workmen's compensation business, would be to fix the premium as a percentage of the payroll, with a modification in respect of maximum and minimum limits of insurance. In determining this percentage, the general age distribution of the employees would of course be taken into account, although a fresh calculation every year based on the exact amount insured

at each age would not be required, as in the case of the usual method of computing the premium. But a census of the staff would be necessary at the outset, and also at intervals during the currency of the policy in order that the percentage might be altered in the event of an appreciable change in age distribution. It is therefore doubtful whether the alternative method would on the whole be simpler than the usual plan; and, moreover, it would represent a considerable departure from the established principles of life insurance practice.

In May last two American actuaries, Messrs. Cammack and Morris, published an investigation of the joint mortality experience during the years 1913 to 1917 of the Aetna and Travelers Insurance Companies in respect of group insurance policies. An interesting feature of the experience is the stability of the business; for out of 949 policies issued in the five years there were no more than 30 terminations, and of these only 21 were cases of actual discontinuance of insurance. There were in all 310,911 years of exposure and 2,434 deaths, the corresponding amount exposed being \$222,026,715 and the amount terminated by death \$2,137,700. High mortality was experienced in the case of electric and street railroads, underground and elevated railways, electric light and power companies, steel foundries, and leather industries, the mortality in tanneries being especially unfavourable. On the other hand, the furniture, woodworking, and textile trades showed favourable mortality, probably due to a relatively small proportion of fatal accidents. The results confirm the view that there is a minimum degree of selection against the office when the whole premium is paid by the employer and all employees are covered. Some years ago group insurance policies were issued to mutual aid associations of police forces, railroad employees, &c., but these show heavy selection against the insurance companies.

The experience was compiled on a calendar year basis, the exposure in year "0" comprising the fractional year's exposure in the year in which a policy was issued. The mortality in policy year "0" is very light, but it afterwards increases considerably, although no regular progression is shown. This increase in the rate of mortality is, no doubt, largely a result of the continued insurance of members who are no longer actively employed. Excluding the experience in extra-hazardous occupations, the actual deaths in policy years one to four for all ages combined were 88 per-cent by numbers and 90 per-cent

by amounts of the expected according to the Medico-Actuarial Table. The experience is very favourable at the younger and older ages, but between 40 and 65 the mortality exceeds the expected. The authors suggest that this is partly due to a strong tendency of middle-aged employees to understate their ages. The rates charged should therefore be sufficiently high to cover these mis-statements. The expected deaths were also calculated by the American Table, but this represents the experience still less closely than the Medico-Actuarial Table. The latter table is evidently more suitable than the American Experience as a basis of group insurance premiums, and some companies have adopted it for that purpose.

Although the Aetna and Travelers experience is immature and not of great volume, it is useful as a measure of mortality during the early years of group insurance. But future mortality may be appreciably affected by changes in labour turnover and by an increased exercise of the option given to employees of continuing their insurance on a permanent basis after leaving service. An early experience should therefore be used with caution as a criterion of rates to be charged in the future ; for considerable loss might result from any general cutting-down of premiums.

#### SOCIAL ASPECTS OF GROUP INSURANCE.

The principal contingencies which may arise during employment, and against which it is possible to make provision, are death from accident, disablement, premature death from natural causes, sickness, and loss of earning capacity through old age. Statutory compensation is awarded to an employee or his dependents in the event of disablement or death by accident in the course of his employment, though there is no legal liability in respect of the other contingencies mentioned. Group life insurance represents a development of the principle underlying workmen's compensation insurance, since it enables an employer to provide for the moral responsibility arising in cases of premature death among members of his staff. In combination with a pension scheme, the system affords to employees a considerable extension of the compensation benefits secured by law. In some cases it takes a more comprehensive form than a mere term insurance, although the latter is the usual plan. For instance, a group insurance policy effected by a New York bank included the following benefits :



- (a) Insurance of one year's salary in the event of premature death ;
- (b) In cases of total disablement from any cause, 100 per-cent of salary to be payable for the first month, 80 per-cent for the succeeding eleven months, and 60 per-cent thereafter during continued disablement ;
- (c) A pension commencing at age 65 of an amount equal to 2 per-cent of the aggregate salary received during continuous service with the bank.

Such policies are rare, but they may perhaps foreshadow a future development of the system which will increase its importance as a social factor.

One of the advantages claimed for group insurance is that it tends to create harmony between the employer and employed, and any system which improves the relations between capital and labour has a special value at the present time. But this advantage cannot be secured unless the whole premium is paid by the employer ; for if a part of the premium has to be contributed by the employees, the rule must apply to those members who are not in favour of the scheme equally with those who approve of it, since it would be contrary to the principles of group selection to allow any individual option. The difficulty would therefore not be avoided by making the adoption of the scheme conditional on the consent of a majority of the employees. The term insurance method is particularly unsuitable for a contribution system on account of the increase in the premium as the age advances and the temporary nature of the insurance benefit, which is not altogether overcome by the option sometimes given of taking up an ordinary policy on leaving service. The automatic termination of the insurance at the retiring age would naturally cause great discontent on the part of an employee who had paid contributions during the whole of his working life. If, however, the entire premiums are paid by the employer, this grievance does not arise, especially if a pension begins when the insurance lapses. Although it is possible to provide for the pension as well as the insurance by adopting the group system on an endowment insurance basis, the cost of such an arrangement would generally be prohibitive, especially in the case of firms with a high labour turnover. The term insurance basis gives the necessary cover at a minimum cost to the employer, and no question arises as to surrender values in the case of employees who leave the firm.



In the opinion of many important American employers, group insurance tends to prevent workers from drifting away through mere unrest ; for, in the absence of a substantial reason for leaving, an employee would probably hesitate before giving up his insurance by joining a firm which did not adopt the system. If, therefore, group insurance reduces the labour turnover, it may save considerable expenses to the employer, as the cost of selecting and training new employees is often very high. No corresponding advantage would result from a small increase in wages equivalent to the group insurance premium.

Group insurance is economical in practice, the expenses of management being lower than in ordinary life business for the following reasons :

1. No medical fees are paid ;
2. Individual solicitation is avoided, and the cost of new business thereby reduced ;
3. The commission payable to agents is on a lower scale than in the case of ordinary policies. Much of the business is obtained direct ;
4. The issue of a single policy to cover a group of lives saves clerical labour. In the book-keeping system, the figures for each group are carried forward from year to year *in toto*, the adjustments resulting from staff changes being kept up to date by a continuous method.

Some of the large firms which have effected group insurance policies with American companies could, of course, undertake their own insurances. There are, however, two difficulties in this. In the first place, it would probably be necessary to set up a specially organized insurance department ; and, in the second place, if any dispute arose in connection with a claim, it might be made an occasion of discontent among the employees, whereas if the matter were in the hands of an independent insurance company the firm could not be held responsible for the decision.

#### INSURANCE OF BRITISH WAGE-EARNERS.

Insurance provision on a small scale is already made by a considerable proportion of the wage-earners of this country by means of industrial policies. Although the individual amounts are low, the aggregate volume of business transacted is very large, total funds of nearly sixty-five millions sterling having been held by British companies in respect of these policies on

31 December 1916. But the industrial system, while serving a useful purpose in placing the benefits of life insurance within easy reach of the working-class population, unfortunately involves an economic waste on account of the heavy cost of making the weekly collections from door to door, which are an essential feature of the system. The result is that the premiums payable by industrial policy-holders are considerably higher in proportion to the sum insured than those payable by ordinary policyholders, and it is therefore very difficult for the majority of wage-earners to make adequate provision against death and old age.

Attempts have sometimes been made by associations of employees to dispense with industrial policies by grafting life insurance benefits on existing pension funds, generally without any actuarial advice; but as the rates of contribution are rarely adequate, this plan in most cases leads to insolvency.

In view of these difficulties, it has often been predicted that group insurance would be widely adopted in England after the war; but although the business has already been commenced, it seems very doubtful whether it will ever become general in this country. English firms, when adopting a scheme of staff life insurance, frequently stipulate that their own contribution shall be conditional upon the payment of an equal amount by the employees included in the scheme; but, for reasons already stated, the group insurance system in its usual form does not lend itself to this arrangement. It would, of course, be possible for the firm to defray the whole cost of insuring the staff on the short term plan, and to collect a contribution from them to be applied in purchasing a more permanent form of insurance, such as a whole life policy with premiums ceasing at the retiring age. But the difficulty would be to fix the amount of the member's contribution, having regard to the fact that in individual cases the firm's contribution increases every year. The principle of equal contributions would obviously have to be abandoned, as it would not be practicable to increase an employee's contribution so long as his salary remained the same. If the firm's contributions, as well as those paid by the employees, were used to purchase limited payment whole life policies, one of the chief objects of group insurance would be defeated, because the wage-earners would no longer obtain the substantial life insurance protection which is possible under the term system at a relatively low cost.

The privilege granted to employees of effecting an ordinary policy without medical examination places group insurance on a convertible term basis, but only in a restricted sense, as the option is limited to employees who leave the service. In the exercise of the option a maximum degree of selection against the office may be expected, as in the majority of cases it is unlikely that a healthy man taking up employment with another firm would wish to avail himself of the option, even if he had the means to do so. As group insurance is still in an early stage, it is probable that the full effect of this selection has not yet been felt; but the whole question of the option is evidently one of the most important points arising in connection with the business. The concession has been justified on the following grounds:

1. The group insurance premiums may be specially loaded to allow for the option;
2. The permanent premium payable is the full tabular rate, medical fee and commission being saved;
3. In a large number of cases the cost of an ordinary policy is prohibitive, and the option is therefore not exercised.

But unless the possibility of a considerable degree of adverse selection is taken into account, it would seem unwise to grant the option. An alternative plan, limiting the company's liability in this respect, would be to issue the group insurance policy on a two-year term basis instead of on a one-year term basis, no permanent policy being offered on leaving service. Under the one-year plan it might be necessary to exclude from the next year's insurance a member who happened to be ill at the end of the first or any subsequent year: a condition which would sometimes cause great hardship. The two-year plan obviates this difficulty to a considerable degree, and the cost is only slightly more than under the one-year plan. In order to obtain the full benefit of the option of renewal contained in the two-year plan, this option would only be exercised in the case of members whose state of health rendered them ineligible to continue the insurance on a two-year basis. The method would, of course, be inadequate in the event of prolonged ill-health.

Messrs. Burn and Symmons, in referring to the group system (*J.I.A.*, vol. xlix, pp. 231-232), suggested that it would be illegal in this country to issue a life insurance policy which did not contain the names of the insured, and moreover that legal difficulties might arise with regard to insurable interest,

although the employer obtained no direct benefit under the policy. Possibly, however, these difficulties might be overcome by issuing the contract on an indemnity basis, and not as a policy of life insurance.

Group insurance is not intended to take the place of ordinary life insurance, but only to supplement it by making provision against premature death. It is said that the adoption of a group scheme has often resulted in a greater appreciation of the value of life insurance in general, and has thereby assisted ordinary business. But it is conceivable that temporary insurance under a group system might sometimes be relied upon in cases where a more complete life policy would otherwise be effected.

Although group insurance possesses several advantages, it seems to leave many of the problems connected with the insurance of wage-earners to a large extent unsolved.

The following is a list of some of the principal papers on the subject :

GROUP INSURANCE.—By V. R. Smith. (*Proceedings* of the Insurance Institute of Toronto, 1913–1914).

GROUP INSURANCE.—By W. J. Graham. (*Transactions* of the Actuarial Society of America, October 1916).

GROUP INSURANCE.—By R. B. Trousdale. (*Annals* of the American Academy of Political and Social Science, March 1917).

GROUP LIFE INSURANCE AND ITS POSSIBLE DEVELOPMENT.—By E. B. Morris. (*Proceedings* of the Casualty Actuarial and Statistical Society of America, April 1917).

LIFE INSURANCE IN GROUPS (1912–1917).—By H. Pierson Hammond. (National Convention of Insurance Commissioners, St. Paul, August 1917).

JOINT MORTALITY EXPERIENCE OF THE AETNA LIFE AND TRAVELERS INSURANCE COMPANIES ON GROUP POLICIES.—By E. E. Cammack and E. B. Morris. (*Transactions* of the Actuarial Society of America, May 1918).

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#### ABSTRACT OF THE DISCUSSION.

MR. F. P. SYMMONS congratulated the author on the clear and logical way in which he had dealt with a form of assurance

that might bulk very largely in the public eye in the near future. Before the war pension and sickness funds and benefit funds of a similar nature were on the increase. The labour position since the war tended to accelerate that increase, and he hoped that benefit funds would help in the solution of the difficult problems with which they were faced. It was evident that group insurance could be granted at what were apparently cheap rates. The cost of collection was negligible, and expenses of administration were comparatively small. Adverse selection was a matter which depended on the care with which the groups were selected, and the *bonâ fides* of the employer. In addition, group insurance was said to form a valuable means of introducing other business to the companies, and Mr. Day, President of the Equitable Life Assurance Society of the United States, in an address delivered in December 1913, remarked that group insurance was a supplement to, rather than a substitute for, other forms of insurance, and it operated not to replace but to create a demand for individual assurance. As to how far these anticipations had been or would be fulfilled, there was no evidence. From the employer's point of view, it was urged that the scheme operated as an inducement to workers to remain with one firm rather than to sacrifice the benefits of the assurance by transferring their services elsewhere. If, however, the practice of group insurance became to any extent widespread, the value of this advantage to employers must considerably decrease, for if all employers adopted schemes of this sort no sacrifice would be involved in changing from one firm to another. The cost to the employer, however, was considerable, amounting, roughly, to 1 per-cent of the pay roll, taking, for example, the case where one year's salary was assured. With the average pension fund the minimum contribution might be taken as 5 per-cent, and frequently 6, 7 or 8 per-cent was paid. From the employee's point of view, he thought it fair to say he received cover which was limited to the period that he remained in sufficiently good health to continue in that particular occupation. The value of this cover could be tested by the rate of mortality experienced while on the active list by the members of any ordinary pension fund. In all the pension funds with which he had had any connection the rate of mortality on the active list could be fairly represented at the earlier ages by the  $O^{[M]}$  rates for the first year of assurance. From forty-five upwards, perhaps, the second year rates applied. The  $O^{[M]}$  rate of mortality for the first year at age twenty-five was 2·8 per thousand, while the lowest group insurance premium quoted in the paper for that age was 6·01 per thousand. At age thirty-five the  $O^{[M]}$  rate was 3·6, as against 6·41; at age forty-five (for the second year of assurance), 7·75, as against 9·17; and at fifty-five 13·29, as against 18·87. If the mortality experience of pension funds were reproduced by group insurances—and the schemes as regarded mortality would appear to be analogous—the premiums charged provided adequate margins, unless they included a charge for the option of continuance. From the employee's point of view, all he received was little more than an accident insurance. When his



health declined to the extent that he was unable to continue in his employment, the cover from the group policy ceased.

In the third paragraph of the paper it was stated that group insurance gave the wage-earners adequate cover, whereas ordinary life assurance, subject to annual, half-yearly and quarterly premiums, was beyond the means of most manual workers, whilst industrial assurance with weekly collections entailed a relatively high cost and only provided sufficient to meet the funeral and other expenses incurred at death. Surely the word "adequate", if applicable at all, was only so to a most limited degree. The amount assured might be adequate, but when it was remembered that the probability of its becoming payable was small, owing to factors apart altogether from rates of mortality, the adequacy of the cover was seen to be to a great extent illusory. He did not overlook the fact that an endeavour was apparently made to meet the objection that cover ceased just when most needed, by granting the option to effect an insurance under any of the tables of the issuing company, irrespective of the state of health of the life in question. If, however, as supposed by the author, the majority of the wage-earners could not afford ordinary insurance when young and in employment, it was evident that when advanced in years and unemployed, or only employed at lower rates of wages—as they would in all probability be just at the time they wished to exercise the option—it would be difficult for them to take advantage of that option and effect an insurance under any ordinary table. It was stated in the paper that the granting of the option had been justified on three grounds, one of which was that the cost of an ordinary policy was prohibitive and the option was therefore not exercised. This might be a factor which the companies were justified in taking into account, but it showed also that the option failed to give group insurance the value of a permanent contract of life assurance, which could, practically speaking, only be cancelled at the will of the assured. He very heartily supported the author's conclusion that, although group insurance possessed some advantages, it seemed to leave many of the problems connected with the insurance of wage-earners to a large extent unsolved.

Mr. W. SCHOOLING, C.B.E., said that as a visitor he would like to express his gratitude to the author for an extremely interesting paper, on a subject which was of far-reaching importance in connection with labour and social questions. Perhaps the most useful contribution he could make to the discussion was to point out the possibility of meeting an objection to which group insurance, as commonly understood, was open. It was contended that a system of insurance of this kind made for good relations between employers and employees, and for continuity of service; that continued service was of greater value than discontinuous service; and that consequently, if group insurance produced continuous service, expenditure upon it by the employer might be an economy and not an expense. Mr. Symmons, however, made the very natural criticism on this contention that, if group insurance became popular



and was very extensively adopted, it took away this advantage. The possibility of making fairly sure of retaining continued service would be attained if there were contributions to permanent benefits, as distinct from merely the temporary insurance protection for, say, one year's service, which was the general feature. If, for example, the employer and the employed accumulated savings regularly, and if, instead of letting the group insurance policies be for one year's salary, the term policies were for the difference between a specified sum, which might be one year's salary, and the savings accumulated by the employer and the employed, that would really be converting the whole thing into endowment assurance, with, perhaps, the investment part of the business done by the employers and the employees more or less on their own account. That could be done by means of War Savings Certificates on better terms than any office, ordinary or industrial, could give.

He believed that group insurance would, in all probability, lead to a very considerable increase in the amount of assurance taken of every kind. If the idea became at all common—and he certainly hoped it might—that no one should carry less than a year's salary as insurance, one would hope that a great many men who had industrial policies for £10 would regard such an amount as totally inadequate, and would want to increase their insurances to £50 or £100. However widely group insurance was extended, there would inevitably be an enormous number of individuals to whom no system of group insurance would be applicable.

SIR ALFRED WATSON said that the paper dealt with one of those superficially attractive schemes of insurance which for many years past they had been accustomed to receive from their cousins across the Atlantic, and which prudence dictated to them should be examined with very great care before they were engrafted upon our system of insurance in this country. He thought possibly that if the scheme were put before the employers here it would meet with a considerably less measure of acceptance than it appeared to have done in the United States, because employers in this country were already under considerable obligations in the matter of insurance, which they could not escape, and for which, as far as he knew, there was no counterpart in America. For example, the employers of 15,000,000 of people here had to pay every week for health insurance, and if one read the signs of the times aright, as gathered from statements at public meetings, and in the press, there was at least some probability that the liability of employers under that head was likely to increase in the near future. Again, four or five millions of people were insured against unemployment, and there again the employer was under the liability of making compulsory contributions; if they were to judge by statements that had been made, more or less officially, that scheme was more likely to be extended than the reverse, especially now that the out-of-work donations had familiarized a large part of the population with the meaning of unemployment benefits. It might, therefore, be that group insurance, if put before employers generally in this country,

would meet with a good deal less appreciation than it seemed to have met with elsewhere. His own feeling was that group insurance only needed examination to be severely criticized. The author made it perfectly clear that such insurance was contingent upon employment, although in some cases arrangements were made whereby people who left employment might continue their insurance at the rates of premium appropriate to their attained age. Let them consider that point in relation to the ordinary working of a superannuation fund. If they looked at the valuation reports they would generally find that the tables exhibited very low mortality rates during active service, a very heavy rate of mortality amongst those who had retired invalided, and a very heavy rate of retirement as the ages between fifty and sixty were reached. That meant that the risk of people falling out of their employment as they advanced in years, on account of inability to stand the constant strain of their work, was very considerable, and that a scheme of life insurance which only covered the active part of a man's life would fall very far short of the real insurance which the working population would think that they were obtaining from the arrangements made by the employer. Looking, again, at the experience of any typical friendly society, they would find that, taking the whole mass of the insured, something like one in seventy-two were in receipt of permanent sickness benefit. In the very nature of the case the great majority of those people must have been struck off the pay-rolls of the firms in whose service they were engaged before their incapacity occurred. A scheme of insurance which said, in effect, that so long as a person was in good health and able to follow his occupation, he was insured against death, but that as soon as the contingency of serious ill-health fell upon him and compelled him to retire from his active occupation, then the insurance ceased, would not, he thought, appeal to the high traditions of life assurance as practised in this country. Take, again, the case of the tuberculous—those unfortunate people who, at the age of thirty or thirty-five, were compelled to relinquish their employment and to go into a sanatorium. As a rule they began their sanatorium treatment some time after they had been compelled to relinquish their employment: the treatment itself extended over a considerable period, and in the great majority of cases he did not suppose that they would be kept upon the employer's pay-roll. He was afraid that in those cases, just when the insurance was most needed, it would fail; and he doubted whether a form of insurance having that result was one which they should encourage. He could imagine the dissatisfaction of the working classes on learning that after years of service in a particular business, followed by the misfortune of ill-health, the insurance had ceased, and that just when it was most needed there was no life assurance protection. He did not overlook the suggestion that a person who left an employment might, under some of the schemes adopted in America, provide for the continuance of his insurance by payment of the ordinary premium appropriate

to his attained age and without any further medical examination. Upon that he would like to say two things. First, as regarded the individual, there was the fact that the great risk of a premature breakdown in health occurred after the age of fifty, when the provision of a policy would be an extremely costly business, especially for a person whose income had ceased by reason of his incapacity. The fact that that option was very little exercised was exactly what one would have anticipated in the circumstances, and really enforced the argument that the option was no real protection. Secondly, in regard to the office which undertook the business, if the option was to be taken advantage of it must represent a very real liability, and it seemed to him almost superfluous to consider what was the appropriate premium on an orthodox mortality basis unless they could in some more or less accurate way gauge the value of the option and protect themselves against its universal exercise. The paper was certainly an interesting one in that it gave them a good deal of food for thought ; but he hoped that, on reflection, it would lead them to think that in this country they were not prepared to adopt the particular form of insurance which it so clearly explained.

MR. C. W. KENCHINGTON felt very considerable doubt whether the adoption of group insurance would benefit industrial policyholders. No doubt insurances under the group system would tend to diminish the expense ratio, but it was not clear to him that there would be any special advantage that would accrue to policyholders whose premiums continued to be paid weekly or at short intervals. The disadvantage of the system was that it did not give permanent insurance, and, although the industrial system might be criticized because in some cases the assured did not find himself in a position to continue the premiums, yet in those cases the lapse was at the free will of the assured and it did not depend upon his falling into ill-health—in fact, they knew that industrial policies were kept up during ill-health. Mention had been made of the option to continue the insurance when a man ceased employment. In that connection he was very much struck by a remark in an American insurance publication, in which the statement was made that it did not matter to any great extent whether the option was included in the policy or not, because even if the option was there it was very unlikely to be exercised.

MR. ERIC B. NATHAN said that in regard to superannuation funds it was generally suggested that even if the employer paid the contribution, the money was really provided by the employees, being of the nature of deferred pay. If that view applied to group insurance they could see the difficulties that might arise if an employer ceased to insure or if the employee left his situation. Such a state of things would, he thought, be regarded by the working classes as a great hardship. Immediately the insurance was effected it would be looked upon as a right, and it would be very difficult to discontinue it ; and it would seem that difficulties of this sort had already arisen in some parts of the United States. He was

rather surprised that Mr. McCormack had not referred to a form of insurance which had been in force in this country for a considerable number of years—certainly before the first group policy was issued in the United States; he referred to a system which obtained in the Co-operative movement, by which on the death of a member his representatives received the amount of his purchases during the previous year.

MR. H. H. AUSTIN deprecated some of the author's remarks with reference to industrial insurance. Great numbers of the working classes did make substantial provision against death by means of ordinary and industrial branch policies. Every facility was given them to make that provision in the ordinary branch by the issue of policies as low as £25; premiums were accepted monthly as well as quarterly and half-yearly, and those facilities were taken advantage of very freely. Although the average industrial policy was only about £10, it was very usual to have two or more policies existing on the same life.

MR. S. G. WARNER, in closing the discussion, said that Mr. McCormack had brought the subject of the paper before them in a way which, for clearness, comprehensiveness and good arrangement, left little room for criticism. He had been happily inspired in his choice of a theme, having regard to the present national position. They were entering on a period in which the relations between Capital and Labour had to be seriously re-considered, and the state of mind thus induced was one suitable for the consideration of experiments in social insurance. Such developments as group insurance could be regarded from two standpoints: as affecting national interests, and as offering attractive paths for private enterprise. The two things might quite possibly be combined, and they might fairly regard the business of life assurance as successfully effecting that combination. The time had come when private enterprise, which did not make adequate contribution to national benefit, would, by that fact, be condemned. By the American companies, which had taken up group insurance, the enterprise was held to be attractive and profitable. Naturally it had been compared with industrial insurance. Industrial insurance was expensive, and could not be otherwise, developed as it had been through the channels of individual appeal and individual collection. Obviously much cost could be saved were it possible to substitute the group for the unit, and place the burden of premium payment upon the employer. Group insurance, in effect, proposed to approach the employer and to insure his employees as a whole and in a group, without medical examination; the substituted safeguards being continuous employment for a specified number of months before assurance and compulsory inclusion in the group of all the employees. This made the group a miniature sample of the whole average population, so that the companies could take the risks one with another and assume that they were getting, on the whole, average lives. Another important point made was that the employer must bear the entire cost. There was, as Mr. Nathan had pointed out,



a danger that this might be considered by the workers an advantage more apparent than real, and that the cost might ultimately fall on the employee. To avoid this, it would be essential that wages and all existing benefits should remain entirely unaffected, so that there could be no possible doubt about the new benefit being an added privilege. Another feature of interest about which not much had been said in the discussion was the grading of rates according to employment hazard. In this respect the scheme, while it touched industrial insurance on one side, might be said to touch workmen's compensation business on the other; and, should it come into general operation, very valuable help would doubtless be found in the experience accumulated by the companies transacting workmen's compensation business. He thought it would be very desirable, if possible, that some common scale of rates for employment risks, based on the experience of the companies, should be agreed upon.

Coming to the important question of how the scheme was likely to suit industrial conditions in this country, it had been pointed out that it would conduce to continuity of service, but that the more it approximated to universal adoption the less this consideration would apply. It was open to question, however, how far such continuity would be an advantage. An able worker might not desire it. Probably the greater advantage would be that the scheme would tend to promote that more friendly and human relation between master and man which, could it be fully attained, would be the true solution of industrial difficulties.

One very important point arose as to what happened when the employment came to an end. The insurance offered was a temporary one, at rates fixed from year to year to cover the risk of each year, the group premium being revised annually on a statistical schedule; thus again linking the methods of the scheme to those of employers' liability business. At its termination the worker was offered a whole-life or endowment assurance of like amount, without medical examination, at the rate for the age then attained. Since this included all who left on grounds of health, it raised a question of some gravity. It has been urged that few of those thus entitled would be able to afford to avail themselves of the option, but that was hardly an argument which it would be worthy or safe to employ. Nor could they base any conclusions upon what might hitherto have happened in this respect, for the whole system was only about seven years old. As time went on what had happened with workmen's compensation would happen here: the workers would become more and more alive to the advantages offered them, and would more freely and generally take them. The origin of the option was the fact that some of the American States would not, without its inclusion, allow the business to be done within their borders. It seemed clear that if the option were offered here it must be paid for, or serious consequences might follow.

One thing more he would say, and it was suggested by the very pregnant and illuminating remarks of Sir Alfred Watson. This scheme came from across the Atlantic and sprang, therefore, from

a national history and national conditions very different from their own. The United States remained, at present, as they had been substantially throughout their history, the home of individualism. Little was done there for the worker by the State. He was highly paid, and left very much to fend for himself. Employment was abundant, and transition from one employer to another easy. In just such circumstances individualism flourished. Under that *régime* great fortunes had been made and a brilliant commercial fabric had been built up. The time would come, as population pressed more on the limits of employment, when some of the difficulties too familiar here might make themselves felt there also ; and already, in fact, there were indications of the change. Here, as they all know, the State had already assumed and was ever more largely assuming a share in looking after the well-being of its people, and that by the pressure of compulsory legislation upon employers of labour. Often unwelcome, regarded with trepidation, entered on with hesitation, it inevitably came and grew. In a country which as yet knew little of these things, it was easy to appeal to employers to take upon themselves new burdens in the interest of the employed, and the movement was beneficent. Some large American companies made it their business to bring before large employers to whom they had granted group insurances, developments of public enterprise in the direction of caring for the health of the people, in sanitation, medical attendance and otherwise. It was done primarily from a commercial motive, but its effect could not fail to be a national benefit. How far the employers of labour in this country, burdened as they already were and were increasingly likely to be, with national responsibilities imposed by the State, might be expected to assume voluntary burdens it was difficult to say. It was to be hoped that if any practical development took place it would be in a form at once attractive to individual enterprise and defensible because a benefit to the people as a whole.

The PRESIDENT, in moving a vote of thanks to Mr. McCormack, said they were greatly indebted to him for his paper and for the interesting discussion which it had brought forth. With reference to Mr. Austin's criticism of the author's observations on industrial offices, he (the President) was partly responsible for those observations, as he mentioned the subject in his Address. He did not believe that Mr. McCormack regarded group insurance as a substitute for industrial assurance, nor certainly did he (the President), either when he first referred to the question or now. He did not regard it as anything more than supplementary to it. He thought, however, that possibly some of the methods which were employed in connection with group insurance might be of value in lessening the expense of industrial assurance. At present that expense was inevitable, but it was going to be a strong point of attack by those who were out to do something which, whatever form it might ultimately take, would not be of benefit to industrial assurance companies as they at present existed.

MR. McCORMACK, in replying to criticisms, said that under



most group insurance schemes employees remained covered during any temporary absence on leave, and sometimes the employer had the option of continuing the insurance in the case of employees who retired or left through ill-health. But this privilege might be conditional upon the continued payment by the employer of at least a part of the salary. This was quite distinct from the option of conversion to an ordinary policy on leaving service; but in spite of these arrangements, he agreed with Sir Alfred Watson and other speakers that serious difficulties remained. It was rather a paradoxical feature of group insurance that not only its advantages but also some of its principal disadvantages were largely the result of the term insurance system, which was the basis usually adopted in practice. He thought the chief difficulty of the scheme from the point of view of the insurance company was to be found in the continued insurance of members who were no longer actively engaged. It therefore seemed very important that when the scale of premiums was fixed careful allowance should be made for the option of conversion to an ordinary policy on leaving service, and it would also be advisable to establish a reserve fund to provide for any losses which might result. With regard to Mr. Schooling's interesting suggestion a difficulty that occurred to him was that in any scheme involving group insurance it was necessary that all the employees in the group should be insured. Therefore the employer would have to insure employees who declined to take part in the scheme on the same footing as those who became members. But as a decreasing short term insurance, such as was suggested in the scheme, would probably be of little value apart from the scheme, the cost of insurance of non-members might represent a practically useless expense to the firm. He had not expressed any definite opinion of group insurance in the paper; but he thought he might safely say that so long as group insurance was regarded merely as an extension of Workmen's Compensation insurance, it might serve a valuable purpose.

*The American-Canadian Mortality Investigation, 1900-1915.\**

[Communicated by MR. ARTHUR HUNTER, A.I.A., F.F.A., Chairman of the Committee on Mortality Investigation.]

SYNOPSIS OF VOL. II.†

IN Volume II of the report on the investigation into the mortality among residents of the United States and of Canada sub-divisions have been made of the statistics according to:

- (a) Plan of insurance;
- (b) Residence in groups of states in the United States:

\* Published by the Actuarial Society of America.

† See p. 259 for synopsis of Vol. I.

(c) Residence in provinces, or groups of provinces in Canada ;

(d) Causes of death ;

(e) Amount of insurance \$50,000 or more.

A synopsis of the results of the investigations is now given.

#### MORTALITY ACCORDING TO PLAN OF INSURANCE.

The mortality according to plan of insurance was confined to men resident in the United States at date of application for insurance, wartime conditions making it not advisable to undertake a similar investigation for men resident in Canada. The former are designated "American Men." The principal plans of insurance investigated were :

1. Ordinary Life, together with Endowment Insurances maturing at age 80 or over.

2. Twenty Payment Life, together with Nineteen Payment Life. and also Nineteen and Twenty Payment Endowment Insurances maturing at age 80 or over.

3. Twenty Year Endowment Insurances, and also Nineteen Year Endowment Insurances, both maturing at ages under 80.

4. Ten Year Term Policies, and Policies changed from Ten Year Term to other plans.

The first three groups largely consisted of policies on the Ordinary Life, Twenty Payment Life and Twenty Year Endowment Insurance plans.

In determining the expected deaths, the graduated rates of mortality for American Men (A.M.), both select and ultimate, were used. (See *J.I.A.*, li, p. 259). The following is a synopsis of the results :

#### *Ordinary Life.*

Duration	ALL AGES AT ENTRY COMBINED		
	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1-5... ..	\$20,505,400	\$19,062,200	107·6%
6 and sub....	288,766,300	283,546,900	101·8
Total ... ..	\$309,271,700	\$302,609,100	102·2%

*19 and 20 Payment Life Plans.*

Duration	ALL AGES AT ENTRY COMBINED		
	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1-5... ..	\$17,310,100	\$18,817,000	92.0%
6 and sub....	74,081,000	78,080,500	94.9
Total ... ..	\$91,391,100	\$96,897,500	94.3%

*19 and 20 year Endowment Insurances.*

1-5... ..	\$5,279,300	\$5,312,500	99.4%
6 and sub....	25,449,000	26,878,600	94.7
Total ... ..	\$30,728,300	32,191,100	95.5%

There is apparently an anomaly in that while the expected deaths for the first five insurance years are 7 per-cent of those for the sixth and subsequent insurance years for Ordinary Life policies, the corresponding percentage for Nineteen and Twenty Payment Life is 24 per-cent. This is due to the fact that the Companies contributed their entire data on plans other than deferred dividend for policies issued prior to 1900, while they contributed only a part of their data for issues of 1900 to 1914, inclusive.

The mortality on the Nineteen and Twenty Payment Life policies is better than on the Ordinary Life plan, and while the mortality on the former is slightly lower than on the Nineteen and Twenty Year Endowment Insurances, this may be due to accidental fluctuations. The experience in the United States with regard to the relative mortality under Limited Payment and Endowment Insurances is approximately the same as shown by the British Offices' Life Tables, 1893. In the case of the American companies, one reason advanced for the difference between these plans is that there has been a tendency to limit slightly impaired or border line risks to Endowment Insurances.

The difference between the relative mortality under the foregoing three principal plans was obtained by a method which took account of the distribution of business at the various ages and in the different insurance years. From this it appears that

the relative mortality is  $12\frac{1}{2}$  per-cent lower on the Limited Payment than on the Ordinary Life plan, the difference being greater in the earlier than in the later insurance years; and that the relative mortality on Endowment Insurance is  $7\frac{1}{2}$  per-cent higher than on Limited Payment Life in the first five insurance years, and practically the same for the sixth and succeeding insurance years.

An analysis is given in the report of the relative mortality by ages at entry for the first five insurance years, and by attained ages for the sixth and succeeding policy years. It is shown that under Ordinary Life policies the percentage of actual to expected deaths calculated by the A.M. Table (Select and Ultimate) was highest in the early policy years for ages at entry 35 to 44, and that under Nineteen and Twenty Payment Life policies the relative mortality was generally better in the earlier than in the later insurance years.

*Term Insurance.*—The mortality under term insurance was divided into two groups. The first was Ten Year Non-Renewable, or Ten Year Renewable Term Insurance exposed during the period of the Term Insurance, but not exceeding ten years. By limiting the period to ten years any selection against the companies at the beginning of the second period under renewable term policies was eliminated. The results were as follows :

Duration	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1-5... ..	\$2,873,200	\$2,878,800	99·8%
6 and sub....	1,878,700	1,769,900	106·1
Total ... ..	\$4,751,900	\$4,648,700	102·2%

For all ages and durations combined the relative mortality happens to be exactly the same as under Ordinary Life policies. This may cause some surprise, because a number of years ago the mortality of several companies was reported to be higher under Term Insurance than under Ordinary Life policies. In recent years, however, many of the companies have exercised a stricter medical selection, and made a more searching inspection on Term than on other forms of insurance.

The second group covered Ten Year Renewable Term policies renewed at the end of the ten years for a further period.

and those converted at that time to Ordinary Life, Limited Payment, or Endowment insurance—all without medical examination at date of renewal or conversion. As the experience begins at the end of ten years, the expected deaths were calculated by the ultimate table of mortality.

	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
All Insurance Years	\$2,036,600	\$1,878,600	108.4%

The mortality is divided according to number of years elapsed after renewal or change. For the first five insurance years following renewal or change the mortality is  $9\frac{1}{2}$  per-cent higher than for the sixth to the tenth years after such renewal or change.

A division of the data into attained ages (*a*) under 60 and (*b*) over 60, showed 100 per-cent of the A.M.<sup>(5)</sup> Table under the former and 131 per-cent under the latter. This was to be expected, because the older the life the higher the cost of renewal or change, and accordingly the greater the incentive to continue the policy on an impaired risk.

*Term Policies Changed to Other Plans.*—An investigation was made of the experience under Ten Year Non-Renewable, or Renewable Term policies which were converted or changed into other forms before the end of the original term period without a new medical examination. As the mortality was likely to be influenced by the number of years elapsed since the original policy was issued, the experience was divided according to the number of years elapsed before change or conversion was made. The expected deaths were calculated by the A.M. Table, making allowance for the time elapsed before change. The following is a synopsis of the results :

	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
Changed during first five calendar years... }	\$482,100	\$573,500	84.5%
Changed during sixth and subsequent calendar years... }	310,500	319,200	97.3
Total ...	\$792,600	\$892,700	88.8%

As the policies were changed to a higher premium plan it was not to be expected that there would be an unfavourable mortality in this class, except, possibly, for those changed in the later years of the Ten Year Non-Renewable Term period. In the case of companies which permitted the conversion within one year of the end of the Non-Renewable Period, a higher mortality than the normal for a few years following the change might be anticipated, but the statistics were not numerous enough to justify such a sub-division.

#### MORTALITY IN DIFFERENT SECTIONS OF THE UNITED STATES.

The data for the United States were divided into ten groups, depending upon the State in which the man resided at the date of application for insurance. These groups took account of the geographical situations and economic conditions of the States. The ten groups were as follows :

1. Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, Vermont.
2. New Jersey,\* New York, Pennsylvania.
3. Delaware, District of Columbia, Kentucky, Maryland.  
North Carolina, Tennessee, Virginia, West Virginia.
4. Illinois, Indiana, Michigan, Missouri, Ohio, Wisconsin.
5. Florida, Georgia, South Carolina.
6. Alabama, Arkansas, Louisiana, Mississippi.
7. Iowa, Kansas, Minnesota, Nebraska, North Dakota  
South Dakota.
8. Oklahoma, Texas.
9. Arizona, Colorado, Idaho, Montana, Nevada, New Mexico,  
Utah, Wyoming.
10. Alaska, California, Oregon, Washington.

The relative mortality in these States for all insurance years and for all ages at entry combined may be seen from the following synopsis, the expected deaths being calculated by the A.M. Table :

Group	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1	59,486,800	62,260,700	95·5%
2	150,749,200	146,499,500	102·9
3	43,133,300	41,815,100	103·2
4	108,017,800	114,032,900	94·7
5	13,703,600	11,073,600	123·8
6	13,181,500	10,626,400	124·0
7	21,533,200	24,289,300	88·7
8	7,143,200	6,557,400	108·9
9	5,952,900	6,005,900	99·1
10	13,241,500	13,185,600	100·4



The lowest relative mortality was in the middle west (7) in the States where the principal occupation is farming. There was very little difference in the mortality between the New England States (1) and the group of Northern Central States (4). The Eastern States, extending from New York to Virginia (2 and 3) showed a slightly higher mortality than the Rocky Mountain States (9), and the Pacific Coast States (10). As was expected, the two groups (5 and 6) of extreme southern States showed the highest mortality—24 per-cent in excess of the average for the whole country.

It is interesting to note that the older settled States and States with large cities, had a lower relative mortality in the earlier than in the later policy years, while the reverse was true in the other States.

#### MORTALITY BY PROVINCES IN CANADA.

An investigation was made by groups of Provinces of the mortality among Canadian men, according to their residence at date of application for insurance. The following divisions were made :

- (A) Ontario.
- (B) Quebec.
- (C) Nova Scotia, New Brunswick and Prince Edward Island.
- (D) Manitoba, Alberta and Saskatchewan ; Yukon Territory.
- (E) British Columbia.

The following shows the comparison of the actual with the expected deaths calculated by the C.M. Table :

Group	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
A	14,435,300	15,284,200	94.4%
B	7,216,800	6,098,800	118.3
C	3,856,900	3,760,200	102.6
D	2,670,700	3,063,600	87.2
E	1,207,200	1,174,900	102.7

The best mortality was found in the wheat-growing States of Western Canada (D). It will be remembered that the best mortality in the United States was in the farming communities of the Middle West. The mortality in the Maritime Provinces (C) and in British Columbia (E) were practically alike. 2½ per-cent higher than the average for the whole of Canada. The Province

of Ontario (A) was  $5\frac{1}{2}$  per-cent below the average. The highest mortality was in the Province of Quebec (B), which was about 30 per-cent higher than for the Western Provinces (D). This is generally ascribed to the higher mortality among the French-Canadian population of the Province of Quebec.

An investigation was made by request of the Canadian actuaries of the mortality among women in the Province of Quebec. The amount of data was small, the actual death losses being \$270,700, and the expected \$241,600—a ratio of 112 per-cent. As the mortality among Canadian women was 10 per-cent lower than among Canadian men, it follows that the mortality among Quebec women was 25 per-cent higher than among all Canadian women.

#### CAUSES OF DEATH.

An investigation was made of the causes of death in the case of both American and Canadian men. The deaths were tabulated in two ways: (a) percentage of deaths from each cause to total deaths; (b) death-rate from each cause per \$10,000 exposed to risk. Nothing unexpected seems to have been developed by these tables. On the basis of the rate per \$10,000 exposed to risk an increasing death-rate with advance in age is exhibited from such diseases as cancer, pneumonia, apoplexy, heart disease and nephritis. The death-rate from typhoid fever decreased until after middle life. It is only possible to make a general comparison of death-rate by causes with those in the Medico-Actuarial Mortality Investigation, and from this it appears that there has been a material reduction in the death-rate from typhoid fever for all policy years, and a marked improvement in the first year's experience in the case of tuberculosis of the lungs.

A comparison of the causes of death among American and Canadian men indicates that there was a markedly higher death-rate among the former than among the latter, from apoplexy, heart disease and nephritis, but in the case of accident the Canadian death rates were the higher.

#### POLICIES FOR LARGE AMOUNTS.

Two investigations were made into the mortality under policies for large amounts: (a) policies for \$50,000 or more, and (b) insurances of \$100,000 or more, taken out in amounts of at least \$50,000 per policy. This investigation was limited to the data on American men. Unfortunately, the amount of data is

small, because a small part only of the statistics of the large companies was included; the larger the company, the smaller was the proportion of data contributed for the years of issue 1900 to 1914 inclusive. The small and young companies contributed all or a large part of their experience, but there were naturally few policies issued by them for amounts of \$50,000 and more.

The expected deaths in the following synopsis were calculated by the A.M. Table :

*Policies for \$50,000 or more.*

Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
\$6,423,700	\$5,502,500	116·7%

The relative mortality for the first five policy years combined was the same as for the sixth and succeeding policy years combined.

On policies with aggregate amounts of \$100,000 or more on individual lives the following was the result :

*Aggregate Amounts of \$100,000 and over.*

Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
\$2,595,500	\$1,978,100	131·2%

The mortality in the first five insurance years combined was lower than in the sixth and succeeding insurance years combined; but too much reliance should not be placed on this result, because of the small amount of material.

From the two foregoing synopses it may be deduced that there was a ratio of actual to expected deaths of 109 per-cent on cases for at least \$50,000, but less than \$100,000.

Information was obtained from three companies with regard to their experience on large amounts of insurance. The actuary of one company stated that, under policies issued from 1880 to 1915 for an aggregate amount on individual lives of \$100,000 or more, issued within a single calendar year, the mortality was about 25 per-cent in excess of the normal mortality of the company on all its insurances. The amount of data in this company's experience was about three times as large as

those investigated in connection with the "American Men" experience.

Another company gave its experience on policies issued for \$50,000 or more from the years 1868 to 1907 inclusive, carried to the anniversaries in 1908. The select table adopted in this experience in determining the expected deaths was probably higher than the general experience of the company, and, accordingly, the ratios now given are too low. On persons who were insured for at least \$50,000 but less than \$100,000 the relative mortality was 107.5 per-cent, and on those insured for \$100,000 and over it was 113.2 per-cent. These ratios are on the basis of "lives"; the corresponding percentages by "amounts" are not materially different.

A third company gave its experience under policies issued in the years 1885 to 1915 inclusive, carried to the policy anniversary in 1916. On policies issued for \$50,000 to \$100,000 inclusive the relative mortality measured by the Medico-Actuarial Mortality Table was 107 per-cent by applications, and 105 per-cent by amounts. For cases about \$100,000 the relative mortality was 110 per-cent by applications, and 148 per-cent by amounts insured.

A recent tendency has been to greatly increase inheritance taxes, and, accordingly, the companies are receiving far more applications than formerly from persons of wealth, who wish to provide for these taxes by means of insurance. Furthermore, it is becoming the practice of companies to insure their principal officers and technical men for the benefit of the company. It is likely, therefore, that there is less "moral hazard" than formerly in connection with applications for large amounts.

It is not the intention to make any further investigations, and the work of the committee is completed, unless the A.M. or the C.M. Tables are likely to become the standards of valuation in the United States or Canada, in which event complete sets of terminal and mean reserves will be prepared and published.

While no additional mortality studies will be made by the committee in charge of the American-Canadian Mortality Investigation, three investigations have been commenced by a Joint Committee of the Actuarial Society and the Medical Directors' Association. That joint committee has already published a report on the mortality due to overweight, and is now engaged in studies of the effect of family history on longevity and the mortality from functional heart murmur and intermittent albuminuria.

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## LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law*.

Voluntary  
assignment of a  
*chose in action*.

ALTHOUGH before the Judicature Act, 1873, the rule was that equity would only grant its aid to compel an assignor to allow the assignee of a legal *chose in action* to sue at law in his name when valuable consideration has been given for the assignment, section 25, sub-section 6, of that Act has not only improved the position of the assignee as regards procedure by enabling him to sue in his own name, but has also enabled him so to sue in regard to a debt which has been assigned voluntarily.

In the case of *In re Westerton, Public Trustee v. Gray*, reported 88 L.J. Ch. 392 (1919), 2 Ch. 104, the validity of a voluntary assignment of a *chose in action* was considered. The facts are as follows :

A testator about a year before his death handed to his landlady an envelope enclosing three documents, saying : " Here is " a present for you ; that is for you. I have given it into your " hand." She was about to open it, when he took it back from her hand, and said : " Remember. I will keep it for you," and then placed it in his despatch box. The envelope contained (a) a letter under the hand of the testator and addressed to the landlady as follows : " You have been very kind to me, and " I desire to make some return by giving you the amount of " £500, now on deposit at the . . . . Bank as per receipt " enclosed " ; (b) a banker's deposit receipt for £500, which he had in 1914 placed on deposit at his bank in his name ; (c) an authority under the hand of the testator to the bank to pay the landlady or her order the sum of £500, then on deposit there. The deposit note was not transferable, and the endorsed form of receipt was not signed by the testator, and no notice of assignment was given to the bank until after his death, in May 1917. when the envelope was taken out of the despatch box and found to contain these three documents.

Sargant, J., held that the letter was a direct assignment to the landlady of the money on deposit ; that upon the delivery of the documents to her there was a completed gift, the subsequent redelivery to the testator being merely for their safe custody on her behalf ; and that by virtue of section 25, sub-section 6, of the Judicature Act, 1873, she could sue for the money



on deposit. although it had been assigned to her voluntarily, and she was therefore entitled to have it paid to her.

The facts in this case should be carefully distinguished from those in the case of *In re Williams, Williams v. Ball*, previously reported in these notes, *J.I.A.*, vol. 1, pp. 118 and 243.

The judgment of Horridge, J., in the case of *In re Stokes, Mellish ex parte*, reported 88 L.J.K.B. 794, deals with a point which frequently arises in office practice, and usefully reviews several previous decisions on the same point, some of which decisions have previously been reported in these notes.

In this case a motion was brought to obtain a decision of the Court as to whether a policy of assurance on the life of the deceased passed under his will to the applicant, or whether it belonged to the Official Receiver as trustee under the following circumstances. On 1 December 1879, the debtor filed his petition for liquidation, and on 13 December 1879, the creditors resolved that the debtor's affairs be liquidated by arrangement, and not in bankruptcy. In January 1880, a trustee was appointed under the liquidation who carried on the business of the debtor for practically the whole period of the liquidation and to enable him to do so employed the debtor as his clerk. On 21 October 1881, the debtor effected a policy on his life for £1,000 at an annual premium of £33 10s. and continued to pay the premiums until his death. On 7 November 1883 the liquidation was closed, and on 13 November 1883 the debtor obtained his discharge. On 26 July 1917 the debtor died at Cape Town, South Africa, having by his will, dated 10 July 1915, appointed his niece, the applicant, his executrix. Letters of administration, as proof of probate of the said will, were granted to the said executrix by the Master of the Court at Cape Town, and were forwarded, together with other documents, by the applicant's solicitors in London to the insurance company; but the company declined to pay the policy moneys to the solicitors on behalf of the applicant until it had been determined whether the same passed under the will of the deceased to the applicant or belonged to the trustee. The trustee stated on the hearing of this motion that he did not know the policy was in existence, and that he knew of no premiums being paid by the debtor, and that the premiums must have been paid out of the debtor's salary. If he had known of the policy he would have disclaimed it.

Policy on life of bankrupt concealed from trustee.  
Conflict of claims to policy moneys.



Horridge, J., said : " This is a case where the policy was effected and the premiums paid from first to last without the knowledge of the trustee or any successive officer representing the trustee. In the case of *In re Tyler, Official Receiver ex parte* (1907) 1 K.B. 865 [*J.I.A.*, vol. xli, p. 411] the Court of Appeal affirmed a decision by Bigham, J., in which he declined to allow the trustee to intervene and take policy moneys without making an allowance to the wife of the debtor who had paid all the premiums ; but in that case the whole matter had been done with the cognizance of the representative of the estate for the time being, although it is true he was not the main representative involved in the case itself. That case entirely proceeded upon the principle that the trustee, having stood by with knowledge and allowed the premiums to be paid by the wife, could not intervene and take the benefit of the policy as an honest and upright man without allowing the wife the amount of the premiums paid by her. It is put quite tersely in the judgment of Farwell, L.J., when he deals with the case of the wife as follows : ' He knew the wife was paying, and he let her go on paying, and said nothing to her to lead her to believe that he was going to claim the policy moneys at the end without repaying the premiums which she had paid.' That was the principle of *In re Tyler*. It appears also in the case of *In re Hall, Official Receiver ex parte* (1907) 1 K.B. 875 (*J.I.A.*, vol. xli, p. 413) in the judgment of Farwell, L.J. He says : ' In the last case (*In re Tyler*) for example, my judgment proceeded on the knowledge of the trustee in the bankruptcy of the existence of the policies, and the necessity for paying the premiums, and the fact that the wife was paying them.' The next case is the case of *Tapster v. Ward* (1909) 101 L.T. 25, 503 (*J.I.A.*, vol. xliv, p. 87), a decision of Eve, J., and the Court of Appeal. Let me compare the facts of that case with the facts of the present case. In that case the policy was effected before the liquidation. The policy was concealed from the trustee, and after the liquidation had been closed, premiums were paid out of his own moneys by the liquidating debtor from the year 1880 to the year 1907—twenty-seven years. Now in this case it is quite true that the policy was effected during the liquidation, and the first premium was paid out of moneys which had been allowed to the debtor by the trustee in respect of his salary. It does not make any difference that the policy in Tapster's case had been effected

“ before the liquidation, and that in this case the first premium  
“ was paid on the policy at a time when the liquidation had  
“ commenced. It is not disputed that every contract made  
“ on and before the liquidation by the liquidating debtor is the  
“ property of the trustee. If that is so, this contract of insurance,  
“ although the liquidation was in fact proceeding, was the con-  
“ tract of the trustee, and if the trustee had known about the  
“ premiums being paid out of the debtor's salary the trustee  
“ could have intervened, and said: ‘I want this policy.’ As  
“ regards the subsequent payments which have been made  
“ from the discharge in 1883 until the day that the assured died  
“ in July 1917, they are exactly the same as the payments for  
“ the policy in *Tapster v. Ward*; they were paid out of the  
“ moneys of the liquidating debtor, and after he had got his  
“ discharge. The Court of Appeal there refused to listen to the  
“ argument that the trustee ought not to intervene and claim  
“ the policy moneys. . . . The last case upon the point is  
“ a decision of my own, *In re Phillips* (1914) 83 L.J., K.B. 1364.  
“ In that case the policy had been effected during the time that  
“ the bankrupt's discharge was suspended under the Bankruptcy  
“ Act, 1883, and subsequent premiums had been paid by him  
“ until he became a bankrupt a second time. The question arose  
“ whether the trustee in the first bankruptcy could claim the  
“ policy moneys without accounting to the trustee in the second  
“ bankruptcy for the premiums which had been paid by the  
“ bankrupt in the interval. I held that, as the trustee in the  
“ first bankruptcy had no notice whatever of the payment of  
“ premiums, he was not bound in the least to make provision  
“ out of the policy moneys. It seems clear law from the case  
“ of *In re Leslie, Leslie v. French* (1883) 52 L.J., Ch. 762, that  
“ a mere volunteer paying premiums has no right to treat them  
“ in any way as salvage remuneration to him for keeping the  
“ policy going. I think this rather a hard case for the lady, but  
“ the law of bankruptcy seems to be perfectly clear. In this  
“ case the policy was effected after the liquidation resolution,  
“ and the policy belonged to the trustee. The bankrupt never  
“ told him that he had effected it; he never told him that he  
“ was paying any premiums. On the authorities I have referred  
“ to I think it is clear that the policy must belong to the Official  
“ Receiver. I declare the policy to be part of the estate and the  
“ Official Receiver entitled to the policy moneys.”

Relief from  
super-tax in  
respect of life  
premium paid  
under contract.

The decision of Mr. Justice Sankey in the case of *Earl Howe v. Commissioners of Inland Revenue* (1918) 2 K.B. 584, reported in these notes, vol. li, p. 135, has now been reversed by the Court of Appeal. The

appeal is reported 88 L.J., K.B. 821—C.A.

The facts are as follows :

The Finance (1909-10) Act, 1910, section 66, sub-section 2, enacts that for purposes of super-tax the total income of an individual shall be estimated in the same manner as the total income is estimated for the purposes of exemption under the Income Tax Acts. The Income Tax Act, 1842, section 163, provides that a person chargeable to income tax shall be exempted if he shall prove, "according to the several rules and directions" of the Act, that his income is less than £150. Section 164 provides that a person claiming exemption under section 163 must deliver a declaration of "every sum of annual interest or other annual "payment reserved or charged" on his income "whereby the "income shall or may be diminished." Section 190, Schedule G, with the rules and directions therein contained, is, in making returns of annual value, to be observed by each person; and Rule xvii, heading 3 thereof, provides that he must make a declaration of the amount of the interest, annuities or other annual payments to be made out of the property assessed. The Finance Act, 1916, section 36, sub-section 1, abolishes the abatements allowed in respect of premiums on life assurance policies for the purpose of super-tax. Earl Howe mortgaged his life interest in certain estates, and also assigned certain policies of assurance on his life by way of mortgage and further charge to an assurance company, covenanting to pay the interest on the sums advanced and the premiums on the policy. In case he should neglect to pay the premiums he empowered the company to pay them and to charge them upon the mortgaged premises. Earl Howe had always paid the interest on the sums advanced and the premiums on the policies, and claimed to be allowed to deduct these premiums, as being annual payments charged on his income from his income for the purposes of super-tax. The Commissioners refused to allow them to be deducted. Sankey, J., contrary to his own opinion, but following the decision of the majority of the Irish Court in *Massy (Lord) v. Inland Revenue Commissioners*, allowed them to be deducted. The Commissioners appealed.

Their Lordships allowed the appeal. They said that the

premiums were not "annual payments" within the Income Tax Act, 1842, sections 164 and 190, Schedule G, Rule xvii, as only annual payments from which income tax could be deducted by the payer were "annual payments" within these sections, and that the premiums could not therefore be deducted.

Are profits or gains on Treasury Bills subject to Income Tax?

A decision on the important question whether a Life Assurance Company, which has not been charged to income tax in respect of its profits or gains under

Case 1 of Schedule D, should be subject to income tax in respect of the profits or gains arising from the sale or maturity of Treasury Bills under the third case of Schedule D, was given in the case of *National Provident Institution v. Brown; Provident Mutual Life Assurance Association v. Ogston*, T.L.R. 35, 690.

This case was an appeal from a decision of the Special Commissioners to the effect that such profits or gains were assessable to income tax, provided this source of income had not ceased before the year of assessment. Both the Institution and the Surveyor appealed against the decision of the Special Commissioners and the case was heard before Mr. Justice Rowlatt, who decided that such profits or gains were assessable to tax, whether or not there had been any similar transaction in the year of assessment. The facts of the case are as follows :

In each of the years ended 5 April 1916 and 5 April 1917 the Institution bought at the Bank of England Treasury Bills. Some of them were held until maturity. Others were sold in the open market during their currency, and the rest, being the whole of the Treasury Bills then held by the Institution, were converted early in 1917 into Five per-cent War Loan, 1929-47, on the terms of the prospectus issued on 11 January 1917. In the year ended 5 April 1918 the Institution did not hold or have any transactions in Treasury Bills.

In each of the years ended 5 April 1917 and 5 April 1918 the Institution received and paid interest, from which income tax was not deducted, on short loans to and from bankers. In the year ended 5 April 1918 the Institution received interest on Five per-cent War Loan, 1929-47, Stock and Bonds, income tax being deducted from the interest on the bonds, but not from the interest on the stock.

The total amount of the difference between the amounts paid and the amounts received by the Institution in respect of the bills held to maturity, and in respect of the other bills which were sold during their currency in the open market in the year ended 5 April 1916 was £5,422 11s. 7*d.* In the year ended 5 April 1917 the Institution bought Treasury Bills some of which they held to maturity and others of which they converted into War Loan. The total difference between the amounts paid by the Institution and received by them in that year was £19,714 8s. 6*d.* There was also an item of £851 19s. 6*d.* the difference between the interest paid and received on short loans to and from bankers. This made a total of £20,566 8s. for the year ended 5 April 1917.

The Institution was not, for any of the years 1916, 1917 and 1918 assessed to income tax under Case 1 of Schedule D on the balance of its profits and gains. For the year ended 5 April 1916 the only direct assessment made upon it under Schedule D of the Income Tax Acts was for income from foreign securities. For the year ended 5 April 1917 the first of the assessments under appeal was made upon the basis of the amount of the differences between the amounts paid and the amounts received on Treasury Bills realized within the preceding year. For the year ended 5 April 1918 the second of the assessments under appeal was made on the basis of the amount of the differences between the amounts paid and the amounts received on Treasury Bills realized, together with the amount of the difference between the interest paid to and the interest received from bankers on short loans within the preceding year. The Institution did not dispute its liability to assessment to income tax for the year ended 5 April 1918 for interest on short loans, on the basis of the net amount received within the preceding year, and the only questions raised by it related to its liability to assessment on the differences between the amounts paid and the amounts received by it for Treasury Bills.

It was contended on behalf of the Institution (*a*) that the differences between the amounts paid and the amounts received for Treasury Bills were an accretion of capital, and not income or annual profits and gains chargeable to income tax, (*b*) that the tax, if assessable at all, was assessable only at payment of the bills on maturity, and on the person holding them at that date, and that the bills sold or converted into War Loan during their currency should be left out of account in computing any liability on the part of the Institution, and (*c*) that in any case



the assessment for the year ended 5 April 1918 could not be maintained, as the Institution did not hold or have any transaction in Treasury Bills in that year.

It was contended on behalf of the Surveyor of Taxes, *inter alia*, (a) that the sums in question, whether received on maturity or on sale or conversion of the bills, were profits on discounts chargeable to income tax under the third case of Schedule D of the Income Tax Acts; (b) that a person was liable to income tax under the third case of Schedule D on the basis of the full amount of the profits or gains arising from the sources comprised in that case within the preceding year, whether any profits arose to him from such sources within the year of assessment or not; (c) that the Institution was in receipt of profits of a description comprised in the third case of Schedule D in each of the years of assessment and was consequently chargeable to income tax under that case in each of those years on the basis of the full amount of the profits or gains from any source comprised in that case within the preceding year.

Mr. Justice Rowlatt, in the course of his judgment, said: "It seems to me in each case that one must look at the real nature of the transaction and see whether the purchase of the future obligation at a discount is really an investment of money at interest or not. Now, in the simple case of the purchase of a Treasury Bill bearing no interest for such a sum as with interest at such and such a rate for the currency of the bill will give the face value, I can feel no real doubt that the transaction is simply one of lending money at interest. If a twelve months' bill for £105 is sold for £100, surely the purchaser simply lends £100 for a year at 5 per-cent. If the face value is £100, and the present value has to be reached by a sum in proportion not giving a round sum in sovereigns as the result the transaction is the same. Unfortunately the problem involved in these appeals is not confined to this simple case. In the first place, these companies have been assessed in the second year, not merely on bills purchased and realized within the preceding year, but on differences between the amount received in that year and the purchase price given before that year. On this principle a twelve months' bill bought in May 1914, and paid in May 1915, would be treated as yielding the whole of the profits and gains represented by the discount in the year ended April 1915. This cannot be right. This is a practical matter, and cannot be treated as adjusted by com-



“pensation over a series of years. In the first place, the rate of  
“tax may change so that it is material to be exact in determining  
“in what year the income is taxable. Furthermore, these  
“Treasury Bills were sold to individuals as well as to corporate  
“bodies, and if one year is to be treated as bearing what is really  
“one and eleven-twelfths of a year’s income, the difference for  
“supertax or abatement purposes may be of great importance.  
“I am not insensible of the difficulty of treating interest accrued  
“but not encashed as profits of the year when the question is  
“not of taxing a trading concern on the profits shown by its  
“balance sheet, but of taxing interest or discounts as such.  
“Still, the step must, as it seems to me, be taken. The difficulty  
“of adjustment as between the years does not lead me to modify  
“my view that the profit in itself is taxable as an annual profit  
“or gain.

“A more formidable complication arises where the Treasury  
“Bill is not bought from the Government or is not held to  
“maturity, but is either bought or sold in the market, or both.  
“I treat the so-called conversion of a Treasury Bill into War  
“Loan as being in substance as it was in form a sale to or a  
“discount with the Government at a price representing the  
“original price plus the proportion of interest or discount  
“accrued. But where a Treasury or any other bill is bought or  
“sold in the market the price depends on the market rate for  
“money. A person who has held the bill for, say, two months  
“may, in a period of pressure affecting himself and the market  
“generally, have to dispose of it at a price involving a loss of  
“the whole or part of, or more than, the interest for the time  
“during which he held it. Conversely, in times of ease, he may  
“sell it at a profit exceeding that interest, consequently, by the  
“time the bill matures, the bill may have passed through the  
“hands of half a dozen persons who have made profits out of  
“it aggregating a larger sum than the difference between the  
“issue price and the face value, such extra profits being, of  
“course, exactly equalled by losses made by other holders.  
“How is this situation to be dealt with? Is the difference  
“between the issue price and the face value to be treated as the  
“taxable income on the bill and to be all assessed on the holder  
“at maturity, or to be divided among the successive holders  
“proportionately to their periods of holding? In neither case  
“would the solution correspond with the truth. They have not  
“received such profits. In my judgment, the matter must be

“dealt with (subject to an adjustment between the years as  
 “already indicated) in the way the Special Commissioners have  
 “dealt with it—that is to say, the difference between the amount  
 “paid on purchase and that received on realization must be  
 “treated as a profit on a discount within the second rule of the  
 “third case. If any holder has made a loss he drops out to the  
 “extent of the loss, and the aggregate of the differences received  
 “will bear tax to the Revenue, even though that aggregate  
 “exceeds the difference between the original purchase or issue  
 “price of the bill and its face value.

“This disposes of the appeals by the companies subject only  
 “to two observations. Some of the bills in the second year in  
 “the case of one of the companies were French Treasury Bills of  
 “a currency, as I gather, of twelve months or less as in the case  
 “of the British bills. Nothing was said in the course of the  
 “argument to suggest that there was any distinction between  
 “French and British Bills for this purpose, and I treat them,  
 “therefore, as covered by my decision. The other point is this.  
 “In the case of the same company and in the second year some  
 “of the documents were not Treasury Bills, but War Expenditure  
 “Certificates. The case gives me no information as to the nature  
 “of these documents, nor do I remember to have been informed  
 “of it at the Bar. I must assumed that they are on the same  
 “footing for this purpose as Treasury Bills. The appeals by the  
 “companies are therefore dismissed.”

As regards the appeal of the Surveyor of Taxes against the  
 Commissioners' decision relating to the assessment for the year  
 ended 5 April 1918, the judge ruled that “in the case of profits  
 “from discounts there is no existing source to be looked for in  
 “the year of assessment, in order to support the tax, and the  
 “appeal of the Crown must therefore be allowed.”

## ACTUARIAL NOTES.

*Graphic Method of obtaining the Yield on a Redeemable  
 Security.* By O. F. DIVER, M.A., F.I.A.

LET OX, OY be rectangular axes (Fig 1), and let AB be a  
 graph of the function  $a_{\overline{n}|i}^{-1}$  for argument  $i$ , the latter being  
 measured from O along OX. Then, if P be a point on the

graph corresponding to a particular value  $i$  of the argument, and PN be drawn perpendicular to the base-line OX,

$$PN = a_n^{-1} \text{ for the value } i = ON.$$

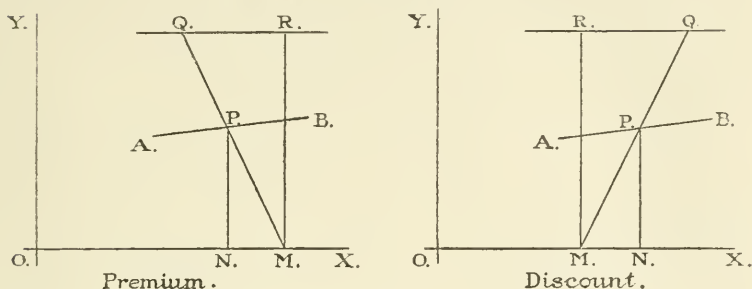


Fig 1.

On OX take another point M, where  $OM = g$ . Join MP, and produce it to meet at Q the line QR parallel to OX at a distance C from it. Draw MR parallel to OY meeting QR in R. Then  $MR = C$ .

By similar triangles MRQ and PNM,

$$QR : MR = MN : PN,$$

or 
$$QR = C(g - i)a_n.$$

Now, if we have a security redeemable at  $C$   $n$  years hence, and paying an annual dividend  $gC$  in the meantime, this equation shows that, when the security is valued at the rate  $i$ , QR is the premium over the redemption price  $C$ , or the discount below it, according as  $g >$  or  $< i$ , or as Q lies to the left or right of R.

Inversely, if QR is the premium or discount, and MR the redemption price, the line MQ will cut AB at the point P, whose abscissa  $ON = i$ , the yield on the security. Thus, if we draw a series of graphs of  $a_n^{-1}$  on squared paper for various values of  $n$ , we can obtain in this way the yield on any security redeemable at a fixed price in any given number of years, and paying any given rate of dividend. If  $a_n^{(2)}$  is used instead of  $a_n$  for the graphs, the method will be applicable to the almost universal case of half-yearly dividends, and the yield will be given in the form of a nominal annual rate convertible half-yearly.

The method also lends itself to the calculation of net rates of yield, after allowing for income tax, as it is equally easy to use with gross or with net rates of dividend.

There is more than one practical method of finding the point P without actually drawing the line QM on the diagram. As it will be found convenient in practice not to measure PN or MR on the same scale as ON, or as each other, let us suppose that on the scale on which ON is measured,

$$PN = l/a_{\overline{n}|i}^{(2)} \text{ and } MR = mC$$

where  $l$  and  $m$  are constants.

Then, if  $k$  is the premium (or discount, if negative), the equation shows that  $QR = \frac{m}{l}k$ . *E.g.*, if  $l=0.1$ , and  $m=0.01$ , then  $QR = k \div 10$ .

Now, if we take first for simplicity the case of a security redeemable at par, QR will be a horizontal line drawn on the diagram at distance  $m$  from OX. If a scale of rates of interest be written in along this horizontal line, corresponding to that along the base-line OX, Q lies at the point  $g - \frac{m}{l}k$  on this scale. A ruler or straight-edged card may now be placed, so as to pass through the point  $g$  on the base-line, and the point  $g - \frac{m}{l}k$  on the horizontal line. It will then cut the graph corresponding to the outstanding term  $n$ , at the point whose abscissa is  $i$ . The yield may then be read off by carrying the eye to one of the interest scales.

If the security stands at a discount,  $k$  is negative, and  $g - \frac{m}{l}k$  is greater than  $g$ , so that Q will lie to the right of R.

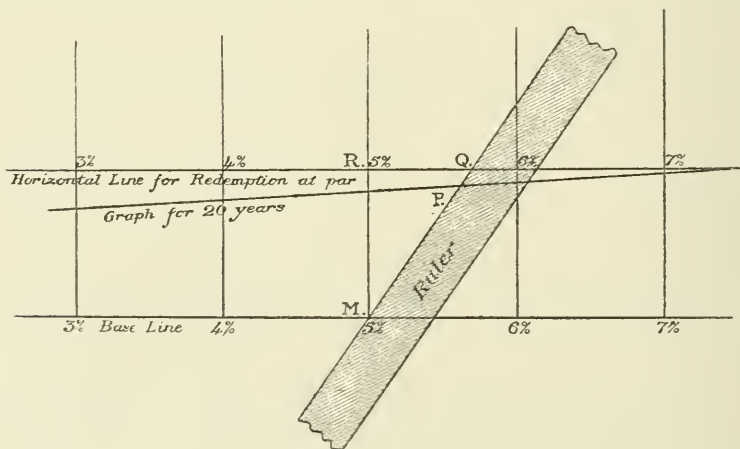


Fig 2.

Fig. 2 gives an example of the case of a 5 per-cent Bond, redeemable at par, and now standing at a discount of 7 per-cent, just after an interest date. In this figure the values taken above for  $l$  and  $m$  are used, so that the ruler has to pass through the point 5.7 per-cent on the horizontal line.

This is the most obvious method, and the modifications necessary in the case of a security not redeemable at par will readily occur to anyone sufficiently interested to try the method practically.

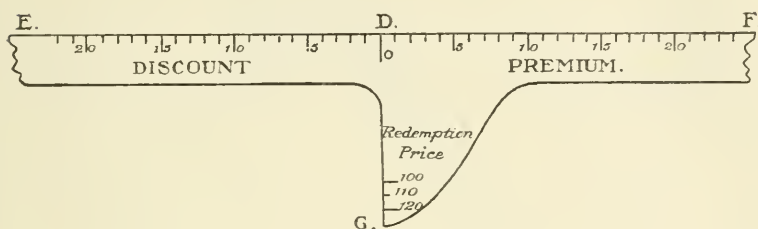


Fig 3.

The writer has found by practical experience that the following alternative method of placing the straight-edge is somewhat quicker, especially if the security is not redeemable at par. For this we require a specially shaped ruler (see Fig. 3), with a straight edge EDF, and a transverse arm DG, the edges EF and DG being at right-angles to one another. Along DE a scale of discount is marked off, along DF one of premiums, and along DG one of redemption prices, all scales being measured from zero at D, the middle point of the edge EF. In this method no horizontal line is required on the diagram, except the base line.

Fig. 4 shows how this method works in practice for the same example as before, namely, a 5 per-cent Bond redeemable at par, and now standing at a discount of 7 per-cent just after an interest date.

The ruler is placed on the diagram, so that the straight edge EF cuts the base line at 5 per-cent on the interest scale as before, but so that the base line also cuts the straight edge at the 7 on the discount scale. The ruler is manipulated till the base line also cuts the scale of redemption prices (DG) at 100 (T). The straight edge now cuts the graph at the required point.





be the point on the scale DG corresponding to the redemption price C, then from the similar triangles TDM and PNM

$$DM : DT = MN : PN.$$

If we suppose, as before, that  $PN = l/a_n^{(2)}$ , and  $DT = mC$ ,

$$\text{then} \quad DM = \frac{m}{l} \times C(g-i)a_n^{(2)} = \frac{m}{l} \times k,$$

so that DM is proportional to the premium or discount.

It will be found convenient in practice to take the horizontal and vertical scales of the diagram in the ratio 10 : 1, in order to reduce it to manageable dimensions. In other words we take  $l = 0.1$ . In this case the scale used on the ruler for discount or premium must also be ten times that used for the redemption price, but there is no need for them to bear any particular ratio to the scales used for the diagram.

The graphs are hardly distinguishable from straight lines for practical values of  $i$ , and after points have been plotted for multiples of 1 per-cent up to about 20 years, and perhaps multiples of  $\frac{1}{2}$  per-cent thereafter, they may be joined with a ruler.

It is unnecessary to include the origin in the diagram, and the graphs may be drawn between two convenient limiting rates of interest. In these days gross rates of yield may be very high, while net rates of dividend, after deducting income tax, may be very low, so that a considerable range is required.

Graphs may be drawn for every half-year for the lower values of  $n$ , and at progressively increasing intervals thereafter, for the curves get closer together as  $n$  increases. If no graph has been drawn for the particular value of  $n$  required, a rough interpolation can be made by the eye. For small values of  $n$  the graphs are wide apart, and it may be found necessary to omit the lowest values, *e.g.*,  $\frac{1}{2}$  and 1 year.

On a diagram drawn to the scales

$$\frac{1}{4} \text{ inch} = \cdot 0005 \text{ horizontally}$$

$$\text{and} \quad = \cdot 005 \text{ vertically,}$$

which the writer has used constantly for many years, the yield can with care be read off with an error that seldom exceeds one penny per-cent. Used rapidly it can probably be trusted to the nearest multiple of 3*d.* per-cent, which is amply

sufficient for ordinary purposes. A smaller scale, in fact, might well be used.

A good scale for use with paper divided into tenths of an inch would be :—

for the diagram :—  $\frac{1}{10}$  inch = .0005 horizontally  
and = .005 vertically.

for the ruler :—10 premium or discount, or 100 redemption price = 2 inches.

A diagram for rates of interest ranging from 2 to 10 per-cent, and for terms of  $1\frac{1}{2}$  years and upwards would then measure 16 inches square, and the ruler could be made to read up to 40 premium or discount without being inconveniently long.

The advantages of the method, when great accuracy is not required, are the rapidity with which it can be used, the absence of tedious computations when a large number of results are required, and its equal applicability to gross and net rates, and to securities redeemable at par or otherwise.

### *The Valuation of Victory Bonds held to pay Estate Duty.*

THE Victory Loan is repayable at par, with interest at 2 per-cent half-yearly, by means of a half-yearly annuity of  $2\frac{1}{4}$  per-cent on the original amount of the loan. The sum available each year for repayment of principal is to be applied in redemption of bonds by annual drawings at the end of the year.

If the drawings were half-yearly, the term of the loan would be almost exactly  $55\frac{1}{2}$  years. Under the actual conditions the operation of the sinking fund will depend to a small extent upon how much interest is earned in the second half of the year on the balance of the first half-year's annuity after payment of interest. It is impossible therefore to state precisely the amounts that will be drawn for repayment each year. For practical purposes it may be sufficiently accurate to assume that the amounts outstanding at the ends of 1, 2, 3, &c., years, will be in the proportions  $a_{\overline{55}|}$ ,  $a_{\overline{54}|}$ ,  $a_{\overline{53}|}$ , &c., calculated at 4 per-cent.

On that assumption the present value—say,  $a_{xVB}$ —at rate  $i$

of an annuity on the joint existence of a person aged  $x$  and a bond will be at date of issue

$$\{vp_x a_{\overline{55}}^4 + v^2 {}_2p_x a_{\overline{54}}^4 + \dots + v^{55} {}_{55}p_x a_{\overline{1}}^4\} / a_{\overline{56}}^4$$

or 
$$\{a_{x\overline{55}} - 1.04^{-56} a'_{x\overline{55}}\} / \{1 - 1.04^{-56}\} *$$

and after  $t$  years

$$\{vp_x a_{\overline{55-t}}^4 + \dots + v^{55-t} {}_{55-t}p_x a_{\overline{1}}^4\} / a_{\overline{56-t}}^4$$

or 
$$\{a_{x\overline{55-t}} - 1.04^{-(56-t)} a'_{x\overline{55-t}}\} / \{1 - 1.04^{-(56-t)}\}$$

where  $a'$  is at the rate corresponding to  $v' = 1.04v$ .

If  $i = .04$ ,  $a'$  becomes  $e$ . Also in many cases the temporary annuity becomes a whole-life annuity.

The following are a few examples of the value of  $a_{xVB}$  at date of issue :

$x$	4%	5%	6%
50	11.694	10.666	...
55	10.440	9.607	...
60	9.071	8.433	...
65	7.639	7.170	...
70	6.207	5.882	5.589
75	4.847	4.634	4.441
80	3.621	3.492	3.371
85	2.574	2.501	2.431

On the assumption that the bondholder will be liable throughout for income-tax at 6s. in the £, the net half-yearly interest on the bonds is 1.4 per-cent, of which the yearly equivalent is 2.828 at 4 per-cent, 2.835 at 5 per-cent, or 2.842 at 6 per-cent.

Hence, if it be assumed that a bond will be applied to pay estate duty (if not previously redeemed) at the end of the year of death, its net 4 per-cent value at date of issue to a subscriber aged 50 is  $2.828a_{50.VB} + 102.828A_{50.VB}$  at 4 per-cent,

\* Mr. Lidstone, who independently investigated the problem on similar lines, points out that the result may be put into the following form, where  $j$  = the rate of interest on the bond and  $z$  the initial annual rate of the sinking fund:—

$$a - \frac{z}{j}(a' - a) \text{ or } a(1 + \frac{z}{j}) - \frac{z}{j}a'.$$

In the present case  $j = .04$ ;  $z = .005$ , and the result at the outset is  $1.125a - .125a'$ .—Eps. *J.I.A.*

which =85·7. Similarly the net 6 per-cent value at date of issue to a subscriber aged 75 is 83·8.

Five years after the date of issue it will be found that the net 4 per-cent value to a holder then aged 55 will be 87·3.

The following examples of approximate net yields at date of issue—on the basis of 6s. tax—may be of interest :

Age	Net Yield	Age	Net Yield
50	4·07	65	4·73
55	4·23	70	5·14
60	4·45	75	5·72

## REVIEWS.

*Pensions for Hospital Officers and Staffs : Report of a Sub-Committee of the Executive Committee of King Edward's Hospital Fund for London.*

[C. & E. Layton and Geo. Barber. 1919. 7s. 6d.]

IN April, 1914, the Executive Committee of King Edward's Hospital Fund for London appointed a Special Committee, consisting of Mr. W. J. H. Whittall, Sir William J. Collins, and Mr. Henry L. Hopkinson, to enquire into the question of pensions for Hospital Officers, and the result of their investigation has now been published in a Report which contains much that is of considerable interest from an actuarial point of view.

The first part of the Report deals with the existing provisions for pensions at London voluntary hospitals to which it is hardly necessary to refer. The second part gives a valuable account of the various ways in which pension funds can be arranged, with a number of examples showing how these plans have worked in practice. This part of the Report is largely historical, but we do not remember to have seen a clearer statement in general terms of the various schemes that have been adopted in one form or another ; it affords a most helpful introduction to anyone who may wish to make a study of the subject.

With the parts of the Report mentioned above all the members of the Committee were in agreement ; but when it came to the choice of a method suitable to the particular case, Mr. Whittall and Mr. Hopkinson recommended that pensions should be obtained through insurance companies, while Sir William Collins preferred a Pension Scheme arranged on a system by which a fund was to be set up for the officers and staffs concerned.

The problem which the members of the Committee had to face

was by no means an easy one, because hospital officers are employed by various authorities under different management, and it was advisable to make arrangements to facilitate transfer from one body to another without loss of pension rights and to make this arrangement so that while the pension is adequately guaranteed either by an insurance office or by some central fund, there was no interference with the employer's authority. But apart from this special difficulty, pension schemes of any kind are sufficiently difficult. If the pension be fixed it can be provided by regular contribution until the pension age is reached ; or it can be left for consideration when it actually has to be paid ; but even although it is comparatively easy to be convinced that the former is the better plan, we are always faced with the difficulty of providing for the pensions that have accrued in respect of past service, and even if we have surmounted this difficulty there is the doubt whether the assumed scale of salary will not be upset by a sudden alteration in salaries, such as has arisen recently owing to increased remuneration as the result of the greater cost of living consequent upon the war. There is also the difficulty in the particular case that the scales of salary by different employing bodies may differ considerably, and it needs but little imagination to see the many difficulties that would arise subsequently if a fund had been set up on such a basis. In the circumstances, therefore, probably every actuary would have agreed that one or other of the courses proposed by the members of the Committee should be adopted.

Perhaps in some respects the simplest plan is to arrange for the pensions through Insurance Offices ; but there are many objections, and although they are in no way shirked in the Majority Report, it is well that they should be explained. The first obvious difficulty is with regard to a choice of the kind of insurance that should be effected, and apparently the scheme which was particularly in the minds of Mr. Whittall and Mr. Hopkinson was that of the Federated Superannuation System for Universities. This system cannot be described as simple. It is arranged with a number of offices and by various kinds of policies ; it necessitates a number of increment policies from time to time as salaries are altered, and annuity rates are guaranteed in respect of endowment assurances when the policies mature ; so that the scheme, while pleasing in its variety, leaves the same feeling of bewilderment as results from attempting to decide upon a suitable form of insurance from an overcrowded prospectus of a Life Insurance Company. We must confess that we have very little sympathy with such complications and there is no need to link them with the purchase of pensions from insurance companies. As the object of a pension scheme is to provide pensions, and as a member of a staff can provide his own insurance through the usual channel the simplest arrangement is to provide pensions by means of deferred annuities, either with or without return of contributions in the event of death. Even such an arrangement, however, leaves out of account the difficulty of making proper provision for pensions in the event of early disablement. If, for instance, an



officer breaks down at age 40, some form of pension has to be provided, and no insurance scheme that has yet been devised will help. Mr. F. L. Collins, who gave some interesting information to the Committee, showed how by "nursing" insurance policies it would be possible for a central fund to make provision in this respect. But if a central fund has to be set up for such a purpose, one is naturally turned in the direction of Sir William Collins' dissentient opinion, that it would be best to use a central fund only and not go to the insurance companies at all.

We can now turn to what is, from the actuarial point of view, perhaps the most interesting part of the work. This is Mr. T. Tinner's suggestion of a "money purchase scheme", which is to provide pensions not only at the retiring age, but also on previous invalidity. The underlying theory is that a sum of £1 paid at any age will provide a pension of £*p* a year at the pension age or earlier retirement, and it shows how the pensions compare with those that are reached by the ordinary salary scale method. Broadly speaking, he found that the retiring allowance on early invalidity was somewhat greater than would be given under the ordinary salary scale arrangement of the old civil service scheme (a sixtieth of final salary for each year of service, with a maximum of 40-sixtieths). He also found that those who came in at a very early age and continued to the latest age at which they can be pensioned ought to receive a larger pension than that paid under the salary scale system. The advantages of such schemes are that they avoid all the difficulties connected with salary scales, and are therefore far less likely to lead into insolvency; they get round all the difficulties that result from a number of employing bodies, and although it is not easy to set out the particulars in the form which conveys quickly the pension to which a person would be entitled, it would be easy to convince any contributor that each payment purchases a reasonable amount of pension. The objections are that people are not accustomed to "money-purchase systems", and prefer pensions to be fixed in amount or fixed as a proportion of salary. Probably similar systems have been suggested by other actuaries, and in the simple form without the invalidity benefit it has of course been adopted in many cases; but we feel that considerable credit is due to Mr. Tinner for the way he has set out the scheme, and it is to be hoped that at some future date he may give a somewhat fuller discussion of his data and methods.

Although there is a Main Report and a Dissentient Memorandum, we are inclined to think that there is not a grave difference of opinion between the various members of the Committee. If we read the Report correctly, Mr. Whittall and Mr. Hopkinson reject the mutual principle largely because they feel that there is an absence of financial guarantee which the hospitals themselves cannot give and which King Edward's Hospital Fund may be either unwilling or unable to provide. The reader will find both sets of views so well expressed that his sympathies will in turn be with everyone concerned.

The whole Report is well worth reading. It has the merit of



being well printed and well set out, and the difference of opinion between the members of the Committee, although it was no doubt not what anyone of them would have wished, adds—it must be confessed—a little zest to the reading.

W. P. E.

*Actuarial Studies: No. 1, Sources and Characteristics of the Principal Mortality Tables. Pp. 79. No. 4, Graduation of Mortality and other Tables. Pp. 82.*

[Actuarial Society of America: 346, Broadway, New York. 1919.]

THE publication of this series of essays is an event of considerable importance in the history of actuarial education, and its progress will be followed with much interest by all English-reading actuaries. Being written by American actuaries primarily for American students, the essays will no doubt present certain differences, in point of view and treatment, from any similar series which might be compiled in the United Kingdom—differences which will probably be more marked in the essays dealing with the practical application of actuarial principles to life assurance business (with valuations and the distribution of surplus, for example) and to social insurance than in those of a more theoretical character—but they must necessarily cover a good part of the ground that would be covered by a Text-Book, Part III.

In No. 1 of the series Mr. Henry Moir and his associated contributors have written as interesting an essay as could reasonably be expected on a rather dull subject. The short accounts of Dr. Sprague's Select Tables (in which four original features are specially commended, namely, the use of a common radix, the junction with the  $H^{(M)}$  Table, the introduction of osculatory interpolation, and the development of a scheme of notation) and of the British Offices' Tables are good and useful. But the sections to which students in this country will turn with most interest are those dealing with the various American Tables. It is rather remarkable, in view of all the work that American actuaries have done on mortality statistics, that there should have been no U.S. population tables worth mentioning until the 1910 Table, and no select tables until the M.A. In connection with the latter table the statement that being based on policies "it should not be applied to the solution of financial problems since mortality rates are higher when based upon amounts insured" seems to us obscure; there are many financial problems to which a table based on policies is at least as applicable as one based on amounts. Another incidental comment that appears open to question is the criticism that in the British Offices Experience the inclusion of the "Old Assurances" had the effect of introducing an increased number of non-select and aged lives, and consequently of making the aggregate mortality curve "unduly steep" after middle life: the true ground of objection (if any) to the inclusion of the "Old Assurances" would seem to be

rather that they belonged to an earlier generation, and that the Experience as a whole would have been more homogeneous if it had been restricted to the "New Assurances."

No. 4, to which Mr. Robert Henderson is the principal contributor, deals with a subject of more academic interest and admitting of more originality of treatment. It is perhaps somewhat too condensed for a text-book, but it will undoubtedly be very helpful to the best students—whether before or after their examinations—and to teachers. The general method of treatment is as follows: After a short introduction on the reasons for graduation, &c., the four classes of methods—graphic, interpolation, summation, and mathematical—are successively discussed, each being applied incidentally for purposes of illustration to the graduation of a limited and very irregular experience, namely, the  $O^{F(50)}$ , and in conclusion the four graduations are compared in respect of smoothness and agreement with the data. The graduations and comparison should of course be understood by the student to be entirely of the nature of an example. Clearly no method of graduation can discover the true law of such scanty data, and in the absence of *a priori* evidence in favour of some specified law such as Makeham's it would probably be best in practice to graduate with reference to some similar data of greater extent.

In the application of the interpolation method to the illustrative experience (p. 22), a 3rd-difference formula including five values of  $w$  (*i.e.*, 25-terms of the original series)\* is used for determining the pivots, instead of a strict interpolation formula. The precise nature of the course adopted might have been explained with advantage, since it involves a material departure from Mr. King's method as defined on p. 7, and as described on pp. 18-21, and also because it introduces the important principle of the reduction of mean square error. American students will probably be familiar with Mr. Henderson's "Mortality Laws and Statistics" (to which, however, no references are given), but by other readers the point of

\* Let  $u_n = pw_0 + qw_{\pm 5} + rw_{\pm 10}$ , where  $w_x = u_{x-2} + \dots + u_{x+2}$ .

(1) Assuming 5th differences of  $u$  to be constant we obtain Mr. King's interpolation formula *iiia* (*J.I.A.*, vol. xliii, p. 114).

(2) Assuming 3rd differences to be constant we can determine  $p, q, r$ , so as to satisfy one additional condition.

(a) Making the sum of the squares of the coefficients a minimum, we obtain the "best" value of  $u$ , namely,

$$(696w_0 + 488w_{\pm 5} - 136w_{\pm 10})/7000.$$

$R^2$  (the reduction of mean square error) = .1018.

The minimum value of  $R^2$  for a range of 25 terms is .090 (*J.I.A.*, vol. xlviii, p. 407).

(b) Putting  $p = q$ , we obtain the formula used by Mr. Henderson  
 $(51w_0 + 51w_{\pm 5} - 14w_{\pm 10})/625$

$$R^2 = .1050.$$

(c) Putting  $r = 0$ , we obtain Mr. King's interpolation formula *Va*

$$R^2 = .2340.$$

the formula may well be missed. The formula is unquestionably a good formula of its kind, but if a combination of the adjusted-average and interpolation methods is to be employed there would seem to be no reason why the best adjusted-average formula should not be used, nor why the restriction of an initial summation in fives should not be removed when the figures for individual ages are available. Possibly this may be contemplated in the brief further reference to the subject on page 76.

Summation-formulas are dealt with effectively by means of a powerful scheme of symbolical operators (in a future edition the use of  $E$  to denote both an error and an operator might be avoided, especially as it occurs in both senses in the same paragraph). The general term is given for the 3rd difference formula of specified range and maximum smoothing power, and the reader is referred to Mr. Henderson's T.A.S.A. paper on Graduation by Adjusted Average. The general proposition is not, however, easy—nor of any importance to the great majority of students—and it would perhaps have been more instructive to give a simple example of finding the formula with maximum smoothing-power for a given combination of operators, *e.g.*, three 5's, and a 5-term operand. We should like to have seen Dr. Sheppard's tables of maximum reductions of error and minimum smoothing-coefficients included. These tables form convenient standards by which summation formulas may be tested.

In the sample Makeham graduation with an arbitrary value of  $c$ , a trial-and-error method is employed for determining the other constants so as to reproduce exactly the total number of deaths and the first moment. The method is simple in application, but the theoretical explanation is rendered somewhat obscure by the introduction (apparently for neatness and convenience) of an intermediary constant which is not absolutely necessary and by a misprint of  $a$  for  $m$  in the first line of p. 62.

We have selected for comment the few points that suggest criticism rather than the many that might justly claim commendation, but we may close with a word of appreciation of the comparison of the four sample graduations. It is a comparison of graduations rather than methods since all four graduations (even those that are sometimes called "mechanical") are markedly individual—they form indeed an object-lesson showing that any method of graduation leaves something to the personal judgment and skill of the graduator—but as an example of the way in which the tests of smoothness and agreement are applied it is most instructive. One other feature which must be specially commended is the inclusion of a useful (although not complete) list of authorities and papers on the subject of graduation.

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## EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE

## (PART I.—SECTION A).

*First Paper.*

1. A sets out to walk from London to Brighton at the same moment that B sets out to walk from Brighton to London. Each walks at a uniform pace. The time that A takes to reach Brighton from the point where they meet is 21 per-cent longer than the time that B takes to reach London from the same point. How much per-cent longer is A over the whole journey than B?

2. Find the condition that the roots of a quadratic equation may be real.

Show, without the use of the differential calculus, that the value of the expression  $\frac{x^2 - 14x + 5}{(x + 3)^2}$  can never be less than  $-\frac{11}{14}$ , when  $x$  is real.

3. Solve the equations

$$(1) \ x^4 - 5x^3 + 8x^2 - 5x + 1 = 0$$

$$(2) \ x^2 + 3x + 2 \sqrt{3x^2 + 9x + 7} = 0$$

$$(3) \ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz} = x + y + z = 7$$

4. Find the values of A which will make the expression

$$8x^2 + 3y^2 - 15z^2 + 10xy + Ayx - 14xz$$

represent the product of two rational factors.

5. If you were required to find the sum of the following series, how would you determine by inspection in each case whether to attempt a solution in the first instance by

- (1) regarding the series as derived from the binomial series  
 or (2) " " " exponential "  
 or (3) " " " logarithmic "  
 or (4) employing the method of differences?

$$(a) \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \frac{1}{7 \cdot 8 \cdot 9} + \dots \text{ to } \infty$$

$$(b) \frac{1}{4 \cdot 6} + \frac{1 \cdot 3}{4 \cdot 6 \cdot 8} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8 \cdot 10} + \dots \text{ to } \infty$$

$$(c) \frac{3}{7} + \frac{3 \cdot 5}{7 \cdot 9} + \frac{3 \cdot 5 \cdot 7}{7 \cdot 9 \cdot 11} + \dots \text{ to } \infty$$

$$(d) \frac{1^2}{3 \cdot 4} + \frac{1^2 + 2^2}{3 \cdot 4 \cdot 5} + \frac{1^2 + 2^2 + 3^2}{3 \cdot 4 \cdot 5 \cdot 6} + \dots \text{ to } \infty$$

The reasons for your decision should be given, but no calculations are required.

6. Show that for certain values of  $x$

$$\log(1+x) - x = \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

and state, without proof, for what values of  $x$  this identity holds.

Given  $\log_{10} e = .43429$  calculate  $\log_{10} 85$  correct to four places.

7. If a number of five figures containing any five of the ten digits once only is written down at random, what is the probability that it is divisible by 9?

8. The 26 letters of the alphabet are placed in a bag. A and B alternately draw a letter from the bag, the letters drawn not being replaced. The winner is the one who draws most vowels. A starts and draws a vowel with his first draw. What is his chance of winning?

9. A and B play a set of games, to be won by the player who first wins four games, with the condition that if they each win three they are to play the best of three to decide the set. A's chance of winning a single game is to B's as 2 to 1. Find their respective chances of winning the set.

### *Second Paper.*

$$1. u_1 = 1; u_2 + u_3 = 5.41;$$

$$u_4 + u_5 + u_6 = 18.47;$$

$$u_7 + u_8 + u_9 + u_{10} + u_{11} + u_{12} = 90.36.$$

Find the value of  $u_x$  for all values of  $x$  from 1 to 12 inclusive

2. The following values of a certain function  $u$  are supplied to you. Which, if any, of these values would you consider affected by a misprint, and how would you, with the material in the question, supply a more correct value?

$t$	$u_t$
1	21.4
2	14.9
3	11.4
4	7.8
5	5.8

3. Given  $u_0 = 58.842$

$$u_2 = 55.257$$

$$u_4 = 51.368$$

$$u_{10} = 37.977$$

complete the series  $u_0, u_1, u_2, \dots, u_{10}$ .

4. Prove that  $\frac{du_x}{dx} = \frac{u_{x+m} - u_{x-m}}{2m}$  approximately.

Give a geometrical interpretation of this approximation.

5. Find from first principles the differential coefficient of  $x^n$ .

Find the differential coefficients with regard to  $x$  of

$$\log x^{\frac{1}{x}}$$

$$\frac{\sqrt{x^2 - 3}}{(x-1)\sqrt{x^2 - 7}}$$

and with regard to  $x^2$  of

$$\frac{\sqrt{x^2 - 2}}{x}$$

6. State and prove Leibnitz' theorem as to the  $n$ th differential coefficient of the product of two functions.

Prove that

$$\frac{\log(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} = x - \frac{2}{3}x^3 + \frac{2}{3 \cdot 5}x^5 - \dots$$

7. A man in a boat at sea, 5 miles distant from the nearest point of a straight shore, wishes to reach a place 12 miles distant along the shore, measuring from this nearest point.

At what point should he land to reach this place in the minimum time, if he can row at 3 miles an hour and walk at 4 miles an hour?



8. Evaluate (i)  $\int e^{x^3} x^3 \cdot dx$

$$(ii) \int_1^2 \frac{dx}{x^2 \sqrt{2+x^2}}$$

$$(iii) \int \frac{dx}{x^2 + 3x - 4}$$

9. Prove that  $\int_2^5 f(x) dx = \frac{1}{12} \{ f(2) + 22f(4) + f(6) \}$  approximately, and hence find an approximate value for  $\int_{1.5}^{2.5} 2^{-x^2} dx$ .

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART I.—SECTION B).

1. Explain what is meant by “nominal” and “effective” rates of interest.

If £100 amounts to £112·73 in 3 years at a nominal rate of interest of 4 per-cent per annum, how often is interest convertible?

$$\text{Given} \quad \log_{10} e = \cdot 434294$$

$$\log 1\cdot1273 = \cdot 052029.$$

\*2. Given  $3a_n = 2a_{2n} = 45$ , find  $n$  and  $i$ .

\*3. A loan of £10,000 due 30 June 1919 is to be repaid by an annuity in  $10\frac{1}{2}$  years, a half payment being made on 31 December 1919 and a full payment at the end of each of the succeeding ten years. Find the amount of the annuity at 4 per-cent per annum and draw up a schedule showing the division of each payment into principal and interest.

\*4. On 1 January 1919 an investor gave £1,200 for an annuity of £100 payable annually from 1 July 1919 to 1 July 1936 inclusive. If he is to earn  $4\frac{1}{2}$  per-cent per annum on his whole capital throughout the period of the transaction, at what rate of interest must he accumulate the sinking fund to replace his capital at the end of the period?

\*5. The following 4 per-cent loans have been borrowed:

- (1) £10,000 on 1 July 1909, repayable in 20 years by means of a level yearly annuity.

- (2) £5,000 on 1 January 1913, repayable in 25 years by means of equal half-yearly instalments of principal, with interest payable half-yearly on the balances outstanding.

It is proposed to consolidate these loans into one, as from 1 July 1919, repayable in 14 years at 4 per-cent per annum by means of a level half-yearly annuity.

What will be the future half-yearly payments?

\*6. A government purchases a railway under the terms of an agreement whereby it must pay £137 for every £100 of ordinary stock. As an alternative to payment in cash it offers to each holder of £100 stock either:

- (a) A terminable annuity of £7. 13s. 8d. per annum payable half-yearly for 40 years, or  
 (b) £120 irredeemable stock bearing interest at 4 per-cent per annum payable half-yearly, together with a triennial dividend based on the net profits of the railway.

What rate of dividend must be paid under (b) in order that the yields under (a) and (b) may be equal?

\*7. A £20 share is bought for £50; the dividends are payable half-yearly and for the first year are declared at the rate of 10 per-cent per annum, for the second year at  $9\frac{3}{4}$ , the third year at  $9\frac{1}{2}$ , decreasing by equal decrements annually to  $7\frac{1}{2}$  per-cent per annum for the eleventh year.

At the end of the eleventh year the share is sold for £45. What yield has the purchaser obtained on his investment?

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

## EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE

### (PART II).

#### *First Paper.*

1. Given that out of 1,000 children at age 0, the deaths under three months old are 40, between three and six months 28, and between six and twelve months 36, find  $L_0$ ,  $m_0$  and  $\mu_{\frac{1}{2}}$  as accurately as possible. Explain why  $\mu_{\frac{1}{2}}$  is less than  $m_0$ .

2. If the probability that exactly three lives out of six all aged  $x$  survive  $n$  years is .08192, find the probability that at least three survive  $n$  years.

3. In the case of three lives all aged  $x$  find

- (1) The probability that the first death will occur in the  $n$ th year ;
- (2) The probability that the second death will occur in the  $n$ th year ;
- (3) The probability that the third death will occur in the  $n$ th year.

4. A staff, including both those in active service and those who have been pensioned, is represented, just after a year's entrants have joined the staff and a year's retirements have taken place, by the  $l_x$  column from age 20 onwards, and has reached a stationary condition.

Retirement on pension takes place at exact ages 61 to 65, and there are no withdrawals except by death. The proportion of those in active service at each age who retire at that age and the corresponding pensions are as follows :

Age	Proportion of those in active service at each age who then retire	Amount of Pension (assume payable momentarily)
61	One-tenth	£140
62	One-sixth	£160
63	One-third	£180
64	One-half	£200
65	All	£220

Assuming that the mortality among pensioners is the same as among those in active service at corresponding ages, find expressions for the following, just after a year's retirements have taken place :

- (1) The total number of pensioners.
- (2) The total pensions payable in a year.
- (3) The sum (ignoring interest) of all future payments in respect of existing pensions.

\*5. A widows' annuity-society pays an annuity of £10 per annum, the first payment being due at the end of the year in which the husband dies. The society is recruited at the beginning of each year by the entrance of 100 married couples, the husband being aged 30 and the wife 25. After five years from the formation of the society, how many widows will there be on the fund, and what will be the value of the pensions then payable ?

Use  $H^M$  functions and value at 3 per-cent interest.

6. If  $\mu_x$  is in the form of  $A + Bx^{\alpha} \left( \log_e x + \frac{1}{\log_e c} \cdot \frac{1}{x} \right)$ , find an expression for  $l_x$ .

7. Explain how you would calculate the annual premium for an assurance payable at the death of  $x$  if he dies after the survivor of  $y$  and  $z$  but before  $w$ .

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering this question.

*Second Paper.*

1. Explain the advantages of commutation columns.

Express in commutation symbols the annual premium for a decreasing term assurance for  $m$  years, the sum assured in the event of death in the  $n$ th year ( $n$  not greater than  $m$ ) being of the form

$$B(m-n) + C$$

and the premium for any year being proportionate to the sum assured for that year.

2. You are asked by a client to quote the present value of a sum of £10,000 receivable immediately upon the death of a man aged 60 on the basis of H<sup>M</sup> Mortality and 5 per-cent interest. Your client has referred to a table of complete expectations of life and has calculated the value himself by assuming 5 per-cent compound interest for the expectation as a term certain.

Set forth in language as little technical as possible the reasons you would give him to account for the difference between your value and his. Show clearly the systematic nature of the difference, and state in which direction the error will prevail.

3. Explain what you understand by the Law of Uniform Seniority.

Does this law apply to a table for which  $\mu_x = A + Bx$ ?

4. If  $a_{x:n} = a - \beta e^x$ , expand  $A_{x:\overline{n}|}$  in ascending powers of  $x$ .

If you were given a complete table of  $a_{x:\overline{n}|}$ , indicate how you would deduce the constants to be employed in the calculation of  $A_{50:\overline{15}|}$ .

5. Find, in terms as far as possible of contingent survivorship assurances payable on  $y$  dying first, the single premium for 1 payable

(a) On the death of  $y$  within 10 years,  $x$  surviving him and both  $u$  and  $v$  having died previously;

(b) On the death of  $y$  after 10 years,  $x$  surviving him, one at least of  $u$  and  $v$  having died previously, and all having survived 10 years;

(c) On the death of  $y$  after 10 years,  $x$  surviving him and one only of  $u$  and  $v$  having died previously.

6. Find the value at 3 per-cent interest of a whole life policy for £1,250 effected at age 30 which has been 20 years in force and to which a reversionary bonus of £500 is attached, having given that  $P_{30} = \cdot 0179$  and  $P_{50} = \cdot 0373$ .

7. How would you verify a table of endowment assurance policy-values by addition?

*Third Paper.*

\*1. If the net single premium for a whole life policy for £100 at a given age is £32. 8s., the corresponding net level yearly premium £1. 12s. 5d., and the corresponding net yearly premium reducible by 50 per-cent after 10 years £2. 5s. 8d., find the net single premium on the same basis for a 10-year endowment assurance for £100.

\*2. A whole life assurance for £500, subject to a yearly premium, was effected on a life aged 20, 10 years ago. Five years ago, just before payment of the sixth premium, it was altered to an endowment assurance at age 60, an increased annual premium being paid in consequence of the alteration. Find the value of the policy at the present time. Use the  $O^{[NM]}$  3 per-cent Table throughout and ignore the question of loading.

\*3. An office has on its books at the beginning of a year 100 annuities of £100 on lives aged 70, each payable by quarterly instalments on 31 March, &c., with proportion to date of death. It makes reserves on the Carlisle 3 per-cent basis and earns 4 per-cent on its funds. If during the year there are four deaths amongst these annuitants find what profit or loss the office will make in the group for the year. Assume the deaths are evenly spread over the year and that there are no expenses.

\*4. Find by the  $H^M$  3 per-cent Table the net single premium for an assurance of 1 payable on the attainment of age 21 by a child now aged 7, or, if this child dies previously, on the attainment of age 21 by a child now aged 5, or, if neither attain 21, then on the death of the survivor of them. Given that  ${}_{14}A_{5:7} = \cdot 101$ .

\*5. A policy for £1,000, under which the sum assured is payable at death after age 21, the premiums paid being returnable with 3 per-cent compound interest at death before age 21, is granted at an annual premium on the life of a child aged 1. It is desired that the option shall be granted at age 21 without alteration in the annual premium to convert the policy either (1) to an endowment assurance for a reduced amount payable at age 55 or previous death, or (2) to an assurance for a reduced amount payable at death with the number of future premiums (including the one at age 21) limited to a maximum of 30. Assuming  $H^M$  Mortality with 3 per-cent interest find the reduced sum assured in each case.

\*6. An under-average life aged 50 takes out an endowment assurance with profits, for a term of 15 years. He is offered two alternatives,

- (1) to pay an extra annual premium of 10s. per-cent; or
- (2) to pay the normal premium, but the bonus to be payable to him only if he should survive the term.

What rate of annual simple reversionary bonus will make these options equal in value to one another?

Assume that the life can be treated throughout with an addition of five years to the age, and work with the  $O^{[NM]}$  3 per-cent Table.

7. On the death of the survivor of  $x$  and  $y$  a sum of 1 is to be divided equally among such of three persons aged  $a$ ,  $b$  and  $c$  respectively, as may then be living, with a further condition that if  $a$  predecease the survivor of  $x$  and  $y$  his share is to be divided equally between such of two persons aged  $u$  and  $v$  as are living at the date of distribution.

Express in the form of an integral the value of  $u$ 's share.

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

## EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW (PART III.—SECTION A).

### *First Paper.*

1. Given the following particulars, how would you deduce the "exposed to risk" for the purpose of (a) an aggregate and (b) a select mortality experience of a life assurance society?

Calendar year of entry. Age next birthday at entry.

Calendar year of exit. Mode of exit and due date of last premium paid.

Assume that the investigation is to cover a period of 15 years and that all premiums are annual.

2. Work out the rates of mortality that can be obtained conveniently from the following information and state what tables in this form would be required to obtain complete select tables for durations 0 to 10 and ultimate tables thereafter:



*Number of policies on the books which, at the date mentioned, were more than 8 and less than 9 years in force.*

Date	AGE LAST BIRTHDAY	
	60	61
1 Jan. 1918 ... ..	3,000	2,850
1 Jan. 1919 ... ..	3,100	3,300

*Number of deaths during 1918, under policies which at date of death were more than 8 and less than 9 years in force.*

Age last Birthday at Death	Number of Deaths
60	100
61	110

3. State shortly the various steps taken in forming the English Life Table, No. 7.

4. The following table has been prepared from the Census Returns of 1901 and 1911 by applying the rates of mortality shown by the English Life Table No. 7 to the population groups at the earlier census. Analyze and explain the divergencies in these figures :

Ages at 1911	MALES		FEMALES	
	Number (in thousands) expected to have survived from 1901	Number (in thousands) enumerated	Number (in thousands) expected to have survived from 1901	Number (in thousands) enumerated
15-19	1,698	1,655	1,706	1,682
20-24	1,620	1,503	1,624	1,673
25-29	1,543	1,456	1,582	1,623
30-34	1,400	1,376	1,579	1,501
35-39	1,246	1,261	1,417	1,352
40-44	1,067	1,075	1,190	1,158
45-49	930	926	1,020	999
50-54	779	768	853	834
55-59	625	608	699	670
60-64	483	477	560	543
65-69	337	366	410	441
70-74	234	237	305	317

5. Describe shortly the methods employed in the graduation of the Whole Life Without Profit Mortality Table (British Offices Life Tables 1893).

At what age at entry is the select table started and what is the general effect of the graduation on the mortality of the first five years of assurance?

6. What are the tests of a good graduation?

*Second Paper.*

1. The following facts are available :

- (a) Number of persons at each age last birthday on 1 January of each year.
- (b) Number of persons at each age last birthday marrying, dying or withdrawing during each calendar year.
- (c) Number of entrants each year with dates of birth.

It is suggested that rates of death, marriage and withdrawal should be obtained by dividing the number of deaths, marriages and withdrawals for each age by the sum of the numbers at each 1 January, decreased by half the deaths and increased by half the entrants (using age last birthday at entrance). Explain whether this is satisfactory, and if unsatisfactory suggest improvements.

2. What are the special objects sought in constructing a mortality table :

- (a) for general use in connection with life assurance work?
- (b) for the purpose of comparative statistics?

3. Explain clearly the method you would adopt in investigating the sickness experience of a large friendly society, and deducing rates of sickness under headings "1st six months," "2nd six months," "after first year."

What inaccuracy is involved in the method you use, apart from that arising from the way in which you arrive at the age?

4. The average deviation, irrespective of sign, is sometimes given as approximately  $\cdot 8 \sqrt{npq}$ . How is this function used in actuarial work, what are the assumptions underlying its derivation, and how far are these justified in the case of mortality statistics?

\*5. Discuss the graduation shown in the following table. Set out the result graphically for ages 20-50, and show on your diagram any improvement you consider necessary.

Age $x$	Exposed to risk from $x-2$ to $x+2$	FORCE OF MORTALITY AT $x$		EXPECTATION OF LIFE	
		Ungraduated	Graduated	Ungraduated	Graduated
10	429	·00233	·00350	47·84	47·60
15	669	·00299	·00355	43·40	43·32
20	990	·00404	·00456	39·13	39·16
25	4,400	·00609	·00585	34·96	35·12
30	14,000	·00715	·00757	30·94	31·22
35	30,000	·01090	·00986	27·25	27·47
40	40,000	·01440	·01225	23·87	23·92
45	60,000	·01795	·01713	20·65	20·58
50	70,000	·02327	·02295	17·60	17·46
55	70,000	·03067	·03100	14·76	14·60
60	70,000	·04125	·04245	12·14	12·00
65	60,000	·05624	·05875	9·76	9·69
70	40,000	·08000	·08150	7·66	7·70
75	20,000	·11636	·11500	5·91	6·00
80	10,000	·16760	·16272	4·53	4·59
85	2,000	·23615	·23078	3·48	3·46
90	200	·32500	·32669	2·69	2·58

You may assume that the deaths are given sufficiently accurately by the product of the figures in the columns headed "exposed to risk" and "force of mortality."

\* Sheets of cross-ruled paper will be supplied for use in answering this question.

## EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW

### (PART III.—SECTION B).

#### *First Paper.*

1. Explain briefly the theoretical assumptions underlying Lidstone's Z method of valuing endowment assurances.

How would you apply the method to the case of an office having a large number of policies maturing on the actual birthday, with premiums payable for  $n+1$  years, where  $n$  is the difference between the age next birthday at entry and the maturity age?

2. State concisely the different methods of valuing whole life policies by limited payments. Which method do you prefer, and why?

3. The question as to the advisability of making a change in the valuation basis is under consideration by your company.

To what extent and under what conditions would you be justified in making use of the tables that have been prepared, based on the "Model Office", showing the relative reserves according to different valuation tables and at different rates of interest? Mention some of the precautions and limitations that must be borne in mind.

Consider as examples :

(1) The change from  $O^M$  3 per-cent to  $O^M$  and  $O^{M(5)}$  3 per-cent.

(2) The change from  $O^M$   $2\frac{1}{2}$  per-cent to  $O^M$  3 per-cent.

4. What method would you adopt in valuing the liabilities of an old established employers' liability insurance company?

5. What considerations would guide you in determining the maximum sum assured to be retained by a life office at its own risk on any one life?

6. Contrast the relative position of with profit policyholders in two mutual offices A and B. The last valuation of each office took place as at 31 December 1914, and since that date both offices are known to have suffered a depreciation equal to 10 per-cent of the invested funds as at 31 December 1914. This depreciation will fall to be dealt with at the next valuation at the end of 1919.

	Office A	Office B
Valuation Basis ... ..	$O^M$ $2\frac{1}{2}$ %	$O^M$ 3 %
Percentage of Business, with Profits...	50 %	95 %
Percentage of Gross Premiums, } reserved for expenses ... ..	15.	21.5
Assurances in force ... ..	£4,500,000	£5,000,000
Bonus last declared ... ..	30s. % per ann. compound	30s. % per ann. simple
Surplus carried forward ... ..	£50,000	Nil.
Funds 31 December 1914 ... ..	£2,000,000	£2,000,000
" " 1917 ... ..	£1,950,000	£2,400,000
Average annual expense ratio, } 1915-1917 ... ..	$12\frac{1}{2}$ %	14 %
Average annual rate of interest } (1915-1917), after deducting } tax ... ..	£3. 17s. 0d.	£4. 0s. 0d.
Percentage of actual to expected } claims ... ..	90	70

*Second Paper.*

\*1. Table I gives a summary of the particulars for valuation of a closed endowment assurance fund as at 31 December 1913.

Table II gives particulars of policies cancelled since that date.

Table III gives certain items of the consolidated revenue account for the five years ending 31 December 1918.

Table IV gives certain values of  $Z_M$ .

The fund is managed at an expense ratio of  $12\frac{1}{2}$  per-cent of the premium income. At the 1913 valuation—on the  $O^M$  3 per-cent basis—a simple reversionary bonus of 2 per-cent per ann. for the quinquennium was declared. £260 undivided surplus was carried forward and it was decided to pay an interim bonus of 30s. per-cent for each complete year's premium paid on policies becoming claims by death. The average net annual rate of interest earned 1914 to 1918 was £3. 10s. per-cent, and at 31 December 1918 it was found that the invested funds had depreciated in value 8 per-cent.

Complete the consolidated revenue account.

Summarize the existing business for valuation as at 31 December 1918.

Deduce the valuation ages. Make the valuation, using the tables of  $v^n$  and  $a_n^-$ , in the absence of tables of  $A_{x:\overline{n}|}$  and  $a_{x:\overline{n}|}$ , at rates of interest respectively  $\frac{1}{2}$  per-cent lower and 1 per-cent higher than the rate which you would adopt were  $O^M$  Tables available.

Draw up a profit and loss account and show what bonus you would recommend.

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering this question.

TABLE I.

Year of Maturity	Sum Assured	Existing Bonus	Office Annual Premium	Balance of Year's Premium outstanding at 31 December	Net Annual Premium	$Z_M \times$ Sum Assured
	£	£	£	£ s. d.	£	
1923	8,500	805	381	18 0 0	255	9,224
1928	7,000	480	292	...	211	6,635
1933	4,750	278	208	13 15 0	140	6,079

TABLE II.

Policy	A	B	C
Date of Entry ...	1 March 1898	1 Sept. 1908	1 Nov. 1903
„ Exit ...	1 February 1916	1 August 1916	1 August 1918
„ Birth ...	1 January 1873	1 Dec. 1875	1 Feb. 1873
„ Maturity	1 March 1923	1 Sept. 1928	1 Nov. 1933
Cause of Exit ...	Death	Death	( Surrendered ( for £240
Sum Assured ...	£500	£1,000	£750
Bonus to 1913 ...	£150	£50	£75
Office Annual } Premium ... }	£21. 5s. 6d.	£52. 10s.	£27. 10s.
How payable and } when due ... }	Half-yearly on 1 Mar. & 1 Sept.	Yearly on 1 September	Half-yearly on 1 Nov. and 1 May
Net Annual } Premium OM }	15.4	41.2	19.8
3 per-cent ...			

TABLE III.

Fund as at 1 January 1914 ...	£9,600
Interest less Tax ...	1,900
Premium Income ...	4,172

TABLE IV.

M	Z <sub>M</sub>	M	Z <sub>M</sub>	M	Z <sub>M</sub>
50	·681	54	·926	58	1·260
51	·735	55	1·000	59	1·360
52	·794	56	1·080	60	1·469
53	·857	57	1·166	61	1·587

2. A life insurance company charges premiums which, allowing for expenses, are exactly sufficient to provide for an uniform compound reversionary bonus. Simple reversionary bonuses are, however, declared up to the full surplus by a strong net premium valuation. What will be the effect on subsequent bonuses of a continuation of this practice, and how may the effect be accentuated or masked by new business?

3. An office has issued a very large number of short term endowment assurances, and wishes to make a separate valuation of them, and to keep a separate fund and accounts in respect of them, the fund having to bear the actual expenses incurred in connection with the business. Many of the premiums are payable by half-yearly or quarterly instalments, the unpaid balance of the year's premium, current at the time of death, being deducted from the policy moneys.

Explain in detail the method of valuation you would adopt.



4. It has been decided to amalgamate two mutual life offices by merging their business and assets, instead of by closing the fund of one office. Explain fully the principles which should be adopted in carrying out the fusion, with special reference to the method of distributing surplus :

- (i) Up to the date of the amalgamation, and
- (ii) Thereafter.

State your views as to the advantages or disadvantages of such an amalgamation as compared with a transfer involving the closing of the fund of the office transferred.

## EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW (PART IV.—SECTION A).

### *First Paper.*

1. What are the regulations laid down in the Assurance Companies Act, 1909, governing the transfer of the life assurance business of one company to another ?

2. What classes of societies may be registered as friendly societies under the Friendly Societies Act, 1896 ?

What limits exist as to the amounts of assurance or annuity which may be granted by a registered friendly society ?

3. Define "industrial insurance company" and "collecting society." What restrictions exist in the case of contracts made by these bodies in respect of (a) Insurance on infants ; (b) Forfeiture of policies ?

4. Give a short account of the provisions of the Assurance Companies Act, 1909, with regard to making deposits for different classes of business. How do these provisions differ from those under the 1870 Act ?

5. In what circumstances may a life assurance company be wound up by the Court ?

State the principles laid down by the Assurance Companies Act, 1909, for valuing the assurance and annuity contracts of a life office in the event of winding up.

6. An office has on its books three classes of short term endowment assurances, under which the sum assured payable at death or maturity is payable respectively,

- (a) In cash ;
- (b) By the transfer of War Loan, 1929-1947 ;
- (c) By the transfer of War Bonds, redeemable 1 October 1927.

What bases would you recommend for the surrender values to be allowed under (a), (b) and (c) ?

7. What are the methods usually adopted by offices in calculating surrender values of :

- (1) Sinking fund or leasehold redemption policies ;
- (2) Simple endowments, with return of premiums at death before maturity ;
- (3) Children's educational annuities, with return of premiums or part premiums at death of child within the annuity period ?

and state why they sometimes differ as between the three classes of policy.

8. How would you calculate the surrender-values of participating policies converted from whole-life to endowment assurances ?

State your views as to the desirability of endorsing on ordinary whole-life and endowment assurances tables of guaranteed surrender-values.

### *Second Paper.*

\*1. A, aged 40, and B, aged 44, are each entitled to a moiety of the undermentioned fund, contingently on their surviving their mother, aged 71. In the event of one only surviving, such survivor takes the whole fund, but if both predecease their mother, the estate passes elsewhere.

£10,000 5 per-cent War Loan, 1929-1947 ;

£5,000 3½ per-cent India Stock ;

A freehold house, stated to have been valued for probate two years ago at £1,800.

What sums would you advise an intending purchaser to give :

- (1) For A's reversion ;
- (2) For the interests of A and B together ?

\*2. A, aged 45, is entitled contingently, on his surviving B, a bachelor, aged 76, and provided the latter leave no issue, to £500 a year for life, secured on an estate of ample value. What sum would you advise a purchaser to give for A's interest, if he wanted to make an investment on the basis of a 4 per-cent yield, free of income tax, throughout?

\*3. Find a value for the following reversionary interest and discuss any points which seem to call for special mention.

A, the reversioner, a male aged 45, is entitled if living at the death of B, a female aged 70, to a fund, out of which an annuity of £100 is payable during the life of a lady aged 65.

The fund is at present invested as follows :

£5,000  $2\frac{1}{2}$  per-cent Consols ;

£5,000 India  $3\frac{1}{2}$  per-cent Stock ;

£120 Great Indian Peninsula Railway "A" Annuity, terminable 1948 ;

£5,000 5 per-cent War Loan, 1929-1947 ;

£2,000 New Zealand 4 per-cent Inscribed Stock, 1929.

4. Discuss generally the suitability of investments in reversionary interests for (1) the funds of insurance companies and (2) for individuals. Say what class of interests are preferable and why, and specify the particular features in such transactions which would influence your opinion as to their suitability.

What is a base fee and how would you deal with the offer of sale of such a security?

\*5. Each of two spinsters, aged 65 and 60, is entitled for life to one-half of the income from

£10,000  $2\frac{1}{2}$  per-cent Consols ;

£10,000 London and North-Western Railway 3 per-cent Debenture Stock ;

£5,000 New South Wales  $3\frac{1}{2}$  per-cent Stock, 1924,

with power to appoint to any husband who may survive her, a life interest to the extent of not more than one-half of her share of income. At the death of the first of the two life tenants to die, the survivor takes for her life the income of the deceased sister also, subject to any life interest appointed in favour of the deceased's husband.

What is the value of the reversion to the whole fund, and what covering policies should be effected?

6. A.B., aged 30, is entitled, if he survives his father, aged 60, to the reversionary life interest in :

£5,000 New Zealand 4 per-cent Stock, 1929 ;

£5,000 Chinese Government 8 per-cent Loan, 1918 (redeemable by 1928, by annual drawings at par).

A.B. desires to sell his interest, but to retain for 5 years the option to repurchase on payment to the purchaser of all sums disbursed by him, accumulated at compound interest. Indicate fully how you would ascertain the value of his interest on this basis.

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

#### PROCEEDINGS OF THE INSTITUTE.—SESSION 1918-1919.

##### *First Ordinary Meeting, 16 December 1918.*

The President (MR. GEOFFREY MARKS, O.B.E.) in the Chair.

The President delivered an Inaugural Address.

##### *Second Ordinary Meeting, 24 March 1919.*

The President (MR. GEOFFREY MARKS, O.B.E.) in the Chair.

A discussion on "Aviation and Life Assurance" was opened by Dr. L. E. Stann. The following gentlemen took part therein:—Mr. A. D. Besant; Major-General Sir W. S. Brancker, K.C.B., Lt.-Colonel Mervyn O'Gorman, C.B., Brigadier-General J. G. Hearson, D.S.O., Lt.-Colonel Flack (visitors); and the President.

##### *Third Ordinary Meeting, 28 April 1919.*

The President (MR. GEOFFREY MARKS, O.B.E.) in the Chair.

Mr. Andrew Rutherford Davidson, F.F.A., was elected an Associate of the Institute.

A paper entitled "Group Insurance" was read in abstract by the Author. Mr. P. H. McCormack.

The following gentlemen took part in the discussion:—Messrs. F. P. Symmons, W. Schooling (a visitor), Sir Alfred Watson, Messrs. C. W. Kenchington, E. B. Nathan, H. H. Austin, S. G. Warner, and the President.

##### *The Seventy-second Annual General Meeting, 2 June 1919.*

The President (MR. GEOFFREY MARKS, O.B.E.) in the Chair.

The proceedings at the Annual General Meeting will be found on page 396.

#### REPORT, 1918-1919.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1918-1919, the seventy-first year of its existence.

There has been a *decrease* of 19 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-three years later, at 31 March 1908, it was 1,009. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1909	254	325	400	19	998
1910	259	335	348	21	963
1911	267	339	308	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896
1918	295	332	215	18	860
1919	288	330	205	18	841

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

*Schedule of Membership, 31 March 1919.*

	Fellows	Associates	Students	Corres- ponding Members	Total
i. Number of Members in each class on 31 March 1918 .	295	332	215	18	860
ii. Withdrawals by					
(1) Death . . .	7	7	7	...	27
(2) Resignation or otherwise . . .	1	2	3	...	
	287	323	205	18	833
iii. Additions to Membership					
(1) By Election . .	...	...	...	...	8
(2) By Examination .	...	...	1	...	
(3) By Re-instatement	1	3	3	...	
	288	326	209	18	841
iv. Transfers					
(1) By Examination:					
<i>from Associates</i>	...	...	...	...	...
<i>to Fellows</i> . . .	...	...	...	...	...
	288	326	209	18	841
(2) By Examination:					
<i>from Students</i>	...	...	4	...	...
<i>to Associates</i> .	...	4	...	...	...
v. Number of Members in each class on 31 March 1919 .	288	330	205	18	841

[Continued on page 392.]

Dr.

Revenue Account for

1918.				1919.			
£	s.	d.		£	s.	d.	£ s. d.
8,045	3	7	Amount of Funds at the beginning of the year—				
			General Fund (including Stock of Publications, other				
			than <i>Journal</i> ) . . . . .	8,184	13	2	
433	18	2	Messenger Legacy Fund . . . . .	446	18	6	
372	16	1	Brown Prize Fund . . . . .	383	19	9	
793	15	3	G. F. Hardy Memorial Fund . . . . .	832	19	5	
9,645	13	1					9,848 10
			G. F. Hardy Memorial Fund (A. J. Cook's Bequest)				135 11
			Subscriptions—				
751	16	0	Fellows . . . . .	804	6	0	
532	7	0	Associates . . . . .	565	19	0	
126	0	0	Students . . . . .	157	10	0	
13	2	6	Probationers . . . . .	32	11	0	
1,423	5	6		1,560	6	0	
3	3	0	<i>Fines on Reinstatement</i> . . . . .	...			
1,426	8	6		1,560	6	0	
			<i>Less</i> Waived and returned to Members and Probationers				
57	15	0	on Naval and Military Service . . . . .	48	6	0	
1,368	13	6		1,512	0	0	
...			One Annual Subscription Compounded for . . . . .	10	10	0	
			Entrance Fees—				1,522 10
4	4	0	<i>Associates</i> . . . . .	...			
3	3	0	<i>Students</i> . . . . .	5	15	6	
...			<i>Probationers</i> . . . . .	18	18	0	
7	7	0					24 13
76	19	11	Balance of Publications Account . . . . .				134 11
			Dividends and Interest—				
295	14	4	General Fund . . . . .	215	19	3	
13	0	4	Messenger Legacy Fund . . . . .	13	8	2	
11	3	8	Brown Prize Fund . . . . .	11	10	5	
39	4	2	G. F. Hardy Memorial Fund . . . . .	19	2	2	
359	2	6					260 0
£11,457	16	0					£11,925 16

Publications Account for

£	s.	d.		£	s.	d.
249	0	2	Stock (excluding <i>Journal</i> ) at the beginning of the year . . . . .	323	19	...
105	12	8	<i>Cost of Reprint of Text-Book, Part II</i> . . . . .	...		
...			<i>Cost of Reprint of Sir G. F. Hardy's Prize Essay on Friendly Societies</i> . . . . .	37	0	
33	3	4	Binding, Advertising, &c. . . . .	31	1	
76	19	11	Balance . . . . .	134	11	
£464	16	1		£526	12	

Balance Sheet

£	s.	d.	LIABILITIES.	£	s.	d.	£	s.	d.	£	s.	d.
8,184	13	2	General Fund . . . . .				8,242	10	7			
233	9	2	Messenger Legacy Fund . . . . .	233	9	2						
213	9	4	Accumulated Dividends . . . . .	226	17	6						
446	18	6					460	6	8			
200	0	0	Brown Prize Fund . . . . .	200	0	0						
183	19	9	Accumulated Dividends . . . . .	195	10	2						
383	19	9					395	10	2			
767	15	9	G. F. Hardy Memorial Fund . . . . .	903	7	0						
65	3	8	Accumulated Dividends . . . . .	84	5	10						
832	19	5					987	12	10			
...			Reserve for Income Tax on War Loan Stock . . . . .							10,086	0	
20	5	11	Sundry unpaid Accounts . . . . .							145	13	
...			Examination Fees, 1919 . . . . .							38	8	
£9,868	16	9								10	10	
										£10,280	11	



1918.				1919.					
£	s.	d.	Journal—	£	s.	d.	£	s.	d.
260	3	4	Printing of Nos. 267, 268	399	10	5			
43	2	6	Editorial Expenses	43	2	6			
303	5	10		442	12	11			
102	5	2	Less Sales during the year	128	8	6			
201	0	8					314	4	5
25	8	3	Library—Binding, Purchases, &c.				49	3	2
25	6	4	Meetings				21	12	0
...			Legal Charges				40	16	0
600	0	0	Office Expenditure—Rent	600	0	0			
520	9	0	Salaries	534	6	2			
49	17	2	House expenses	55	5	1			
30	7	2	Fire and other Insurance	32	14	10			
78	17	11	Stationery and Printing	149	7	1			
20	5	0	Postage and Telegrams	38	2	11			
57	13	8	Sundries	4	4	8			
1,357	9	11					1,414	0	9
9,848	10	10	Amount of Funds at the end of the year as per Balance Sheet				10,086	0	2

*Examined and found correct, 28 April 1919.*

11,457 16 0	W. MOUAT JONES, E. W. HUMPHRY, STANLEY HAZELL,	} <i>Auditors.</i>	£11,925 16 7
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Year ending 31 March 1919.

£	s.	d.		£	s.	d.
140	16	11	Sales (excluding <i>Journal</i> )	277	5	8
323	19	2	Stock (excluding <i>Journal</i> ) at the end of the year	249	6	

*Examined and found correct, 28 April 1919.*

£464 16 1	W. MOUAT JONES, E. W. HUMPHRY, STANLEY HAZELL.	} Auditors.	£526 12 0
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1 March 1919.

£	s.	d.	ASSETS.	£	s.	d.
1,875	0	0	£3,000 Natal 3 per-cent Inscribed Stock	1,875	0	0
726	5	0	£1,000 Dominion of Canada 3½ per-cent Registered 1930-50 Stock	726	5	0
700	0	0	£1,000 New South Wales 3½ per-cent Inscribed 1930-50 Stock	700	0	0
357	15	0	£600 Belgian Government 3 per-cent Sterling Loan of 1914	357	15	0
4,275	0	0	£4,500 5 per-cent War Stock, 1929-47	4,275	0	0
400	0	0	£400 National War Bonds, 4 per-cent, 1927	400	0	0
323	19	2	Stock of Publications (excluding <i>Journal</i> ) in hand	249	6	0
...	...	...	Cash on Deposit Account	250	0	0
355	18	10	Cash on Current Account and in hand	480	18	0
87	3	0	Subscriptions in Arrear	63	0	0
767	15	9	£954. 5s. 6d. 5 per-cent War Stock, 1929-47 (G. F. Hardy Fund)	903	7	0

*The Stock Exchange Securities are taken at the values at which they stood on 31 March 1916, and at cost price in the case of any acquired since that date.*

*Examined and found correct, 28 April 1919.*

£9,568 16 9	W. MOUAT JONES, E. W. HUMPHRY, CHAS. D. HARRIS.	} Auditors.	£10,280 11
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There are also 169 candidates admitted as Probationers, and 73 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes since 31 March 1913 have been as follows :

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1914	200	67	1917	173	67
1915	188	72	1918	156	70
1916	172	73	1919	169	73

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of five Fellows, Messrs. Vyvyan Marr, J. G. Priestley, H. T. Kay Robinson, C. Stevens, and H. Archer Thomson; seven Associates, Messrs. W. Borland, E. C. Coote, F. Defries, J. C. Hurley, H. B. Keable, A. G. Portch, and W. G. Titmuss; and five Students, Messrs. G. L. L. Carter, T. Middleton, H. J. Mills, E. A. Newland, and A. R. D. Watson. Eight of these Members, namely, Lieut.-Col. H. T. Kay Robinson, D.S.O. and Bar, Captains F. Defries and T. Middleton, Lieutenants H. J. Mills and E. A. Newland, Lance-Corporal J. C. Hurley, Lieutenant H. B. Keable, R.N.V.R., and Naval Instructor G. L. L. Carter, rendered the supreme sacrifice in the Nation's cause. The Council have in addition to record with much regret the deaths of twenty-one Probationers of the Institute, Captains E. C. K. Clarke, M.C., and J. H. Orr, Lieutenants J. W. E. Alexander, E. M. Dove, M.C., R. G. Gale, A. S. Gregory, M. H. Grigg, F. G. Goodyear, M.C., and D. A. Roberts, Lance-Corporal F. L. Bristow, Gunner C. G. Giffkins, and Privates G. F. T. Ascott, V. J. Austin, W. Bradley, H. W. Brown, R. Cole, T. G. Cunliffe, A. R. Harriss, C. P. Maddox, C. S. Shilson, and A. D. Steed, whose lives were also given in the service of their King and Country. The Council hope at an early date to place in the Hall of the Institute a memorial of the Members and Probationers who have fallen in the War.

By the death of Mr. Vyvyan Marr the profession has lost one of its most loyal and valued Members. Mr. Marr had served on the Council of the Institute, and at the time of his death was Chairman of the Board of Examiners.

Messrs. C. Stevens and J. G. Priestley were known to but few of the present generation of actuaries. They were the last surviving Members of the Actuaries' Club who became Fellows of the Institute under the Charter.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £1,547. 3s. 6d., as compared with £1,376. 0s. 6d. in the previous year. The Income and Expenditure for the year were £1,941. 14s. 6d. and £1,839. 16s. 4d. respectively.

At the close of hostilities the total number of Members and Probationers who had been on service with the Army and Navy was 428, of whom 75 had been killed in action or died of wounds.

On behalf of the Institute the Council have submitted to His Majesty The King a Loyal Address on the signing of the Armistice.

The amended Bye-laws regulating the election of Members of the Council, duly passed and confirmed at Special General Meetings of the Fellows and Associates of the Institute, have received the sanction of the Privy Council.

The Council have accepted with much regret the resignation of Mr. J. Spencer as Joint Honorary Librarian. Mr. Spencer held the office for ten years, and the thanks of the profession at large are due to him for the valuable services thus rendered. Mr. A. D. Besant has been appointed to fill the vacancy.

The Council desire to place on record their appreciation of the work of the Board of Examiners in preparing a course of reading for the guidance of students. The course of reading has been published in the *Journal*, and is issued as part of the revised Regulations and Syllabus of Examinations.

As already announced, the Council have decided to resume the Examinations of the Institute. The first to take place under the new Regulations will be held from the 23 to the 25 June next.\*

During the Session a special course of Lectures was arranged for Students and Probationers preparing for Parts I and II of the Examinations. The Council take this opportunity of thanking those who have undertaken to deliver the Lectures.

The Council have in contemplation one or more courses of Lectures, to be given at the Institute during next Session, on subjects included in the Syllabus for Parts III and IV of the Examinations as well as on subjects outside the Syllabus but of general interest to the profession.

The Council have been approached by the Royal Patriotic Fund Corporation to undertake the valuation of the Funds under its control. In continuing this work, which was performed by the Institute some years ago, the Council believe that their action will meet with the hearty support of the Members.

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
29,716 . . . . .	Parts of <i>Journal</i> .
726 . . . . .	Index to Vols. 1 to 40.
1,582 . . . . .	<i>Text-Book</i> , Part I (Revised Edition).
375 . . . . .	<i>Text-Book</i> , Part II (Second Edition).
630 . . . . .	Government Joint-Life Annuity Tables.
728 . . . . .	Select Life Tables.
14 . . . . .	A Short Collection of Actuarial Tables (New Edition).
844 . . . . .	Frequency-Curves and Correlation, with Addendum and Errata (W. P. Elderton).
38 <i>in cloth</i> )	( Lectures on Finance and Law (Clare and Wood Hill).
2,308 <i>in paper</i> )	
1,525 . . . . .	Lectures on the Companies Acts (A. C. Clauson).
1,187 . . . . .	Lectures on the Law of Mortgage (W. G. Hayter).
693 . . . . .	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,375 . . . . .	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
842 . . . . .	Lectures on Stock Exchange Investments (J. Burn).
1,492 . . . . .	Lectures on Friendly Society Finance (Sir A. W. Watson).
315 . . . . .	South African War Mortality (F. Schooling and E. A. Rusher).
241 . . . . .	Life Assurance Law (A. R. Barrand).
644 . . . . .	British Offices' Valuation Tables.
642 . . . . .	British Offices' $2\frac{3}{4}$ per-cent Temporary Annuity Values.
135 . . . . .	Transactions of the Second International Congress of Actuaries.
781 . . . . .	Index to Transactions of Seven International Actuarial Congresses.
1,500 . . . . .	Examination Questions, 1912-15.

12 May 1919.

\* For Results see pp. 394-5.

## EXAMINATIONS, JUNE 1919.

Examinations were held on the 23rd, 24th and 25th June, 1919, at London, Liverpool, Norwich, Edinburgh, Dublin, Melbourne, Sydney, Wellington, Montreal, Toronto, Ottawa, Winnipeg, Bombay, and Calcutta, with the following results:

## PART I.—SECTION A.

Seventy candidates sent in their names, of whom fifty-nine presented themselves, and twenty-six passed, namely:

Allen, F. D. C.	Ibbotson, L. E.	Stark, A. W.
Chapman, H. V.	Johnston, W. N.	Tharp, H. W.
Cope A.	Knowles, M. B.	Thakur, B. T.
Douglas, F. C. R.	Long, J. A.	Thomas, J. H.
Fassel, E. G.	Milnes, H. L.	Unthank, H. W.
Freeman, H.	Murray, J. R. C.	Walker, J. R.
Goodfellow, P.	Perryman, F. S.	Walker, R. B.
Hallett, H. J.	Reynolds, P. C.	Willows, C. E.
Hooker, P. F.	Southwell, M. G.	

## PART I.—SECTION B.

Forty-five candidates sent in their names, of whom thirty-seven presented themselves, and eighteen passed, namely:

Allen, F. D. C.	Ibbotson, L. E.	Perryman, F. S.
Cope, A.	Johnston, W. N.	Polden, L. S.
Fassel, E. G.	Knowles, M. B.	Thomas, J. H.
Gibberd, J. A.	Long, J. A.	Thakur, B. T.
Goodfellow, P.	Mann, A. H.	Unthank, H. W.
Hooker, P. F.	Murray, J. R. C.	Warren, L. A. H.

## PART II.

Thirty-seven candidates sent in their names, of whom thirty-five presented themselves, and sixteen passed, namely:

Capon, G. W.	Innes, F. F.	Rider, W. W. H.
Clarke, J. H.	Knowles, M. B.	Rutherford, C. D.
Coleman, H. D.	Mabon, J. B.	Shrewsbury, A. H.
Fassel, E. G.	McLean, P. S.	Thorpe, A. H.
Hocking, W. S.	Maddex, G. H.	
Houston, C. C.	Marriott, A.	

## PART III.—SECTION A.

Twenty-five candidates sent in their names, of whom twenty-four presented themselves, and sixteen passed, namely :

Barrett, C. C. C.	Denmark, R. J.	Weyer, D.
Blake, W. T. C.	Johnson, A. N.	White, O. D.
Brown, S. P.	Klagge, O. C. J.	Wickens, C. H.
Carpmael, C.	Moore, W. R.	Wilson, A. B.
Chatham, E. F.	Savory, D. S.	
Davidson, A. R.	Watson, A. D.	

## PART III.—SECTION B.

Eighteen candidates sent in their names, all of whom presented themselves, and four passed, namely :

Barrett, C. C. C.	Klagge, O. C. J.
Chatham, E. F.	Savory, D. S.

## PART IV.—SECTION A.

Twenty-seven candidates sent in their names, of whom twenty-six presented themselves, and fifteen passed, namely :

†Brenton, W. P.	†McCormack, P. H.	†Smither, H. B.
†Brown, P. G.	†Menzler, F. A. A.	†Traversi, A. T.
†Evans, A. W.	†Paton, A. G.	†Tyler, V. W.
†Fielder, T. L.	†Robertson, F. W.	†Underwood, R. E.
†Hustwitt, W. E.	†Searle, A. J.	†Wilton, H. G.

Those marked (†) have now completed the Examination for the Class of Fellow.

By Order of the Council,

A. C. THORNE,

*Chairman of Board of Examiners.*

W. PALIN ELDERTON,

H. M. TROUNCER,

*Joint Honorary Secretaries.*

## PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventy-Second Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 2 June 1919. Mr. Geoffrey Marks, O.B.E. (President), in the Chair. Mr. A. Levine (Hon. Secretary) read the Notice convening the Meeting. The Minutes of the preceding General Meeting were read and confirmed.

The PRESIDENT said that the first thing which attracted attention in the Report was the progressive decrease in the number of their members. That could not be attributed to the War, because the decrease began as far back as 1909 and had been going on ever since. When he looked at the figures he asked himself whether it could possibly be the result of the comparatively limited outlook that there was for the profession in the days when the decline in membership first began. By limited outlook he meant in the sense that the number of Actuaries was getting greater than the possible openings for their activities and energy. If that were the real explanation of the decrease in numbers, he hoped that the action which the Council had taken in the last year or so would lead to a comparatively rapid restoration of the numbers to the high-water mark which they reached in 1909.

Naturally, a prominent place in the Report was taken by their record in the country's cause. He thought they might congratulate themselves that the Institute had done its duty nobly. A large proportion of their eligible members had gone on service, and of that number a high proportion had laid down their lives. That, he thought, was a testimonial to their loyalty and bravery, and it would be for the Council to consider, in the course of the coming year, what form of memorial should be placed in the Hall to their memories. He hoped that the memorial which they put up might have some reference, not only to the service which those men had rendered, but to the end which they had attained in that service.

The only deaths to which he would refer were those of Vyvyan Marr and Archer Thomson, both of them loyal servants of the Institute—the former cut off in the height of his usefulness and the latter giving in after a long struggle with indifferent health. Marr was a conscientious man in all the aspects of his life and Archer Thomson had almost a genius for friendship, which endeared him to all those with whom he came in contact.

The consideration of the numbers of those who had lost their lives led him to mention another point, and with some emphasis. It was obvious to him, as well as to the older men amongst them, that the old order with which they were familiar was changing, not only in the Institute, but everywhere outside it. For the older among them, whether they were individuals or institutions, it must be a painful process, and whether or not they could get any good out of it depended on the amount of ease and unselfishness with which they subordinated their private interests to the common good. There was no doubt that many of the changes in the circumstances in which they were living and were going to live were such that old institutions such as their Institute and old individuals like himself must either adapt themselves or perish. The individual perishing did not much matter, but it was not thinkable that an Institute such as that should perish or that its usefulness should be in any way abated, and his conviction was (and he hoped they would share it with him) that out of the turmoil and trouble through which they had passed the Institute might arise stronger and more powerful than it had ever been before, touching more important and wider aspects of life and dealing with them as their training and experience ought to enable them to do. He did not know that they considered often enough that the Institute was the centre of their profession all over the world. The members, as a body, had not so many opportunities of realizing that as had the members of the Council and the President. It was an aspect of the matter, however, which he would ask them to bear in mind, because he could assure them that what they did there was closely scrutinized throughout



the world, and the example which they set would be followed wherever their profession and its tenets had penetrated, and that area was practically the whole world—China and Japan, as well as more advanced countries like America and the Colonies.

In considering to what extent the usefulness of the Institute might be increased, the Council had devised certain methods and plans which were gradually maturing. The modification of the syllabus and the resumption of the examinations in a different form, and at more frequent intervals, was one method which the Council felt would help to make up the ground which had been lost by so many of the younger members during the last four or five years. The Special Lectures which the Council had instituted for the benefit of candidates in Parts I and II had been a great success, and in the name of the Council he would like to thank those gentlemen who gave them. Further courses of a more advanced character, and of a scope somewhat outside the range of their previous interests, were being arranged. The Council had decided to have at least two courses of Lectures, one dealing with economics and banking and one with general statistics, in contradistinction to their vital statistics.

As regarded the first, he was authorized to approach Professor Foxwell, and he was glad to say that he had agreed to give a course of Lectures which would be somewhat more extensive than the Council originally intended, because Professor Foxwell told him that from his experience at the School of Economics, if any really good results were to come out of such a course as they proposed, it must be a somewhat extended course. As at present advised, Professor Foxwell thought that there should be three sessions, in each of which ten Lectures should be delivered. The Lectures would be delivered once a week at some convenient time, and they would be open to all members of the Institute. Neither in that course nor in the statistical course, the details of which were not yet settled, would there be included any examinations, so that would free the minds of a good many of them from a certain amount of anxiety. The Council felt that the members who attended those Lectures, and desired to show to their fellows and to the world that they had attained a certain amount of proficiency in the subjects dealt with, should have an opportunity given them to prove that proficiency, and they therefore had agreed that, in order to encourage original work, Papers should be invited, not necessarily on any specific point dealt with in the Lectures, but on some subject chosen by the student himself. Papers which were considered to be of sufficient merit would be rewarded by prizes out of the funds at the disposal of the Institute, and the Papers would be published in the *Journal* if they were appropriate. Personally, he thought that if they were not suitable for the *Journal*, either owing to their length or some other cause independent of their merit, but, in the opinion of the Council, were of sufficient merit to deserve publication, then they might be published at the expense of the Institute. However, that was only his own personal opinion and, so far, he had not submitted the idea to the Council. He would like to emphasize the fact that the continuance of those opportunities for a higher and wider education must depend very much upon the favour which was accorded them by the members. It was going to be a considerable cost to the Institute to initiate those courses, and the only justification for the expense would be the interest shown and the results secured.

Another indication that the Council was alive to the tendencies of the times was to be found in the new regulations for the election of members of the Council. They had amended their bye-laws on the democratic basis which was now fashionable. He thought the method which the Council had devised tended to stimulate interest in their domestic concerns, and that was all to the advantage of the Institute as a whole. The Council had also introduced regulations as to the nominations for the Presidency, which were as democratic as those which pertained to the election of members of the

Council. Although by their bye-laws they could not go outside the Council to secure nominations for the Presidency, yet the Council had arranged that any member of that body might nominate any other member who was qualified to act as President, quite independently of seniority or any other considerations than those of merit. If more than one nomination were made, the question as to who should occupy the Presidential Chair would be settled by ballot.

He thought he had said enough to show that the governing body was doing its part in the new order of things which they anticipated. The members must help. There should be no body of men engaged in business or professional work better qualified than they were to achieve success, not only in their ordinary work, but in any other vocation of life. He had often thought, and he believed it was true, that accuracy and imagination were the prime factors in a successful career. Since their early mathematical training must tend to accuracy, and seeing that the fact of so large a portion of their training and work being devoted to the application of the theory of probabilities to the affairs of life must stimulate imagination, members of the Institute, if their early training was followed up and the qualities which they had learned were properly developed, ought to secure success in every walk of life to which they chose to turn. He hoped members would bear that in mind and try to realize that the policy of the Council, which he also hoped would be the policy of the Institute, was to encourage a wider and more practical outlook, not only upon the affairs with which they had had to deal in the past, but on affairs which were of vital importance in the life in which they moved.

MR. R. TODHUNTER, in seconding the motion, said he believed the report would be regarded as a satisfactory record for a period which, perhaps, of all periods in the history of the Institute, had the best claim to the over-worked title of a transition period. Satisfactory, however, as the report was, it was not an entirely complete record of the activities of the Council or of its executive members; a few unconsidered trifles had been passed over. First and foremost, he would mention the public services rendered by the President as a member of two Royal Commissions—the Income Tax Commission and the Commission on Decimal Coinage. It must, he thought, be as unprecedented in the history of the Institute as it was gratifying that the head of the profession in this country should be serving on two such bodies. Then, again, the Council had taken no credit for the very interesting and instructive meetings which were held in March and April. Those who had had experience in the arranging of meetings would know that their warmest thanks were due to the executive officers for having given them those two gatherings. One other incident worth noting was the remarkable conjuring trick by which the hon. librarians had succeeded in accommodating all the books of the library in the cloak room. In recording the various happenings of the session they must remember that the really great event distinguishing it from every other session in the history of the Institute was the coming back of their members from Service. While they deeply deplored the loss of those who had not come back, they welcomed and congratulated those who were once again with them.

The PRESIDENT mentioned that one of the earliest questions which the Institute would have to tackle when they re-assembled concerned the admission of women. He imagined that, by the time they resumed, most of their members would have returned from the Front, and they would be a full body, able to deal with that question effectively, and he hoped, finally.

#### ELECTION OF OFFICERS.

The report having been unanimously adopted, a ballot was then taken for the election of five members of the Council. The President subsequently

announced that the Officers and Council for the ensuing year would be as follows:—

*President.*

GEOFFREY MARKS. O.B.E.

*Vice-Presidents.*

ARTHUR DIGBY BESANT, B.A.  
JOSEPH BURN, C.B.E.

ABRAHAM LEVINE, M.A.  
JAMES DOUGLAS WATSON.

*Council.*

SAMUEL JOHN HENRY WALLIS  
ALLIN, C.B.E.  
\*HERBERT HENRY AUSTIN.  
\*HENRY JAMES BAKER.  
\*LOUIS ERNEST CLINTON.  
LEWIS FREDERICK HOVIL.  
CHARLES WILLIAM  
KENCHINGTON.  
OWEN KENTISH.  
GEORGE JAMES LIDSTONE, F.R.S.E.  
\*HAROLD EDWARD WILLIAM LUTT.  
REGINALD GEORGE MAUDLING.

SIR GEORGE ERNEST MAY, K.B.E.  
HENRY EDWARD MELVILLE.  
WILLIAM CHARLES SHARMAN.  
EDWARD ROBERT STRAKER.  
\*ERNEST CHARLES THOMAS.  
ALFRED CHARLES THORNE.  
EDWARD WILLIAM TOWNLEY.  
SAMUEL GEORGE WARNER.  
SIR ALFRED WILLIAM WATSON.  
ARTHUR THOMAS WINTER.  
ERNEST WOODS.  
WILLIAM ARTHUR WORKMAN.

*Treasurer.*

WILLIAM PEYTON PHELPS, M.A.

*Honorary Secretaries.*

WILLIAM PALIN ELDERTON. | HAROLD MOLTKE TROUNCER, M.A.

\* New Members of Council.

On the motion of Mr. R. L. ELDERTON, seconded by Mr. H. L. TRACHTENBERG, Messrs. E. W. Humphry, Stanley Hazell, and D. M. Carment, were elected auditors for the ensuing year.

Mr. A. R. BARRAND, M.P., in proposing a vote of thanks to the President, Vice-Presidents, Council and Officers, including Mr. Jarvis, the Assistant-Secretary, for their services during the past year, said that very few words were needed to ensure a unanimous vote. They all knew what the President had done for them during the past year; his work spoke for itself. They congratulated themselves most heartily that, in the difficult and critical times they had passed through, they had had such a President as Mr. Geoffrey Marks. They were also grateful to the Vice-Presidents and the members of the Council for what they had done. As to the Assistant-Secretary, they might regard Mr. Jarvis as the permanent civil servant who did the major part of the work, and he thoroughly deserved their warmest thanks.

Mr. HENRY COCKBURN seconded the vote, which was carried.

The PRESIDENT, in acknowledging the resolution, remarked that he was particularly pleased to hear the references to Mr. Jarvis, for he had fully deserved them.

A vote of thanks was accorded to the auditors on the proposition of Mr. E. F. SPURGEON, seconded by Mr. H. E. W. LUTT, and was acknowledged by Mr. HAZELL.

The proceedings then terminated.

*Additions to the Library.*

The following works have been added to the Library since the publication of the *Journal* for October 1918:

- |   | <i>By whom presented<br/>(when not purchased).</i> |
|---|--|
| Actuarial Society of America.   |  |
| Transactions, 1918.   | <i>The Society.</i>                                |
| Containing, <i>inter alia</i> —   |  |
| “An Analysis of Claims for Total and Permanent Disability Benefits”, by A. Hunter.  |  |
| “Valuation of Bond Holdings of a Life Insurance Company”, by F. H. Johnston.  |  |
| “Note on Mortality by Habits Representation”, by P. H. Evans.   |  |
| “Note on Mean Population”, by J. S. Thompson.   |  |
| Actuarial Society of America and Association of Life Insurance Medical Directors.   |  |
| Report of Joint Committee. Standard Mortality Ratios incident to variations in the Height and Weight among Men. New York. 1918.   | } <i>The Actuarial Society of America.</i>         |
| Actuarial Society of Scandinavia.   |  |
| Transactions, 1918-19.  | <i>The Society.</i>                                |
| American Academy of Political and Social Science (March 1917). Svo. Philadelphia.   | <i>Purchased.</i>                                  |
| Containing, <i>inter alia</i> —   |  |
| “Life Annuities”, by M. A. Linton.  |  |
| “Insurance on sub-standard lives”, by A. Hunter.  |  |
| “The problem of cash surrender-values and cash loans”, by J. B. Lurger.   |  |
| “Mutualization of Life Insurance Companies”, by M. M. Dawson.   |  |
| “Conservation of life by Life Insurance Companies”, by Dr. L. K. Frankel.   |  |
| “Group Insurance”, by R. B. Trousdale.  |  |
| “The exemption of Life Insurance Funds from taxation”, by Dr. B. D. Mudgett.  |  |
| “The Disability Insurance Policy”, by A. P. Woodward.   |  |
| “Methods of insuring Workmen's Compensation”, by H. E. Ryan.  |  |
| “The Calculation of Workmen's Compensation Premium Rates”, by C. E. Scattergood.  |  |
| American-Canadian Mortality Investigation.  |  |
| Based on the Experience of Life Insurance Companies of the United States and Canada during the years 1900-1915, inclusive of Policies issued from 1843 to 1914, inclusive. La. 8vo. New York. 1918. | } <i>The Actuarial Society of America.</i>         |
| American Mathematical Society.  |  |
| Transactions, 1918-19.  | <i>The Society.</i>                                |

*By whom presented  
(when not purchased).*

American Statistical Association.

Transactions, 1918-19.

*The Association.*

\*Anson (Sir W. R., Bart.).

Principles of the English Law of Contract and of }  
Agency in its relation to Contract. 14th edit. }  
Svo. 1917.

*Purchased.*

\*Ashburner (—).

Concise Treatise on Mortgages, Pledges and Liens, }  
by W. F. Webster. 2nd edit. Svo. 1911. }

*Purchased.*

Association des Actuaire Belges.

Bulletin, 1914.

*The Association.*

Austria.

Bericht der Arbeiter-Unfall-Versicherungs-anstalt für }  
das Königreich Böhmen. 1916. }

*Austrian  
Government.*

Belgium.

La Caisse Générale d'Epargne et de Retraite pendant }  
la Guerre, 1914-18. Svo. Brussels. 1919. }

*Fl. Hankar.*

"Biometrika."

Vol. XII, Parts I and II.

*Purchased.*

Containing, *inter alia*—

"On the Standard Deviation of Adjusted and  
Interpolated Values of an observed Polynomial  
Function and its Constants and the guidance  
they give towards a proper choice of the  
Distribution of Observations", by K. Smith.

"On the Product-Moments of Various Orders of  
the normal correlation surface of two Variates",  
by K. Pearson and A. W. Young.

"The Correlation Coefficient of a Polychoric Table",  
by A. Ritchie-Scott.

"On a Formula for the Product-Moment Coefficient  
of any order of, a normal Frequency Distri-  
bution in any number of Variables", by  
L. Isserlis.

"On the Mathematical Expectation of the  
Moments of Frequency Distributions," by  
A. A. Tchouproff. Part I.

Bowley (A. L.).

The Division of the Product of Industry. An Analysis }  
of National Income before the War. Svo. 1919. }

*Purchased.*

Brockbank (Dr. E. M.).

Life Insurance and General Practice. Svo. 1908.

*Purchased.*

Carnegie Foundation for the Advancement of Teaching.

Rules for the Admission of Institutions and for the  
granting of retiring allowances. 4to. New York.  
1918.

Twelfth Annual Report of the President and Treasurer.  
4to. New York. 1917.

*W. J. H. Whittall.*

Castelnovo (G.).

Calcolo delle Probabilità. 1a. Svo. Milan. 1919.

*The Author.*

*By whom presented  
(when not purchased).*

**Casualty Actuarial and Statistical Society of America.**

Proceedings, 1918.

*The Society.*

Containing, *inter alia*—

“Economic Problems of the World War”, by J. D. Craig.

“The relation between the Actuary and the Statistician”, by J. D. Craig.

“Mortality from external causes among Industrial Policyholders in the Metropolitan Life Insurance Co.”, by Dr. L. I. Dublin.

“Essentials of Family Statistics”, by E. W. Kopf.

**Chartered Insurance Institute, Journal of the**

Vol. XXI. 8vo. 1918.

*The Institute.*

**Dougharty (H.).**

Notes on Deposit Pension Schemes. 1919.

*The Author.*

**Dublin (Dr. L. I.).**

Mortality Statistics of insured wage-earners and their families. Experience of the Metropolitan Life Insurance Co. (Industrial Dept.), 1911–1916 in the United States and Canada. 8vo. New York. 1919.

*The  
Metropolitan Life  
Insurance Co.*

**Duguid (C.).**

The Stock Exchange. 3rd edit. 8vo. 1913.

*Purchased.*

**Economic Society (Royal).**

Journal of the, 1918–19.

*Purchased.*

**Frost (P.).**

An Elementary Treatise on Curve Tracing. 4th edit. 8vo. 1918.

*Purchased.*

**Galton (Sir Francis), F.R.S.**

Hereditary Genius. An enquiry into its laws and consequences. 8vo. 1914.

*W. Palin  
Elderton.*

Memories of my Life. 3rd edit. 8vo. 1909.

*Purchased.*

Record of Family Faculties. 4to. 1884.

**Greenwood (Dr. M.), Jr., and Frances Wood.**

On changes in the recorded mortality from Cancer and their possible interpretation. 1914.

*Dr. Greenwood.*

The Relation between the Cancer and Diabetes Death-rates. 1914.

**Greenwood (Dr. M.), Jr., and G. Udny Yule.**

The Statistics of Anti-typhoid and Anti-cholera inoculations, and the interpretation of such Statistics in general. 1915.

*Dr. Greenwood.*

**\*Halsbury (Rt. Hon. Earl of).**

The Laws of England. Being a complete Statement of the Law of England. With Index and Supplement. 32 vols. 8vo. 1907–1918.

*Purchased.*

**\*Hanson (A.).**

The Acts relating to Estate, Increment Value (on Death), Legacy, Succession, and Probate Duties. With Supplement. 6th edit. 8vo. 1911–15.

*Purchased.*



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(when not purchased).*

- \*Hart (H. L.).  
The Law of Banking, with an Appendix on the Law  
of Stock Exchange transactions. 3rd edit. 8vo. } *Purchased.*  
1914.
- Hawtrey (R. G.).  
Currency and Credit. 8vo. 1919. *Purchased.*
- Henderson (R.), and Others.  
Graduation of Mortality and other tables. 8vo. New York. 1919. } *The Actuarial  
Society of  
America.*
- Hoffman (Dr. F. L.).  
Pauper Burials and the Interment of the Dead in  
large cities. 8vo. Atlantic City, N.J. 1919. } *The Author.*
- Holland.  
Archief voor de Verzekerings-Wetenschap. 1918-19. *The Society.*
- Hunter (A.).  
An Analysis of Claims for Total and Permanent  
Disability Benefits. New York. 1918. } *The Author.*  
Gastric and Duodenal Ulcers. Mortality after opera-  
tion. New York. 1919.
- \*Indermaur and Thwaites.  
Principles and Practice in matters of, and appertaining  
to, Conveyancing. By C. Thwaites. 3rd edit. } *Purchased.*  
8vo. 1910.
- Insolera (F.).  
Sulla misura dell' aumento di mortalità per effetto  
indiretto della guerra. Rome. 1919. } *The Author.*
- Institute of Bankers.  
Journal of the, 1918-19. *The Institute.*
- Insurance Institute of America.  
Proceedings of the Tenth Conference. 8vo. 1918. *The Institute.*
- Insurance Institute of New South Wales.  
Transactions, 1918. *The Institute.*
- Insurance Institute of Toronto.  
Proceedings, 1918-19. *The Institute.*  
Containing, *inter alia*—  
“Present-day tendencies in Insurance”, by  
G. D. Finlayson.  
“Group Insurance”, by W. J. Graham.  
“Methods of Valuation for Life Insurance  
Securities”, by D. E. Kilgour.
- Isserlis (L.).  
The application of Solid Hypergeometrical series to  
Frequency Distributions. 1914. } *The Author.*  
The variation of the multiple Correlation Coefficient in  
samples drawn from an Infinite Population with  
normal Distributions. 1917.  
On the value of a man as calculated from a sample.  
1918.

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(when not purchased).*

**\*Jelf (A. E.).**

Where to find your Law. Being a discursive bibliographical essay upon the various divisions and sub-divisions of the Law of England and the Statutes, Reports of Cases, and Text-Books containing such Law. 3rd edit. 8vo. 1907. } *Purchased.*

**Joffe (S. A.).**

Calculation of eighteen more, fifty in all, Eulerian numbers from Central Differences of zero. 1919. } *The Author.*

**King Edward's Hospital Fund for London.**

Report of a Sub-Committee of the Executive Committee on Pensions for Hospital Officers and Staffs. Fol. 1919. } *W. J. H. Whittall.*

**Laughlin (Dr. J. L.).**

Credit of the Nations. A study of the European War. 8vo. 1919. } *Purchased.*

**Life Offices' Association.**

Minutes, etc., of the Meetings of the Standing Committee, 1918-19. } *The Association.*

**London Mathematical Society.**

Proceedings, 1918-19. } *The Society.*

**Marshall (A.).**

Industry and Trade. 8vo. 1919. } *Purchased.*

**Moir (H.), and Others.**

Sources and Characteristics of the principal Mortality Tables. 8vo. New York. 1919. } *The Actuarial Society of America.*

**National Health Insurance.**

Reports of Medical Research Committee. 8vo. 1918-19. } *Purchased.*

**Parliamentary Papers.**

Acts of Parliament. Income Tax, 1918. } *Purchased.*

Assurance Companies. Returns to the Board of Trade. 1918. } *The Board of Trade.*

**Colonies.**

**Canada.**

Report of the Superintendent of Insurance for the year 1917. } *The Government Insurance Dept.*  
Insurance Companies. Abstract of Statements for the year 1918. }

**New South Wales.**

Friendly Societies, &c. Report of the Registrar for 1918. } *The Government of N.S.W.*  
Official Year Book, 1918. }  
Statistical Register for 1917-18 and previous years. }  
Vital Statistics. Report for 1918 and previous years. }

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(when not purchased).*

Parliamentary Papers—*continued.*

Colonies—*continued.*

New Zealand.

- |   |   |                                   |
|---|---|-----------------------------------|
| Government Insurance Department. Triennial Investigation, 1917. | } | <i>The Government<br/>of N.Z.</i> |
| Annual Report, 1917.  |   |                                   |
| Accident Branch. Report, 1917.                                  |   |                                   |
| Census, 1916, Parts II and IV. Fol. Wellington. 1919.           |   |                                   |

Victoria.

- |   |   |  |
|---|---|--|
| Friendly Societies. Fortieth Annual Report of the Government Statist, 1917. | } | <i>The Government<br/>of Victoria.</i> |
| Twenty-eighth Annual Report of the Registrar, 1917.                         |   |  |

Western Australia.

- |   |   |                                   |
|---|---|-----------------------------------|
| Friendly Societies. Report of Proceedings by the Registrar for the year ended 30 June 1918. | } | <i>The Government<br/>of W.A.</i> |
| Statistical Register, 1917-18.  |   |                                   |

Currency and Foreign Exchanges.

- |  |                   |
|--|-------------------|
| First Interim Report of Committee on Currency and Foreign Exchanges after the War. 1918. | <i>Purchased.</i> |
|--|-------------------|

Registrar-General. England.

- |  |   |                                    |
|--|---|------------------------------------|
| Eightieth Annual Report of Births, Deaths, and Marriages, 1917.  | } | <i>The Registrar-<br/>General.</i> |
| Supplement to the 75th Annual Report of Births, Deaths, and Marriages. Part III. Registration Summary Tables (1901-1910), Fol. 1919. |   |                                    |
|  |   |                                    |

India.

- |   |                         |
|---|-------------------------|
| Life Assurance Companies. Returns of Companies doing business in British India, 1917. Fol. Simla. 1918. | <i>H. G. W. Meikle.</i> |
|---|-------------------------|

Periodicals.

- |  |         |                                      |
|--|---------|--------------------------------------|
| Accountants' Magazine.                         | }       | <i>Purchased.</i>                    |
| American Bankers' Magazine.                    |         |                                      |
| American Economic Review.                      |         | <i>The Editor.</i>                   |
| Bankers' Magazine.                             |         | <i>Purchased.</i>                    |
| Commercial and Financial Chronicle. 1895-1913. | }       | <i>O. T. Falk.</i>                   |
| 38 vols. Fol. New York.                        |         |                                      |
| Economist.                                     |         | <i>Purchased.</i>                    |
| Insurance Record.                              | }       | <i>The Editor.</i>                   |
| Journal of Political Economy. (Chicago).       |         |                                      |
| Post Magazine.                                 | }       | <i>Institute of<br/>Secretaries.</i> |
| Post Magazine Almanack.                        |         |                                      |
| The Secretary.                                 | <i></i> |                                      |

\*Ringwood (R.).

- |  |                   |
|--|-------------------|
| The Principles of the Law of Bankruptcy. Embodying the Bankruptcy Act, 1914. 12th edit. Svo. 1915. | <i>Purchased.</i> |
|--|-------------------|

Royal Astronomical Society.

- |  |                     |
|--|---------------------|
| Memoirs of the. Vol. LXIII, Part III. 4to. 1918. | <i>The Society.</i> |
|--|---------------------|

Statistical Society (Royal).

- |                          |                     |
|--------------------------|---------------------|
| Journal of the, 1918-19. | <i>The Society.</i> |
|--------------------------|---------------------|

*By whom presented  
(when not purchased).*

## Sweden.

Enskilda Försäkringsaustalter ar 1917, av Kungl. } *The Swedish*  
Försäkringsinspektionen. } *Government.*

"Thule" Lifförsäkrings-aktiebolaget. Fyrtio års } *The Company.*  
verksamhet, 1873-1912. Also Souvenir Volume. }  
4to. Stockholm. 1917-18.

## Switzerland.

Rapport du Brueau Fédéral des Assurances sur les } *The Swiss*  
Entreprises privées en matière d'Assurances en } *Government.*  
Suisse. 1916.

## Teachers' Insurance and Annuity Association of America.

Handbook of Life Insurance and Annuity Policies for } *W. J. H. Whittall.*  
Teachers. Sm. 8vo. New York. 1918.

## \*Underhill (A.).

The Law relating to Trusts and Trustees. 7th edit. } *Purchased.*  
Svo. 1912.

## United States of America.

## Official Publications.

## Connecticut.

Fifty-fourth Annual Report of the Insurance } *The Commissioner.*  
Commissioner, 1919.

## Massachusetts.

Sixty-third Annual Report of the Insurance } *The Commissioner.*  
Commissioner (Life and Miscellaneous), }  
1917.

## Wisconsin.

Report of the Special Committee on Social } *Dr. F. L. Hoffman.*  
Insurance. Madison, Wis. 1919.

## Vega (G. F. von).

Logarithmisch-trigonometrische Tafeln. 8vo. Leipzig. } *Mrs. E. E. Kimber.*  
1914.

## \*Walker (Hon. W. G.) and E. J. Elgood.

A Compendium of the Law relating to Executors and } *Purchased.*  
Administrators. 4th edit. 8vo. 1905.

## \*Wharton (—).

Law Lexicon. Forming an epitome of the Laws of } *Purchased.*  
England under Statute and Case Law. By E. A. }  
Wurtzburg. 12th edit. 8vo. 1916.

## Williams (Caroline E.)

A Welsh Family. 8vo. 1893. } *Mrs. R. H. T.*  
 } *Williams.*

## \*Williams (Joshua).

Principles of the Law of Real Property. 22nd edit. } *Purchased.*  
Svo. 1914.

## Withers (Hartley).

War-time financial problems. 8vo. 1919. } *Purchased.*

*By whom presented  
(when not purchased).***Yule (G. Udny).**An introduction to the Theory of Statistics. 5th edit. }  
Svo. 1919. }*Purchased.*

\* For Reference only.

Additional copies of works already in the Library :

**Farr (Dr. W.).**

Vital Statistics. Svo. 1885. }

*Mrs. Vyryan  
Marr.***Hammond (H. P.).**Life Insurance in Groups, 1912-17. St. Paul, Min. }  
1917. }*The Author.***Karup (J.).**Reform des Rechnungswesens der Gothaer Lebens- }  
versicherungsbank. La. Svo. Jena. 1903. }*Mrs. Vyryan  
Marr.***Todhunter (I.).**A History of the Mathematical Theory of Probability. }  
Svo. 1865. }*Mrs. Vyryan  
Marr.*

## THE INSTITUTE OF ACTUARIES.

## ENCOURAGEMENT OF ACTUARIAL RESEARCH.

THE Council of the Institute has resolved that prizes may in future be given out of the Prize Funds for any work of exceptional merit. To help those who may wish to undertake original work, the appended list of subjects requiring investigation has been prepared.

Papers submitted should be concise ; many of the subjects indicated are capable of sub-division and a part only of any one chosen may be discussed.

Prizes may also be given for original work on other subjects. To prevent a number of persons working at the same subject, or perhaps in suitable circumstances to arrange for co-operation, it would be well for those who contemplate undertaking work on the subjects indicated to communicate privately with one of the Honorary Secretaries.

*List of Subjects.*

## MATHEMATICAL.

Rigid proof of circumstances in which the method of *J.I.A.*, vol. xlv, pp. 293, &c., holds.

Relative accuracy of various two-variables interpolation methods.

## MORTALITY AND OTHER TABLES.

How is selection modified by the withdrawal of healthy lives and other influences ?

A discussion of the various aggregate tables that can be evolved by various aggregations of select rates of mortality and the uses to which such work could be put.

The effect of catastrophes on mortality, sickness, &c., and their treatment in constructing mortality and other tables (*e.g.*, War, Plague in India, &c.).

Sickness and mortality rates according to disease.

## SOCIAL.

The effect of occupation on mortality, sickness, birth rate, &c.

Studies in population density, emigration, &c., in various countries.

Fertility in relation to occupation, locality, duration of marriage, ages at marriage, &c.

Infantile mortality in relation to birth rate, housing, wages, order of birth, &c.

Effect of heredity on mortality and disease.

Wages and cost of living compared with size of family. &c.

Taxable capacity of a country and its connection with social schemes and present liabilities.

## INSURANCE VALUATIONS, PRACTICE, &amp;c.

If selection be exaggerated by a table of mortality, what errors will be introduced into the premiums charged or into a valuation, (a) by the select, (b) by the full aggregate table ?

An assurance company charges premiums to give a certain form of bonus and its experience is exactly as anticipated. What surpluses would be revealed by valuations assuming various mortality tables based on that assumed in the premiums (*e.g.*, full aggregate table, ultimate table, full aggregate made up from the select table in different proportions) ?

General basis of non-profit rates in view of growing expenditure tax, &c.

Statistics and treatment of under-average lives.



The effect on surpluses of continuous fall (or rise) in interest or of gradual changes in expenses, or mortality or other factors.

The effect on successive surpluses of a life assurance or other fund of unduly stringent valuations.

#### PENSION FUNDS.

A discussion of cases in which mortality may be heavy, then decrease and become normal, and its effect on pension funds, &c.

The effect of changes in salary scale on liabilities of pension funds.

The dangers of using aggregate tables based on past experience for valuing pension funds, &c. (*i.e.*, aggregate tables of rates of lapse, retirement, mortality, &c.).

Size of family and ages of children, &c., in relation to orphans' pensions.

The application of various money-purchase systems to the provision of pension funds.

#### TABULATION.

What is the best system of tabulating functions ?

The tables based on British Offices mortality experience omit many useful rates of interest and helpful functions. Are individual ages necessarily the best division ? Special points arise in connection with varying insurances and benefits dependent upon a number of lives (*e.g.*, last survivor annuities for equal ages). The object is to ascertain the most useful tables within certain defined limits.

#### FINANCIAL.

Statistical studies on the relationship between changes in currency conditions and the rate of interest.

Statistical studies of the changes in the relations between prices of securities of different types and in different markets.

#### GRADUATION.

The tests of a good graduation. Can a mathematical test be given to show that a suggested graduation is so far from the facts as to be "impossible" ?

Studies in graduation by new methods or variations of old methods—specially as regards select mortality, sickness, &c.

Examples of types of material which particular graduation methods fail to meet.

The relative merits either from a practical or theoretical standpoint of various systems of frequency curves.

#### GENERAL.

New approximate methods of valuation, calculation, &c.

## LECTURES ON FINANCE.

THE following is the Syllabus of the Lectures to be delivered by Prof. Foxwell at Staple Inn Hall during the current session :

Markets, Speculation and the Function of the Dealer: an omitted chapter in the text-books of economics. Nature and definition of a Market.

Conditions of the efficiency of a great speculative market. Standardisation. Warrants. Dealing in Futures.

Nature and general economic utility of speculative dealing: how distinguished from gambling. Abuses of speculation. Publicity as a remedy.

Recent attacks on the speculative markets. The German Bourse Law of 1896. Legislation against Time dealings.

The Stock Exchange the greatest of all markets: except that the machinery of standardisation, not required, is absent. An essential part of the banking system and money market. Constitution of the London Market. Provincial exchanges.

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Classification of Securities. Bonds and Industrials. Trustee Securities *v.* Speculative Counters. Debentures and Preferred Stocks, &c. Bearer Bonds *v.* Registered. Terminable *v.* Non-Terminable Loans. Saleability on Foreign Markets. Accrued Interest. New York parity, &c.

General causes that determine the rise and fall in the values of Securities. Habits of Saving and Consumption. Field of Employment. Effect of movements in the general level of prices. Effects of Money Market and foreign exchange conditions.

Other considerations relating to the problems of investment. Effect of Cyclic Variations. Effect of War conditions. Effect of Market movements. Relative stability of securities. "Floaters" sensitive. Policy of "Nursing" loans.

Theoretical presumptions less valuable than they used to be : usually over-discounted. Intrinsic *v.* Market values. The Market in some respects less efficient than it was : probable reasons. Is Government control as exercised in some foreign countries desirable here ? The market has shown itself at least as patriotic and well-informed as the Government. Publicity the simplest and most effective form of control.

BOOKS.—*The most essential are starred. Others will be referred to in the course of the Lectures.*

\*Prof. H. C. EMERY.—“Speculation on the Stock and Produce Exchanges of the U.S.” *New York*. (1896).

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E. S. MEADE.—“Trust Finance.” *New York. Appleton*.

W. J. GREENWOOD.—“Foreign Stock Exchange Practice and Company Laws.” 36, *Camomile Street, E.C.3.* (1911).

## Obituary.

GEOFFREY YATES HEALD, Probationer of the Institute, Captain, 15th Battalion, Lancashire Fusiliers.

*Killed in Action 1 July 1916.*

## THE NATIONALITY OF TETENS.

We regret that, owing to an oversight, for which Dr. Steffensen was in no way responsible, the birthplace of Tetens was stated on p. 309 to be Kiel. From the late Mr. Frederick Hendriks's article in *J.I.A.*, vol. i, p. 14, it appears that he was born at Tetensbüll, Eiderstädt, in the Duchy of Schleswig.—Eds.

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## ERRATUM.

*J.I.A.*, vol. li, p. 134, line 30,  $i+d$  should be  $\frac{i+d}{2}$ .

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# JOURNAL

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"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament therunto."—BACON.

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# JOURNAL

OF THE

## INSTITUTE OF ACTUARIES.

*The Reversionary (or Prospective) and Collective Methods of Valuing Widows' Funds, with some Notes on the Valuation of the Church of Ireland Widows' and Orphans' Fund.*  
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[Read before the Institute, 24 November 1919.]

### INTRODUCTORY.

EVERY Church of Ireland clergyman whose stipend is payable through the Representative Church Body (the body in whom the funds of the Church were vested at its dis-establishment) must become, and is designated, a compulsory member of its Widows' and Orphans' Fund, which was founded in 1876. If the member be unmarried and not more than 35 years of age at date of entry he is charged the normal subscription of £6 per annum, payable quarterly; while, if he be married or aged more than 35 years at date of entry, he must pay such subscription as the Board of the Fund, by the advice of its actuary, may determine.

But in any case where the quoted subscription exceeds £6, the member may elect to pay the smaller contribution, becoming, if he so elects, a "partial" contributor, whose widow and orphans are entitled to receive annuities reduced in such proportion to the full annuities as £6 bears to the quoted contribution.

Clergymen whose stipends are not payable by the Representative Body are allowed to join as voluntary contributors on such terms as the Board may require, and members entering the service of the Church of England, or going abroad as missionaries, are transferred to the voluntary class.

Members joining the voluntary class, and compulsory members married or aged over 35 years at entry, must submit to medical examination.

There are no entrance fees, marriage taxes or disparity fines.

The benefits are :

- (a) A widows' annuity, terminating on re-marriage ;
- (b) An annuity to each orphan, ceasing at age 21, or, in the case of a girl, at earlier marriage. If there be no widow alive at the death of the member, each orphan's annuity falls to be increased by one-fifth of the amount which would have been payable if there had been one, not more than five children, of course, participating in this increase at any one time.

The progress of the Fund may be shown as follows :

Date	Number of Members	Number of Widows	Widow's Annuity	Number of Orphans	Orphan's Annuity*
			£		£
31 December 1877 ...	24	...	...	...	...
"    1886 ...	240	3	33	8	5
"    1891 ...	427	6	33	18	5
"    1896 ...	600	9	33	21	5
"    1901 ...	754	20	38	45	5
"    1906 ...	906	30	38	50	5
"    1911 ...	1,111	47	40	46	5
30 September 1916 ...	1,216	72	45	49	5

\* Subject to increase as stated.

Of the 1,216 members of the Fund in 1916, there were 191 in the voluntary class.

Eight valuations of the Fund have been made, the first in 1881, the last in 1916. I have been responsible for the last three.

In addition to a valuation by its own actuary, the Board has on occasions requisitioned outside advice, and thus it comes that the late Mr. Meikle reported on some of the earlier investigations, and that Mr. George King has reported on the 1901, 1906 and 1911 valuations.

I cannot say what the origin of this somewhat unusual



practice was ; but, as to its continuance, the Board may have wished to support itself by outstanding authority when it found that, in consequence of the normal increase of the Fund, it was subjected in the General Synod, the Parliament of the Church, and in the clerical press to hostile and ignorant criticism.

The consequence of this practice is that I have had the privilege of having been associated with Mr. King in important work, and all the processes employed by me in my valuations, and to be described hereafter, have been approved by him.

It is perhaps due to Mr. King that I should state here that this paper has been prepared without any reference to him, and that I am solely responsible for any opinions expressed or comments made.

#### VALUATION BY REVERSIONARY (OR PROSPECTIVE) METHOD.

A Widows' Fund may be valued by the reversionary annuity method or by what is indifferently known as the collective or varying assurance method.

The procedure to be followed under the former plan may perhaps best be illustrated by reciting the benefits to be valued when it is adopted, and by indicating formulas, suitably modified to meet in small matters of detail the circumstances of each particular fund, which may with substantial accuracy be employed in their valuation.

Apart from annuities to existing widows, which if we ignore, as is usually done, the contingency of re-marriage of the annuitants, present no difficulties, the following benefits have to be valued :

*Benefit (i).*—Annuities to existing wives who may be left widows. Valuation formula :

$$\bar{a}_y - \bar{a}_{xy}$$

where  $y$  is the nearest age of the wife, and  $x$  the nearest age of the husband at date of valuation.

*Benefit (ii).*—Annuities to widows left by first marriages of present bachelors. Valuation formula :

$$\frac{1}{(bl)_x} \sum (bm)_{x+n} v^{n+\frac{1}{2}} \bar{a}_{x+n+\frac{1}{2}/s}$$

where  $(bm)_{x+n}$  denotes the number of bachelors out of  $(bl)_x$  alive at age  $x$  who marry in the year of age  $x+n$  to  $x+n+1$ .  $s$  denotes the average age at date of marriage of the wives of bachelors who marry in the year of age  $x+n$  to  $x+n+1$ .

The assumption is made that marriage takes place on the average in the middle of the year.

*Benefit (iii).*—Annuities to widows left by next marriages of widowers. Valuation formula :

$$\frac{1}{(wl)_x} \sum (wm)_{x+n} v^{n+\frac{1}{2}} \bar{a}_{x+n+\frac{1}{2}/r}$$

where  $(wm)_{x+n}$  denotes the number out of  $(wl)_x$  widowers alive at age  $x$  who re-marry in the year  $x+n$  to  $x+n+1$  on the average in the middle of it.  $r$  denotes the average age at date of marriage of the wives of widowers re-marrying in year of age  $x+n$  to  $x+n+1$ .

*Benefit (iv).*—Annuities to widows left by the next marriages of presently married men. Valuation formula :

If  $p$  be the average age of the existing wives of members aged  $x$ , and if we denote the value of Benefit (iii) in the case of a widower aged  $x$  by  $(wm\bar{a})_x$ , the formula for this benefit may be written

$$\frac{1}{l_x l_p} \sum l_{x+n+\frac{1}{2}} d_{p+n} v^{n+\frac{1}{2}} (wm\bar{a})_{x+n+\frac{1}{2}}$$

A formula can be devised on lines already indicated for annuities to widows who may be left by second marriages of bachelors, but even then there still remain annuities to widows left by second and subsequent re-marriages of widowers, and by third and subsequent marriages of bachelors and husbands.

But while there should be no great difficulty in writing down theoretical formulas for the valuation of these benefits, the actual compilation of tables based thereupon would be extremely laborious. Moreover, the data which any fund can afford as to these subsequent marriages must be so insignificant as to make it useless to work them up into tables. If, therefore, in a valuation by the reversionary method any estimate of the cost of these remote liabilities is made, it would seem to be inevitable that it be made by some arbitrary plan.

An examination of the preceding formula shows that, if it be decided to value a fund by the reversionary method, and by the data which it can itself supply, a series of very lengthy operations must be undertaken. And it will be equally clear that if the fund be not very large, or very old, and if in either case its records have not been completely and accurately kept, the reversionary method necessitates the employment of the

published tables of Huie and Hewat, which may have little in common with the features of the fund.

This valuation plan certainly has the outstanding merit that it enables prospective annuities to existing wives (generally much the largest portion of the liabilities of any fund) to be valued with ease and celerity by a formula which eliminates all contingencies save mortality.

All the valuations of the Church of Ireland Fund up to 1901 inclusive were made by the reversionary plan with Carlisle mortality and Huie's tables, replaced as far as possible in 1901 by Hewat's Scottish Ministers' tables.

For many years, therefore, the periodical testings were made by the employment of an alien experience, and no use whatever was made of any features which the fund itself possessed.

### COLLECTIVE METHOD.

The last three valuations of the fund were made by the collective method.

This does not concern itself with individual cases, ignores class distinctions and does not occupy itself with the separate consideration of first, second or  $n$ th marriages.

Starting with the proposition that if a member be a married man at the date of his death he will leave a widow, and that, if he be a married man or widower when he dies, he may leave children, it elaborates the argument by combining the probability of marriage (or parenthood) with the chance of death, multiplies the product into the value of the benefit as at the date when it may be entered upon, discounts the entire at interest, and brings out a result which is a function of the present age of the member only.

If  $W_x$  denote the value of an annuity of 1 to the widow who may be left by a member now aged  $x$ , then the processes of the Collective Method may perhaps be most succinctly shown by quoting the formula given by King in his paper on Family Annuities (*J.I.A.*, vol. xxx, p. 308).

$$W_x = \frac{1}{l_x} \int v^t l_{x+t} \cdot \mu_{x+t} h_{x+t} \cdot \bar{a}_{y+t} dt$$

where  $h_{x+t}$  is the probability that at the moment of age  $x+t$  a member will be a husband and  $y+t$  is the then age of his wife.

In evaluating this integral by any formula of approximate

summation it is to be noted that, from the manner in which the average ages of the wives are obtained—either by recording them as at the date when they become widows, or by taking the average of the ages of the wives of existing members—it does not necessarily, or even probably, follow that, if  $y$  be the age of the wife of a husband aged  $x$ ,  $y + w$  will be the age of the wife of a member aged  $x + w$ .

In practical work it will be better, perhaps, to discard the calculus and to employ a formula which will lend itself to the formation of commutation columns.

Widows' annuities in the compulsory class were valued by the following formula at the last three investigations :

$$W_x = \frac{v^{\frac{1}{2}}}{l_x} \{d_x k_x \bar{a}_b + v d_{x+1} k_{x+1} \bar{a}_c + \dots\} \quad (I)$$

where

$x$  = age nearest birthday of member at date of valuation,

$k_x$  = probability of his being married at age  $x + \frac{1}{2}$ , the age at which, on the average, death will take place if it occur in the year  $x$  to  $x + 1$ ,

$b, c, d, \&c.$  = the average age plus one half-year at date of widowhood of the widows left by members dying at age  $x, x + 1, x + 2, \&c.$ , or of the existing wives of members aged  $x, x + 1, x + 2, \&c.$ , nearest birthday.

This formula may readily be thrown into commutation form by multiplying numerator and denominator by  $v^x$ .

In strictness the several values of  $\bar{a}$  employed in (I) should be those of annuities ceasing at re-marriage ; but in his paper on *The Valuation of a Widow's Annuity* (T.F.A., vol. ii, p. 307), the late V. Marr has shown that the substitution for an annuity ceasing at re-marriage of an ordinary life annuity makes very little change in the value of  $W$ . Such small change as the substitution does make is in the direction of safety, a direction always very popular with the actuarial mind.

Having provided ourselves with a complete table of  $W$  for every integral age of  $x$ , all that is necessary in order to make a complete valuation of the liabilities of a Widows' Fund (apart from existing annuities) is to multiply the total number of members on the fund at each nearest birthday, bachelors,

married men and widowers combined, into the appropriate values of W.

#### COMPARISON OF METHODS.

The superiority of the new over the old method would seem to be great, and it is not easy to discover any line of criticism which would be destructive if applied to one and harmless if addressed to the other.

Examination of (I) shows that, apart from mortality, the assumptions involved are either (i) that in every future year the proportion of bachelors, husbands and widowers dying at any age to total members dying at such age, and the average ages of wives on becoming widows, will be the same as were shown in the table used in the valuation, or (ii) that the ratios bachelors, married men and widowers to total members existing at each age at date of valuation, and the average ages of the wives of members existing at that particular period shall not vary in succeeding years.

It may be advanced in criticism of the collective method that these ratios and average ages may not obtain in the future. But, surely, a very similar objection may be urged against the reversionary method, which requires for its complete validity that the proportion of bachelors marrying, and widowers re-marrying, in the future shall not differ from that disclosed by past experience, as well as that the average ages at marriage of the wives to be taken by existing bachelors and widowers shall be the same as those of the wives taken by past bachelors and widowers.

Any canvassing of the merits of the reversionary method should, to be pertinent, undertake to show that its underlying assumptions are more likely to be realized than are those implicit in the collective plan. This undertaking does not heretofore appear to have been essayed, and would seem to be a difficult task.

#### DESCRIPTION OF ACTUAL VALUATION AND EXPERIENCE.

All the collective valuations of the Church Fund were based upon "living" experiences, formula (I) being used, mortality only being assumed.

In 1906  $O^{M(5)}$  mortality was assumed for male lives and B.O.A. select for females. At the succeeding valuations the rates of mortality adopted were those shown by King in his Report to the Treasury on the Elementary Teachers' Deferred



Annuity Fund (1907). The rates of mortality for male lives shown in this Report were based, for ages below 40, upon the experience of abstainers and Church of England clergymen, for ages 40 to 70 upon the actual experience of the Deferred Annuity Fund. for ages 71 to 75 they are 90 per-cent of the B.O.A. ultimate rates, and from 76 on, they are the full B.O.A. ultimate rates.

As regards female mortality "below age 40 the recorded "female rate of mortality was increased. . . . For ages 40 "to 70 the experience of the fund itself was taken, and for "ages above 70 the curve was joined on to that of the "British Offices." (*Vide* King's report.)

The following are specimen annuity values at  $3\frac{1}{2}$  per-cent, based upon the rates of mortality above referred to.

Age $x$	VALUES OF $a_x$	
	Males	Females
25	20.952	21.792
35	18.949	19.825
45	16.030	17.035
55	12.592	13.662
65	8.899	9.921
75	5.398	6.091

I may mention that, remarkably low as are the rates of mortality shown in King's Report, a comparison I made between the actual male deaths in the five years 1907-1911 and those expected showed the mortality amongst Church of Ireland clergymen to be lower still.

The preliminary data as to marital condition available at the last three valuations of the Widows' Fund are shown in Table I. The figures in the "married" column of this table at any particular age divided by the total number of members at that age, give us the ungraduated probability of a member being a married man. Thus, the 1916 figures show that in the compulsory class, the probability of being a husband at age 40 is  $\frac{27}{38}$  or .7105.

The ungraduated average age last birthday of the wife of a member of nearest age  $x$  at date of valuation was found by summing the ages of all the wives of husbands aged  $x$  and dividing the sum by the number of husbands. Thus, in the 1916



compulsory class, the total ages of the 23 wives of husbands aged 50 amounted to 1,023 years, which number divided by 23 gives the ungraduated average age of the wife of a husband aged 50 as 44·5 years.

Some of these ungraduated average ages are shown in the following table :

TABLE A.

*Showing the ungraduated average Age last birthday of the Wife of a member of nearest age  $x$  deduced from the data of the Church of Ireland Fund.*

Age of Member ( $x$ )	AVERAGE AGE OF WIFE					
	Compulsory Class			Voluntary Class		
	1906	1911	1916	1906	1911	1916
25	26·0	33·0	26·0	...	...	...
30	26·7	32·0	29·9	...	...	20·
35	31·6	32·3	33·3	31·0	31·5	...
40	37·6	33·8	35·9	38·7	40·0	36·0
45	40·9	41·0	38·3	42·7	44·0	45·3
50	46·0	44·4	44·5	54·3	48·3	48·7
55	52·2	51·3	48·0	47·6	50·3	49·7
60	47·7	51·6	53·6	59·0	52·5	51·7
65	...	59·0	53·2	53·0	...	54·8
70	...	...	52·0	69·0	58·0	73·0

All the data provided by the Church of Ireland Fund were graduated by the summation method, the procedure followed being that described by King in *J.I.A.*, vol. xlii, p. 225, and vol. xliii, p. 109, and by his formula (*va*) the following graduated "pivotal" ages of wives were derived (1916 Valuation, Compulsory Class) :

Husband's Age (nearest)	Wife's Average Age (last birthday)
32	30·7
37	33·3
42	37·5
47	40·9
52	46·2
57	49·5
62	56·7

For valuation purposes it is, of course, necessary to have the probability of being a married man,  $h_x$ , and the average age of the wife, for every age of the member up to the limiting age

of the life table. So far, however, the Irish Church Fund only supplies these particulars up to late middle-age, and those for higher ages had, perforce, to be fixed arbitrarily.

In 1916 it was, therefore, assumed that in the compulsory class  $h_x$  would remain constant from age 62 at '85, and that from age 67 on there would be a constant difference of six years in the ages of husbands and wives.

Apart from the consideration that, thereby, the stringency of the valuation is increased, there is certainly no reason why  $h_x$  should be fixed at a high constant from middle-age to the end of life. *A priori* reasoning suggests that the curve of this probability should gradually ascend to a maximum and then gradually descend, and confirmation that this is so is afforded by Schjoll's "Norwegian Statistics" (*J.I.A.*, vol. xl, p. 209), and by Marr's Tables.

Equally we should not expect that over any considerable period the difference between the ages of husbands and wives could be represented by a constant, and here, again, confirmation is lent by Schjoll and Marr, who show this difference to be an increasing one throughout life.

The effect of the assumptions made would therefore seem to be that one made for overstating the liability and the other for understating it, with the result, it is hoped, that the true liability has been stated with substantial accuracy.

I have found Buchanan's interpolation formula (*J.I.A.*, vol. xlii, p. 369), to be productive of considerable economy of time and labour, and I have therefore used it for all the interpolations necessary in the preparation of this paper.

An outstanding merit of the collective method is contained in the readiness with which data supplied on preceding occasions can be incorporated with those furnished by a present investigation whereby the experience can be made to become cumulative and the bases whence we reason can be enlarged and strengthened. This is a matter to which no previous reference would appear to have been made, although it is of considerable importance.

The insistent demand of actuarial science is for data, and I think that the only ground upon which an actuary would be relieved of the duty of incorporating in a present investigation data furnished on preceding occasions is that the latter have ceased to be homogeneous.

In the following table are shown quinquennial ungraduated

and graduated wives' ages and similar values of the probability of being a married man (husband), all based upon the combined experience of the last three valuations of the compulsory class of the Irish Church Fund.

It was assumed that at age 67 the graduated wife's age was 60; that from age 72 inclusive of husband the latter would be eight years older than his wife, and that from age 67 inclusive the probability of being a married man would be constant at .7.

Commencing the summations in fives at age 25 graduated quinquennial values of wives' ages for husband's ages 32, 37 and so on, were obtained by King's formula ( $va$ ), and all the interpolations were made by Buchanan's formula already referred to.

By assuming that the age of the wife of a husband aged 27 years was also 27—an assumption which the data show to be a safe one—I was enabled to interpolate the wives' ages from age 33 on; and by means of the graduated values at ages 27, 32, 33, 34 and 35 and a fourth difference formula the values between 27 and 32 were obtained.

At ages below 27 the values are arbitrary.

The interpolations for the marriage probabilities were made in precisely the same manner, the assumption being made that at age 27 the graduated probability was .15.

So far as it goes to show the goodness of the graduation, it may be mentioned that up to age 67, after which the data are too scanty to be reliable, the "expected" number of married men is 1648.87, as compared with 1,648 actual cases.

TABLE B.

*Showing average Ages of Wives and probability of being a Husband ( $h_x$ ) based upon the combined 1906, 1911, and 1916 Valuations of the Church of Ireland Fund.*

Age $x$	WIFE'S AGE LAST BIRTHDAY		$h_x$	
	Ungraduated	Graduated	Ungraduated	Graduated
27	29.86	27.0	.1250	.1500
32	31.00	30.1	.4707	.3998
37	33.91	33.4	.6629	.6093
42	37.35	37.0	.7723	.7327
47	41.14	41.6	.8102	.7792
52	45.38	46.2	.7704	.8116
57	50.39	50.6	.7568	.7950
62	52.68	55.6	.7600	.7292
67	61.50	60.0	.2222	.7000

The following statement of graduated values is submitted in support of the contention that the data of the Church of Ireland Fund continues to be homogeneous and, incidentally, to lend support to the collective method, as indicating that, so far as this particular fund is concerned, its underlying assumptions have been reasonably fulfilled :

Husband's Age $x$	AVERAGE AGE LAST BIRTHDAY OF WIFE $y$				PROBABILITY OF BEING A MARRIED MAN $h_x$			
	1906	1911	1916	Whole Experience	1906	1911	1916	Whole Experience
32	30.2	30.5	30.7	30.1	.4200	.3182	.4420	.3998
37	33.4	33.5	33.3	33.4	.5250	.6510	.6450	.6093
42	37.6	36.4	37.5	37.0	.7351	.7117	.7557	.7327
47	41.7	42.1	40.9	41.6	.8018	.7786	.7608	.7792
52	45.6	46.6	46.2	46.2	.8314	.7835	.8338	.8116
57	...	51.9	49.5	50.6	...	.7829	.8161	.7950
62	...	...	56.7	55.6	...	...	.7765	.7292

Remembering that if a husband aged  $x$  die within the year, his average age at death will be  $x + \frac{1}{2}$ , and that, consequently, the average age of the widow he will then leave will be  $y + 1$ , the following comparison between the wives' ages, as shown by the whole experience of the fund and the widows' ages at date of entry into widowhood as shown in Marr's paper, is interesting:

Husband's Age last Birthday $x$	WIDOW'S AGE	
	Irish Clergy	Scotch Ministers
32	31.1	32
37	34.4	35
42	38.0	40
47	42.6	42
52	47.2	47
57	51.6	50

The smallness of the differences between these figures is remarkable, and this is all the more so when it is borne in mind that Marr's method of obtaining his ages differs from mine in that he dealt with the values of annuities that became payable by the deaths of husbands, and thereby, as it were, "weighted" the widows' ages.

Table II shows the values of W at  $3\frac{1}{2}$  per-cent, based on the combined experience of the last three valuations of the compulsory class of the Irish Church Fund.

I close my remarks on the Widows' Annuity portion of this paper by submitting the results of two valuations, made as at 1916 and with interest at  $3\frac{1}{2}$  per-cent, of the compulsory class of the Fund, the one being made by the reversionary annuity plan and the other by the collective method.

For the reversionary valuation I employed Hewat's Scottish Bankers' Tables, with Huie's for second marriages, and for the collective one the Fund's own 1916 experience with King's Teachers' mortality.

	Reversionary Method	Collective Method
Present value of Annuities of £45 to Widows to be left by present Members ... ..	£194,157	£193,071
Present value of Future Subscriptions	95,283	101,924
Net Liability ... ..	£98,874	£91,147

#### VALUATION OF ORPHANS' BENEFITS.

In the valuation of the orphans' benefits of the Irish Church Fund there are no competing methods. Here the collective plan is in complete possession, being, in fact, the only scientific mode of dealing with the subject.

It may be noted that Huie makes no attempt to value an annuity of the Church Fund's main type, and that he dismisses any attempt to estimate the values of annuities to children to be left by members still unmarried as involving "much uncertainty and far more labour than the object is worth."

All the valuations of liabilities before 1906 under Prospective Benefits to children appear to have been estimated by a process wholly empirical, the underlying assumption of which was that every member that married would procreate four children. King has, however, put this branch of actuarial practice on a scientific basis, and his method has been followed from 1906 inclusive with the Irish Fund.

Annuities to existing orphans were valued by the late H. W. Manly's tables (*J.I.A.*, vol. xxxviii, p. 116), and these tables

also formed the basis of the tables prepared for valuing annuities to prospective orphans.

If  $O_x$  denote the value of a continuous annuity of 1 payable up to a specified age to each child of a family left by a member now aged  $x$ , then

$$O_x = \frac{1}{l_x} \int_0^{\infty} v^t \cdot l_{x+t} \cdot \mu_{x+t} (hw)_{x+t} (fa)_{x+t} dt$$

$$\text{or} \quad = \frac{1}{v^x l_x} \sum_{t=0}^{t=\omega} v^{x+t+\frac{1}{2}} \cdot d_{x+t} \cdot (hw)_{x+t+\frac{1}{2}} (fa)_{x+t+\frac{1}{2}}$$

where  $(hw)_{x+t}$  is the probability of being a husband or widower at age  $x+t$ , and  $(fa)_{x+t}$  is the value at that moment of age of the member of an annuity of 1 to each child he may then possess.

The second form for  $O$  lends itself to the use of the commutation method.

The material in Table I enables us to calculate the values of  $(hw)_{x+\frac{1}{2}}$ , the mean of  $(hw)_x$  and  $(hw)_{x+1}$ , and in the following table are shown values of  $(hw)_x$  based upon the combined experience of the last three valuations, compulsory and voluntary classes together. The graduated values were obtained as heretofore indicated, the value at age 27 having been assumed to be .1575, and that from age 67 on being treated as a constant .9.

TABLE C.

*Showing the probability of being a Husband or Widower.*

Age $x$	$(hw)_x$	
	Ungraduated	Graduated
27	.1147	.1575
32	.4593	.4179
37	.6907	.6470
42	.7904	.7634
47	.8137	.8245
52	.8131	.8610
57	.8269	.8769
62	.8379	.8965
67	.7500	.9000

The "family" experience of the fund, as shown at the last three valuations of compulsory and voluntary classes combined



is amalgamated in Table III, and therefrom we are enabled to obtain unadjusted values of  $(fa)_x$ .

At any age,  $x$ , the unadjusted value of  $(fa)_x$  will be the sum of the values of temporary annuities to all the children possessed by members aged  $x$  divided by the sum of the husbands and widowers living at that age. Thus, if there be  $r$  children aged 0—1,  $s$  children aged 1—2, and  $t$  aged 2—3, &c., the issue of the marriages of all or some of the marriages of the  $n$  husbands and widowers now aged  $x$ , and if  $a'_0$  be the value of a temporary annuity on 0—1,  $a'_1$  that on 1—2, and  $a'_2$  that on 2—3, &c., we shall have

$$(fa)_x = \frac{1}{n} (r \cdot a'_0 + s \cdot a'_1 + t \cdot a'_2 + \text{&c.})$$

In the graduation of  $(fa)_x$  "pivotal" values were obtained for ages 32 to 67, both inclusive, and the mean of Manly's figures in *J.I.A.*, vol. xxxviii, p. 147, were adopted as the adjusted values at ages 72 and 77, at which latter age the series was assumed to close. The mean of the unadjusted values at ages 27 to 29, or 7'163, was taken as the graduated value at age 27, and intermediate values were interpolated.

In Table IV, in addition to a complete series of values of  $O$ , are given the unadjusted and graduated values of  $(fa)$ , all at  $3\frac{1}{2}$  per-cent interest.

A comparison of my values of  $(fa)$  with those of Manly lends strong support to the conviction that the families of the Irish clergy are remarkably small.

And if my values be compared with those furnished by King in the discussion upon Manly's paper, and based upon the 1901 "family" experience of the Irish Church Fund, it will be found that the tendency of families to diminish in size, which S. J. H. W. Allin discovered to exist in those of English Presbyterian ministers (*J.I.A.*, vol. xxxix, p. 337), is in full operation with the families of Irish Church clergymen.

The value of the increase in the orphans' benefit which will take place in the Irish Fund if the father be a widower at the date of his death must be small, and it will no doubt be sufficient to provide for it by some small percentage addition to the value of the main orphans' benefit. If, however, it be thought necessary to value this benefit separately, this may readily be done by compiling  $(fa)_x$  from the record of widowers' children and

replacing the probability of being a husband or widower by the probability of being a widower only.

### MARRIAGE RATES.

In the discussion on Allin's paper Mr. Lidstone showed that the ratios Bachelor : Total Members, say K, may be made to yield the central marriage rates.

By taking out the complements of my probabilities of being a married man or a widower I was enabled to procure graduated values of K. and hence to obtain central marriage rates deduced from the Church Fund's own experience. Specimens of these values are shown below, together with, for purpose of comparison, Hewat's Scottish Bankers' rates :

Age	CENTRAL MARRIAGE RATE	
	Irish Church	Scottish Bankers
30	·0736	·0690
35	·1063	·0693
40	·0727	·0521
45	·0541	·0371
50	·0433	·0246
55	·0204	·0149
60	·0360	·0104

It will be seen that my figures run very irregularly, and this may, perhaps, to some extent be due to secessions, which, although small, are not inconsiderable. Of course it would be quite possible by graduation of my rates to obtain a smooth series. But whether a table of central marriage rates deduced from a "living" experience, no matter how smoothly it be graduated, can be entitled to the same implicit confidence that we extend to a table compiled in the old way, is a matter upon which further investigation is required before a definite opinion can be given.

TABLE I.

*Bachelors, Married Men and Widowers on Church of Ireland Fund at  
1906, 1911 and 1916 investigations.*

Age near- est birth- day	COMPULSORY CLASS								
	1906			1911			1916		
	Bachelors	Married	Widowers	Bachelors	Married	Widowers	Bachelors	Married	Widowers
23	2	...	...	2	...	...	3	...	...
24	12	...	...	7	...	...	8	...	...
25	6	1	...	14	1	...	14	1	...
26	18	2	...	12	...	...	12	...	...
27	14	3	...	23	2	...	12	2	...
28	21	...	1	17	5	...	15	3	...
29	11	2	1	18	6	...	16	4	...
30	20	12	...	20	2	...	12	9	...
31	16	14	...	21	6	...	18	6	...
32	11	5	...	10	13	...	15	14	...
33	17	17	1	21	11	...	15	14	...
34	15	11	...	16	11	1	10	13	1
35	10	12	...	16	20	...	9	12	...
36	24	12	...	10	20	1	10	19	1
37	12	20	...	10	14	...	7	23	...
38	9	22	...	12	31	...	14	15	...
39	11	9	1	9	21	...	10	23	1
40	10	16	1	4	19	...	11	27	...
41	10	21	...	15	24	...	12	25	...
42	4	20	...	10	31	1	8	27	...
43	6	22	...	5	26	1	6	34	...
44	6	19	...	9	13	1	7	21	...
45	3	14	...	5	22	1	5	21	...
46	9	23	2	11	22	2	8	27	2
47	2	14	...	5	18	...	8	32	...
48	4	16	1	4	26	1	5	20	1
49	3	19	...	4	22	...	5	17	2
50	6	12	...	1	12	1	1	23	...
51	1	18	...	8	24	3	8	23	2
52	4	12	...	5	12	...	4	23	1
53	1	10	...	4	17	...	3	26	1
54	1	6	...	3	18	...	3	17	1
55	1	9	1	5	12	...	6	16	...
56	1	5	...	1	18	...	...	25	2
57	2	5	...	3	12	1	3	11	...
58	2	8	1	1	7	1	2	21	1
59	...	1	1	1	5	1	3	10	...
60	...	3	...	3	11	...	4	16	1
61	...	1	...	...	3	...	...	11	1
62	...	...	...	2	6	...	3	13	1
63	...	4	1	2	4	5	...	10	...
64	1	...	...	...	1	1	2	4	2
65	...	...	...	...	2	...	2	11	...
66	...	...	...	...	1	...	2	6	1
67	...	...	...	...	...	1	3	2	3
68	...	...	...	...	3	...	...	4	1
69	...	...	...	1	...	...	...	1	1
70	...	...	...	...	...	...	...	1	...
71	...	...	...	...	...	...	...	1	...
72	...	...	...	...	...	...	...	1	...
73	...	...	...	...	...	...	...	1	...
74	...	...	...	...	...	...	1	...	...

TABLE I—continued.

Age near est birth day	VOLUNTARY CLASS								
	1906			1911			1916		
	Bachelors	Married	Widowers	Bachelors	Married	Widowers	Bachelors	Married	Widowers
23	...	...	...	...	...	...	...	...	...
24	...	...	...	...	...	...	...	...	...
25	...	...	...	1	...	...	...	...	...
26	...	...	...	...	...	...	2	...	...
27	1	...	...	1	...	...	3	...	...
28	...	1	...	...	...	...	2	...	...
29	...	1	...	2	1	...	2	1	...
30	1	1	...	1	...	...	2	1	...
31	...	...	...	1	1	...	1	2	...
32	2	1	...	2	1	...	...	...	...
33	2	3	...	1	1	...	...	3	...
34	2	5	...	...	2	...	3	1	...
35	1	3	...	...	4	...	...	...	...
36	4	3	...	2	2	...	...	4	...
37	...	6	...	...	2	...	1	2	...
38	1	4	1	...	4	1	...	2	...
39	...	4	...	1	8	...	...	3	...
40	1	6	...	1	5	...	2	4	...
41	1	8	...	3	5	...	...	4	...
42	3	8	...	...	8	...	1	3	...
43	...	3	...	...	4	...	1	6	1
44	...	4	...	...	7	...	...	6	...
45	...	6	...	2	3	...	1	3	...
46	...	6	...	...	9	...	1	9	...
47	1	2	1	2	8	...	1	8	...
48	...	8	...	...	2	...	...	6	...
49	...	3	...	1	5	...	...	5	...
50	...	3	2	1	3	1	...	3	1
51	...	4	...	...	8	1	1	10	...
52	1	5	...	...	2	...	...	6	...
53	...	1	...	...	7	...	...	5	...
54	...	3	1	...	4	...	...	3	1
55	...	5	...	1	4	1	1	3	...
56	...	4	...	...	1	1	...	5	1
57	...	5	...	1	6	...	...	2	1
58	...	3	...	...	1	...	...	6	...
59	...	9	...	...	3	1	...	3	1
60	...	1	...	...	4	...	...	3	...
61	...	1	...	...	6	...	...	2	...
62	...	3	...	...	3	...	1	4	1
63	...	2	...	...	4	...	...	2	...
64	...	...	...	...	6	1	...	2	...
65	...	1	...	...	...	...	...	4	...
66	...	1	...	...	1	...	...	6	...
67	...	...	1	...	2	...	...	...	...
68	...	...	...	...	2	...	...	5	...
69	...	1	...	...	...	...	...	5	...
70	...	1	...	...	1	...	...	1	...
71	...	1	...	...	1	...	...	1	...
72	...	...	...	...	1	...	...	2	...
73	...	...	...	...	...	...	...	...	...
74	...	...	...	...	1	...	...	...	...
75	...	...	...	...	...	...	...	1	...
76	...	...	...	...	...	...	...	1	...
79	...	...	1	...	...	...	...	...	...

TABLE II.

Showing the present value ( $W_x$ ) of an Annuity of 1 to the Widow who may be left by any member (now a Bachelor, Married Man, or Widower), to commence at his death. Interest  $3\frac{1}{2}$  per-cent.

Age of Member (nearest birthday) $x$	Age of Wife (last birthday) $y$	$\bar{a}_{y+1}$	$W_x$
(1)	(2)	(3)	(4)
23	...	...	2.158
24	...	...	2.243
25	25	22.126	2.332
26	26	21.954	2.425
27	27	21.776	2.513
28	27.4	21.601	2.595
29	28	21.590	2.676
30	28.7	21.456	2.755
31	29.4	21.318	2.832
32	30.1	21.178	2.908
33	30.8	21.033	2.984
34	31.4	20.906	3.059
35	32.1	20.756	3.134
36	32.7	20.622	3.209
37	33.4	20.463	3.284
38	34.2	20.277	3.358
39	34.8	20.134	3.432
40	35.5	19.961	3.505
41	36.3	19.760	3.577
42	37	19.580	3.647
43	37.9	19.340	3.716
44	38.8	19.092	3.782
45	39.7	18.836	3.846
46	40.6	18.575	3.908
47	41.6	18.277	3.966
48	42.5	18.003	4.022
49	43.4	17.723	4.074
50	44.4	17.407	4.121
51	45.3	17.118	4.163
52	46.2	16.825	4.200
53	47.1	16.530	4.232
54	47.9	16.263	4.260
55	48.8	15.961	4.283
56	49.6	15.691	4.303
57	50.6	15.349	4.317
58	51.5	15.038	4.327
59	52.6	14.655	4.336
60	53.6	14.304	4.344
61	54.7	13.912	4.351
62	55.6	13.588	4.359

TABLE II—*continued*.

Age of Member (nearest birthday) $x$	Age of Wife (last birthday) $y$	$\bar{a}_{y+1}$	$W_x$
(1)	(2)	(3)	(4)
63	56.5	13.262	4.363
64	57.4	12.932	4.367
65	58.2	12.636	4.366
66	59	12.338	4.362
67	60	11.961	4.352
68	60.7	11.695	4.339
69	61.4	11.427	4.317
70	62.3	11.081	4.286
71	63.1	10.771	4.247
72	64	10.421	4.197
73	65	10.029	4.139
74	66	9.636	4.073
75	67	9.243	3.998
76	68	8.850	3.916
77	69	8.460	3.826
78	70	8.072	3.728
79	71	7.690	3.626
80	72	7.313	3.520
81	73	6.947	3.412
82	74	6.591	3.302
83	75	6.248	3.190
84	76	5.919	3.077
85	77	5.603	2.964
86	78	5.301	2.852
87	79	5.012	2.741
88	80	4.735	2.630
89	81	4.467	2.521
90	82	4.211	2.408
91	83	3.965	2.300
92	84	3.729	2.237
93	85	3.503	2.073
94	86	3.288	1.982
95	87	3.083	1.858
96	88	2.889	1.740
97	89	2.704	1.608
98	90	2.529	1.456
99	91	2.363	.603



*Showing the number of Married Men and Widowers and the Ages of their Children, as disclosed by the 1906, 1911 and 1916 Valuations of the Church of Ireland Fund.*

Age	No. of Husbands or Widowers	NUMBER OF CHILDREN AND AGES																				
		0-	1-	2-	3-	4-	5-	6-	7-	8-	9-	10-	11-	12-	13-	14-	15-	16-	17-	18-	19-	20-
25	3	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
26	2	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
27	7	1	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
28	10	3	2	1	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
29	16	5	5	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
30	25	9	3	4	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
31	29	6	3	4	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
32	34	6	5	4	3	1	2	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...
33	50	20	10	9	10	5	4	3	2	...	...	...	...	...	...	...	...	...	...	...	...	...
34	45	11	9	9	5	3	2	3	...	...	...	...	...	...	...	...	...	...	...	...	...	...
35	51	13	15	7	11	3	8	3	2	...	3	...	...	...	...	...	...	...	...	...	...	...
36	61	17	8	16	5	3	6	8	3	...	1	1	...	...	...	...	...	...	...	...	...	...
37	67	9	15	7	15	4	9	11	5	8	3	3	3	...	...	...	...	...	...	...	...	...
38	80	16	14	17	12	15	17	8	8	8	6	2	2	1	...	...	...	...	...	...	...	...
39	68	7	9	7	11	8	10	11	9	10	8	4	2	1	2	...	...	...	...	...	...	...
40	77	13	13	13	12	11	12	13	10	8	7	5	6	4	1	2	...	...	...	...	...	...
41	87	16	11	14	14	12	10	8	16	5	8	7	8	6	3	2	1	2	...	...	...	...
42	99	13	9	9	11	17	12	16	8	18	10	11	15	11	8	9	4	4	1	2	2	...
43	97	10	12	9	12	10	17	13	18	6	15	16	7	7	7	7	1	2	1	1	...	...
44	70	9	6	16	8	11	12	13	10	15	10	10	9	8	11	5	5	4	2	2	...	...
45	70	6	6	3	12	9	14	9	10	12	13	12	8	11	9	5	5	6	2	5	2	...
46	102	3	8	11	10	14	13	11	12	12	14	13	12	14	6	12	10	9	6	6	5	4
47	83	6	8	3	8	6	8	6	11	8	10	15	13	9	16	10	12	12	13	7	7	4
48	81	4	10	7	3	10	10	13	9	13	8	15	8	14	7	12	10	6	3	5	3	...
49	73	3	3	2	4	5	7	7	13	9	4	11	11	10	14	8	10	8	12	11	7	7
50	61	...	2	1	2	11	4	6	...	8	9	8	6	11	10	12	12	7	12	11	5	7
51	93	2	6	1	7	7	1	6	12	9	11	10	12	17	8	17	16	14	10	14	11	12
52	60	3	3	2	3	5	6	9	5	4	7	6	2	7	9	8	11	5	8	11	10	7
53	67	3	2	2	3	5	3	7	2	5	5	8	10	9	8	6	13	11	10	9	11	10
54	54	1	1	1	2	2	2	2	2	5	9	7	2	9	10	7	8	5	8	11	5	8
55	51	2	...	1	2	2	...	1	1	5	4	3	2	3	4	9	3	3	13	8	10	5
56	62	...	...	...	1	1	...	5	...	4	4	3	8	7	9	7	7	9	9	8	8	10
57	41	2	...	...	3	1	2	1	1	3	6	5	3	1	5	2	5	4	7	9	6	10
58	49	2	...	2	...	2	2	3	2	2	4	...	4	3	2	4	6	5	5	8	4	2
59	35	1	...	...	...	1	1	1	1	2	...	1	3	...	2	2	2	2	6	5	5	3
60	39	1	...	...	1	...	...	...	...	1	2	...	1	...	2	3	3	2	3	5	4	2
61	24	...	...	...	...	...	...	1	...	...	1	1	1	...	2	4	1	1	1	4	3	4
62	31	...	1	2	1	...	1	...	...	1	1	1	2	3	2	5	3	1	4	1	3	3
63	32	...	...	...	...	1	1	...	1	1	1	3	2	1	3	...	3	2	4	...	5	5
64	17	...	...	...	...	...	...	...	...	...	1	...	1	...	1	...	1	...	...	1	...	...
65	17	...	...	...	...	...	...	...	...	...	...	1	...	1	...	2	...	...	2	1	1	...
66	16	...	...	...	...	...	1	...	...	1	...	1	...	...	1	...	1	2	3	3	2	1
67	9	...	...	...	...	...	...	...	...	...	...	...	1	...	1	...	1	...	1	...	1	2
68	15	...	...	...	...	...	...	...	...	...	1	1	...	1	...	...	1	2	1	1	1	2
69	8	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
70	4	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
71	4	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
72	4	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	1	1	...	...
73	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
74	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
75	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
76	1	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

TABLE IV.

Values of  $(fa)_x$  an Annuity of 1 to Age 21 to each Child of a Member now aged  $x$  and  $O_x$  an Annuity of 1 to Age 21 to each Child of a Member (Bachelor, Married Man or Widower) now aged  $x$  and to commence on his death. Based upon the 1906, 1911 and 1916 Valuations of the Church of Ireland Fund, Compulsory and Voluntary classes combined. Rate of interest  $3\frac{1}{2}$  per-cent.

Age $x$	$(fa)$		O
	Ungraduated	Graduated	
(1)		3)	(4)
23	...	...	1.18
24	...	...	1.23
25	...	...	1.28
26	...	3.582	1.33
27	3.789	7.163	1.38
28	9.295	8.000	1.43
29	8.404	8.525	1.47
30	9.032	8.997	1.52
31	6.396	9.593	1.57
32	8.359	10.416	1.61
33	16.191	11.491	1.66
34	12.070	12.765	1.69
35	16.070	14.107	1.73
36	13.996	15.385	1.76
37	16.376	16.468	1.79
38	18.854	17.388	1.81
39	16.511	18.232	1.83
40	19.218	18.952	1.84
41	18.175	19.500	1.84
42	19.010	19.828	1.84
43	18.260	19.904	1.83
44	23.565	19.760	1.82
45	21.174	19.445	1.79
46	17.530	19.005	1.76
47	17.995	18.486	1.72
48	18.670	17.885	1.67
49	16.507	17.166	1.62
50	15.954	16.341	1.55
51	14.776	15.423	1.48
52	15.060	14.415	1.41
53	13.578	13.242	1.33
54	12.270	11.895	1.25
55	8.752	10.496	1.17
56	8.661	9.160	1.10
57	10.396	8.003	1.03
58	7.568	7.035	.97
59	5.109	6.177	.91
60	3.713	5.417	.85
61	3.843	4.745	.79
62	6.547	4.149	.74
63	5.271	3.653	.69
64	1.782	3.264	.65
65	1.929	2.947	.60
66	4.404	2.669	.55
67	2.638	2.395	.51
68	3.109	2.131	.46
69	...	1.898	.42
70	...	1.692	.37
71	...	1.499	.32
72	1.414	1.315	.28
73	...	1.140	.23
74	...	.981	.17
75	...	.832	.14
76	...	.693	.09
77	...	.560	.05

## ABSTRACT OF THE DISCUSSION.

MR. TINNER said that the author had placed them under a double obligation by putting new data at their disposal, and by bringing before them the question of the "prospective" versus the "collective" method of valuation which, although it had been discussed at meetings of the Institute, had not previously been the main topic of debate.

He would in the first place touch upon a few minor points. The author stated that the prospective method had the advantage of enabling "prospective" annuities to existing wives to be valued with "ease and celerity" by means of the formula  $a_x - a_{xy}$ . That statement needed perhaps a little qualification if the formula was to be applied strictly to every case, seeing that it involved the construction of fairly extensive tables of joint-life annuities based on special rates of mortality, and that each case had to be dealt with individually except in so far as there might be more than one husband aged  $x$ , whose wife's age was  $y$ . It would seem, therefore, that although, given the necessary joint-life tables, the work was simple, the amount of work involved would be such as to defeat the claim to celerity. With regard to the other benefits, the annuities to widows of the first marriage of present bachelors were comparatively easy to value, but when one came to value the annuities to widows of second and third marriages one would get into deep water because of the complicated analysis required and the paucity of data. One was obliged, therefore, to seek for some simpler method of valuation which would make allowance, without prohibitive labour, for the different forces involved in the calculations, and that led naturally to what was known as the "collective" method, which consisted in deducing from the statistics of members living the combined effect of two or more of the various forces operating to produce the events—other than the death of the member—which caused the benefits to be payable and affected their value.

He observed that the author, in valuing the widows' annuities, found the average age of the wives of all the husbands of age  $x$ , and assumed that the annuity at that average age represented the value of an annuity to the widow of a man dying at age  $x$ . It would seem as though a more accurate result would be obtained by taking the sum of the annuity-values corresponding to the different ages of the wives of men aged  $x$  and dividing that sum by the number of wives. It might be that the difference between the results of the two methods was so small as to be negligible, and perhaps the author could throw some light on that point. Then the author gave a formula for the valuation of widows' annuities by the collective method somewhat similar to that for valuing orphans' annuities. Both these formulæ had the feature that they were applied to the total number of members aged  $x$ , regardless of whether they were married, widowers, or bachelors. He did not altogether like the plan of valuing the benefits in respect of all members of a given age without regard to whether they were married or bachelors, as it

involved the assumption that whatever might be the actual proportion of bachelors among the members of a given age, the proportions among the survivors at all higher ages would be the same as those given by the graduated ratios of bachelors to total; and in particular, that if all the members at age  $x$  were married, some of them would be bachelors at every subsequent age. It was true that if the graduated ratios of bachelors to total members were based on the experience of the fund, the errors involved would tend to cancel one another, but it was not clear that the final balance of error would be negligible. He thought that bachelors and married men should be treated separately. There was a definite risk attaching to every husband seeing that in case of his death his wife would receive an annuity, and also to most of the widowers as the majority of them would have one or more children for whom the fund must provide if the fathers died before reaching a comparatively advanced age. The bachelor did not, however, represent an actual so much as a potential risk, and that fact, in his opinion, made it advisable to distinguish between bachelors on the one hand and married men and widowers on the other. That might be done by valuing the liabilities as if all the members were married or widowers, and then making a deduction in respect of the bachelors who would not marry. If one denoted by  $B_{x+\frac{1}{2}}$  the value of an annuity commencing at the death of a married man or widower between ages  $x$  and  $x+1$ , where

$B_{x+\frac{1}{2}} = \frac{1}{2}(B_x + B_{x+1})$  the value of the first part would be

$\Sigma i^{n+\frac{1}{2}} \cdot d_{x+n} \cdot B_{x+n+\frac{1}{2}} / l_x$ . The probability of a bachelor of age  $x$  remaining unmarried and dying between  $(x+n)$  and  $(x+n+1)$  would be  $(bd)_{x+n} / bl_x$ , where  $(bd)_{x+n}$  was the number of bachelors dying at the same age, and  $(bd)_x$  was equal to  $\frac{1}{2}[(bl)_x + (bl)_{x+1}]m_x$ ,

$m_x$  being the central death rate. The deduction, therefore, could be represented by the expression  $\Sigma i^{n+\frac{1}{2}} \cdot (bd)_{x+n} \cdot B_{x+n+\frac{1}{2}} / bl_x$ . The number of bachelors at each age according to the table would of course be obtained by means of the graduated ratios of bachelors to total members at each age. Both the foregoing expressions could be put into commutation form and denoted by  ${}_B M_x / D_x$  and  ${}_B M'_x / D_x$  respectively. If now the actual number of members at age  $x$  was  $l'_x$  and the number of bachelors among them was  $(bl)'_x$  the liability would be  $l'_x {}_B M_x / D_x - (bl)'_x {}_B M'_x / D_x$ . Investigation might show that that refinement was unnecessary, and that the difference in the results brought out by the two methods was so small that it might safely be neglected, but until that was done the use of the method to which he had ventured to take exception appeared to require justification. He had used the symbol  $B_{x+\frac{1}{2}}$  to denote the average value at the date of death of the benefit payable in respect of a husband or widower dying in the year of age from  $x$  to  $x+1$ . There seemed no real need to deal separately with widows' and children's annuities. If they were combined it might be sufficient to obtain  $B_x$  by finding the value of the annuities to all the wives of men

aged  $x$  and of the annuities to all the children of married men and widowers, add the two together and divide by the number of married men and widowers, thus obtaining an ungraduated value per man who was not a bachelor. This method would appear to give the desired result without loss of accuracy and with a minimum of labour: also it avoided the necessity of distinguishing between married men and widowers and graduating their ratios, although that distinction would be made implicitly in the process.

The table showing the wife's average age last birthday corresponding to certain ages of the husband had led him to refer to some statistics he had prepared some years ago relating to a very different body of lives, namely, the officers and men of the London Fire Brigade. Few of those men remained in the brigade after age 50, but as the figures might be interesting he would give them for comparison with those furnished by the author, together with the corresponding proportions of bachelors to total members at certain ages:

MEMBER'S AGE (NEAREST)	WIFE'S AVERAGE AGE LAST BIRTHDAY		RATIO OF BACHELORS TO TOTAL	
	Church of Ireland	London Fire Brigade	Church of Ireland (Table C)	London Fire Brigade
27	...	25.2	.8425	.566
32	30.7	29.2	.5821	.227
37	33.3	33.7	.3530	.081
42	37.5	38.1	.2366	.033
47	40.9	41.8	.1755	.022

When taking out the figures, he had thought they might throw some light on the results of different economic conditions, but the great differences between the ratios of bachelors might be due in part to the desire for a celibate life felt by some at least of the clergy.

In view of the importance of having as stable a basis as possible for the ratios of bachelors to total members at the various ages, it would be advantageous and would not involve excessive labour, if the statistics of bachelors and other members were taken out not only as on the valuation date, but as on every preceding anniversary of that date in the inter-valuation period. That, by giving a smoother series of values, would simplify the task of graduation and at the same time make the final results somewhat more reliable. The principle of combining the figures for different years was recognized by the author, and he thought it was a question whether an extension of the principle as suggested would not have the advantage of giving results more up to date and possibly more useful because of that fact.

The results of the valuations made by the prospective and collective methods respectively must, he supposed, be regarded as being fairly confirmatory of each other; but as they were based



on such widely different data, it did not appear that any definite conclusion could be drawn from their agreement or disagreement.

Tables II and IV giving the values of annuities to the widows and children of members of each age were very useful. General considerations led one to expect the value of the benefit in each case to increase to a maximum and afterwards diminish; that expectation was confirmed by the tables which showed that age 64 was the turning point in the case of the former benefit and about 41 in the case of the latter.

Passing now to the general question from which the paper derived its title, he understood that by the "prospective" method was meant the method of deducing the probabilities of the happening of the various events which on the one hand led to the payment of the various benefits and on the other determined the period for which they were payable. Thus in connection with a fund of the kind now under consideration, one would require to know the probabilities of marriage, of death, of having children, and of becoming a widower. To apply the method exactly one would require to know for each age the probabilities of children being born and of their dying. That, again, would depend on the age of the wife and the duration of the marriage. All that refinement was absolutely impracticable, and so one saw that there could be no such thing as a valuation by the "prospective" method in its strict sense. The term "collective" was used to describe a process by which one obtained as far as possible from the statistics of living members and their dependants the required particulars of the numbers and ages of the dependants left by a man dying at a given age. But that was only a part of the necessary procedure. The members' rates of mortality must in any case be ascertained by the "prospective" method, and likewise the probabilities of widows and children living from year to year, as those were required for the computation of the annuity-values, and none of those probabilities could be obtained otherwise than by compounding the elementary probabilities  $p_x$  and  $q_x$ . If he was right in that, it followed that neither "prospective" nor "collective" could be regarded as completely descriptive terms inasmuch as a valuation which was wholly "prospective" was impracticable, and one which was wholly "collective" was impossible. Apart from that, it was perhaps regrettable that the words "prospective" and "collective" should have become so firmly established in actuarial terminology, as neither of them indicated, without previous explanation, the meaning it was intended to convey.

He thought there was no doubt that the author had established his case, which was that the collective method was superior in practice; in fact, its superiority to some extent consisted in the fact that, as a rule, it was the only possible method. Nevertheless it was not altogether satisfactory, because it represented the result of forces which had operated very many years ago, and, although in one sense the experience was up to date, in another sense it was really more behindhand in some respects than that used in the



application of the prospective method. For instance, the percentage of men aged 70 who are married is very largely the outcome of the marriage rates for ages 20 to 40, and is therefore from 30 to 50 years out of date. The rate of marriage obtained by the prospective method represented the mean of the rates which had operated during the whole of the period covered by the investigation and, therefore, if this period is short the rates may be regarded as practically up to date. He could not help thinking that rates so obtained would lead to a better forecast of the future than that given by the collective method. The fundamental objection to the prospective method, however, was that the work of valuation, if done accurately, could not be brought within practicable compass.

MR. W. PENMAN said that a few years ago he had valued a Fund similar to that dealt with in the paper. The Fund in question was similar to the Irish Clergymen's Fund in that it did not matter how many times a man had been married; provided he left a widow, an annuity was payable to her. But it had an additional complication in that if there was a difference of ten years between the husband and wife the rate of annuity was cut down to 75 per-cent, and if there was a difference of 15 years the rate of annuity was further cut down to 50 per-cent. That, of course, if one had attempted to use a prospective method would have presented formidable difficulties, but by the collective method it could be dealt with quite simply. Essentially the problem was to find the value of a varying assurance dependent on the husband's death. If the problem was approached from that point of view, he thought that the difficulties in connection with bachelors, married men and widowers re-marrying, &c., disappeared. In fact, in some cases one could go further and add to the variable factor, payable at the death of the husband, something in the nature, approximately, of a constant, to provide for the contingency of the annuity being continued to the children. In some Funds—certainly in the one with which he had had to deal—the actual cash paid out in connection with the children's benefit was surprisingly small, and in such cases it seemed unnecessary to go into an immense amount of detail over a feature of the Fund which was of no financial importance; in the Fund in question he had added a half-year's purchase to the variable for the purpose of providing for the extension of the widows' annuity to the children. There were some differences between the problem with which he had had to deal and that of the author; but the valuation factors brought out were sufficiently close to confirm both the author's results and his own. He thought that the differences arose mainly from the same cause as the difference between the author's two valuations, the results of which were given in the paper, namely, the use of a standard mortality table in the one case and a special mortality table in the other. In his judgment there was a great deal to be said for using a standard rather than a special mortality table.

MR. J. BACON said he had not had time to study the paper so

carefully as he would have liked but he desired to point out that in the use of the collective method there was one point particularly which should be borne in mind. The functions deduced by that method, as Mr. Tinner had said, were the result of the operation over many years of a number of forces, some of them dependent upon the human will, and not simply of the force of mortality. It followed that, to use the method with complete confidence, the whole past experience must have been homogeneous. If there had been any change in economic conditions or in the conditions governing the Fund, then the ratios brought out by the collective method must indeed in his opinion be used with considerable care and due allowance must be made for any changes that might have resulted from past variations in conditions or probable variations in future. For that reason he preferred the prospective method, especially as future changes could be readily estimated and allowed for by that method. It must be admitted, of course, that in some cases the collective method was the only method available, but, wherever possible he preferred to use the other. The main benefit in widows' and orphans' Funds was usually the annuity to the future widow of a man who was now married. That was particularly the case where the Fund was not on a compulsory basis. Where the Fund was a voluntary Fund the annuity to the youngest child of the  $n$ th wife of a man who was now a bachelor was interesting to talk about but of no importance in fact, and the annuity to the wife of a man who was a bachelor was of comparatively small importance. Nearly the whole liability was in respect of annuities to the existing wives. That did not involve, in his opinion, a great deal of trouble in valuation. It was not necessary to take every combination of ages, but he thought it would be found to be quite accurate enough in practice to take for each age of the husband the age of the wife to the nearest difference of 5 years, as was done in the usual tabulations of joint life functions. That involved the calculation of only five or six columns of joint life and survivor annuities, which was not a heavy task even where the experience of the Fund itself was used. Probably also one was dealing with mortality drawn from the experience of the Fund or from a standard Table which had been found, by a comparison of expected and actual deaths, to represent the experience of the Fund fairly closely, and the data were comparatively recent, whilst as regards the future one was dealing, so far as this, the main benefit, was concerned, with something that was not dependent on the element of human will. But when one came to use the collective method one had not only to consider the actual existing facts of the difference of age between husband and wife, but to use functions which might even result in a man who was now married being treated at some future time as a bachelor unless caution was exercised in the progression of the rates of married men, bachelors and widowers. It was quite well known that there had been changes in the average age at marriage and that for some time there had been a tendency for it to rise with the result that the difference in the ages of husbands and wives was increasing. The

ratios by the collective method at advanced ages would depend, *inter alia*, on the ages and rates of marriage of forty, fifty, or sixty years ago, and in respect of young members it would be assumed that they would still hold as many years hence though it was humanly certain this would not be the case. He simply desired to give that as an example of the sort of thing which should be looked out for and corrected where necessary in using the collective method. Of course if salaries were involved even greater caution was necessary. He also thought that where the collective method was used it was certainly preferable to use the combined ratios at, say, the valuation date with those of the previous five years, rather than to go back five, ten or fifteen years.

There was one other point. It had been put forward by advocates of the collective method as a convincing argument that the valuation was based on the latest ratios and was always up to date. He thought that contention had been disposed of, but he would also like to add that if so based it was perhaps a little too much up to date, the functions varying with changes that might be purely temporary or due to paucity of data. He suggested that even when one adopted a method of that sort it was as well to establish standard rates and to keep to those standard rates as long as possible, until it was quite evident that they must be departed from owing to something inherent in the statistics and not due to merely temporary changes. He advocated that purely on the practical ground that it was inadvisable to change the valuation basis and so produce very divergent results at succeeding investigations. This served no useful purpose but did give people with no technical knowledge the feeling that actuarial surpluses or deficiencies were unreal and not worth consideration.

MR. A. D. BESANT said, with reference to the previous speaker's remarks, that in dealing with a Fund which had not been in existence for many years he thought it would sometimes be found that the collective method might lead to mistakes. He had had an opportunity some time ago of valuing a Bank Fund with a membership of about 2,000. The Fund had been valued five years previously by the late Mr. Vyvyan Marr, who was well-known as an expert in such matters. Mr. Marr had valued the orphans' benefits by both the prospective and the collective methods, and the reserve required by one method was something like four times that required by the other. That, he thought, was largely due to the fact that at the time Mr. Marr had made his valuation the Fund had only been in existence some 15 years. The number of orphan children had been quite small and the proportion of bachelors in the Fund had been abnormally large. When he (the speaker) valued the Fund, he found that the position had changed in a somewhat unexpected manner. First of all, owing to the war, some 200 bachelors had been killed and the proportion of bachelors on the Fund at the present time was much smaller than would have been expected under normal conditions. Curiously enough, although a certain number of husbands had been killed in the war, the number of orphan children

at the present time was very small, and was actually less than it had been five years ago. One would have expected that the number of orphans would have increased considerably in the last five years. He found that by the collective method he would have to make quite a small reserve. He had hesitated to make a smaller reserve than had been made five years ago, and he had followed Mr. Marr (who also had followed his predecessor, the late Mr. Archibald Hewat) in making a considerably larger reserve than that required by the collective method. He would ask the author whether in his own experience when dealing with a comparatively young Fund he had felt satisfied that the collective method formed a satisfactory basis.

MR. E. C. THOMAS, referring to the point raised by Mr. Besant, said that there seemed to him to be a little confusion as to the difference between the two methods under discussion. It was by no means necessary, although it was very useful and sometimes very advantageous, in employing the collective method, to use the facts derived from one's own experience. The collective method simply meant that one took a certain number of probabilities and combined them in a very simple way to produce a result which it would be almost impossible to obtain by the prospective method. Those probabilities one could obtain from one's own experience, or from a standard table if one thought the latter would give better results. To his mind the greatest advantage of the method was that one could go to one's own experience, and, even though it might be a very limited and small one, if, in considering all the special circumstances of the case, one could be satisfied that it was by no means abnormal, one could then use the figures of the Fund with complete confidence and produce a result which would bring out the special features of that Fund and not merely the features of some other Fund which might be entirely different from it.

He was in some doubt as to when the collective method was first used. Up to the time when the late Mr. Manly's monumental paper on Widows' Funds was read, he thought he was right in saying that Mr. Huie's and Mr. Hewat's methods and tables held the field. Mr. Manly's own paper had been based upon a combined marriage and mortality table, and there was no hint, as far as he could remember, in the discussion which followed, of any easier methods. In the discussion Mr. Manly's paper had been described as embodying the collective method, but, whatever else it had been, it certainly had not been the method of the present paper. The idea of using collective methods for national and other investigations had been in the air for many years; but he thought the germ of the idea underlying the present paper was contained in Mr. King's paper on Family Annuities. The idea of going to the statistics of the living instead of the dead had been first suggested, at least to himself, by Mr. Allin's paper. He had happened at the time to have been grappling with a problem similar to the one now under discussion, involving the question of salary combined with widows' and orphans' benefits in such a way that it had been almost impossible to tackle it by the



old methods. Mr. Allin's paper had been put into his hands with a view to his opening the discussion on it, and he therefore had had to study that paper rather carefully. By the light of Mr. Allin's suggestions he had been enabled to solve his own difficulties. Mr. King, to whom he had shown his own solution, had thought sufficiently well of it to suggest that he should prepare a short note for publication in the *Journal*. While he had been preparing that note, Mr. Ackland, who was then Editor of the *Journal*, had written to him enclosing a paper by Mr. Oscar Schjoll, and suggesting that he should prepare a suitable abridgment of it for publication in the *Journal*. When he had read that paper he had found it was an exhaustive description of that very method at which actuaries in this country had only just arrived. Mr. Ackland had explained that Mr. Schjoll's paper had been in possession of the Institute for many years, but that, owing to an extraordinary combination of circumstances, it had been put aside and, if not actually forgotten, at least overlooked. The result had been that the abridgment of Mr. Schjoll's paper and his own note had been published side by side in the same number of the *Journal*. It thus appeared that, while British actuaries had been groping towards what might now be considered the obvious method of dealing with the question, that very method had been in use many years before by Mr. Schjoll, and a full description of it had been in the hands of the Institute all the time. Of course nobody had mastered the contents of the paper, and those who had made contributions on the subject in the meanwhile had been absolutely ignorant of its existence. He thought that the method had come to stay. In his opinion it would very often, if not usually, give more dependable results, and the gain in simplicity, ease and labour-saving was beyond all question.

MR. S. J. H. W. ALLIN, in closing the discussion, said the most interesting portion of the paper to him was that relating to the statistics of the Fund with which the author had dealt. New data were always valuable, not only as such, but also for the lessons which they taught when compared with statistics obtained from similar Funds. There appeared to be some special features in the statistics which the author had given, as compared with those of a Fund which he (the speaker) had been connected with for many years as actuary, a Fund very much on the same lines as the author's, namely, the "Church of England Presbyterian Ministers' Widows and Orphans Fund." No comparison could be made in respect of mortality as the author had given practically no information on that point. The other main functions which had to be dealt with in the valuation of these Funds were the proportion of married men to the total, the relative ages of husbands and wives, and the size of families. The proportions of married men in the Church of Ireland Fund were extremely low compared with those in the Presbyterian Ministers' Fund. For example, under age 30 the author found 7 per-cent married as against the Presbyterian Fund's 40 per-cent; at ages 31 to 40, 57 per-cent as against 79 per-cent; at ages 41 to 50, 78 per-cent as against 86 per-cent; at ages 51 to 60, 81 per-cent

as against 85 per-cent ; at ages 61 and over, 75 per-cent as against 85 per-cent. These seemed very large differences, but he did not think they were altogether surprising, and there appeared to be a satisfactory explanation which would also account for the divergence from the figures obtained from the Firemen's Fund to which Mr. Tinner had referred. A large number of the clergymen of the Church of Ireland Fund were probably curates and their salaries were so small that they did not permit of their getting married. In the Presbyterian Ministers' Fund he believed there were practically no assistant clergy and every minister got a salary upon which he was able to marry. The position was probably very much the same in the case of the Firemen's Fund, where a man on reaching maturity got a full working wage, on which, in his position, he could marry. He thought that probably was the explanation. With regard to the ages of wives, from age 45 to age 50 the wives of Presbyterian ministers seemed to be a little older, but generally the figures of the two Funds were in very close agreement. With regard to the size of family, however, there was considerable difference. In the Presbyterian Ministers' Fund the orphans received pensions up to age 18 only. Allowing for this, he found that the average number of children per married man or widower was 1.5 in the Church of Ireland, whereas in the Presbyterian Ministers' Fund it was 20 per-cent lower, namely, 1.2. He could not put forward any explanation for that very low figure, but he might mention that he had found a continual decrease in the average number of children from year to year since 1902, when he had first examined the Fund. Such differences in Funds which were apparently similar showed, he thought, that past experience spread over a number of years was not reliable and that one must go to the Fund itself. He thought Mr. Thomas had really put the matter in a nutshell : although the data of the Fund should be employed by use of the collective method it should not necessarily be used without adjustment. It should be adjusted if it was considered that the present facts would not be reproduced in the future. He thought it was distinctly preferable to use the material obtained from the living members of the Fund rather than to go back to experience which was many years old. It was true that by using the present data one was involving forces which had come into play many years ago, but surely any error was accentuated if one used tables which were constructed many years ago, like Mr. Hewat's. Of course, if the Fund were large enough to obtain rates of marriage and issue based on the experience of the last few years, it would be the most satisfactory course to employ such data, but in the bulk of the cases the data were too limited.

A comparison of the author's valuations by the prospective method and by the collective method seemed to him of very little value, since there was only one factor in common, namely, the rate of interest. The results were obscured by the different tables of mortality which had been used. The difference in the value of the contributions was entirely due to the rate of mortality, whilst as



regards the value of the widows' benefits there were three factors coming into play. First there was the mortality of the husbands. The rates used by the author for the collective valuation were very much lower than those employed for the reversionary valuation. Secondly, the probabilities of being a married man were also less, and thirdly, the value of the annuity to the widow on the death of the husband was based on a mortality which was very much lighter in the case of the collective valuation. Surely for a proper comparison the rates of mortality employed should have been the same for both methods. Then the effect of employing marriage rates obtained by Hewat's tables on the one hand and the probability of being married obtained from the experience of the Fund itself could have been seen. With regard to the rate of mortality which the author had used, he did not understand why the author had employed King's Elementary Teachers' Table. That table was a hybrid one and had been formed for a special purpose, and he could see no reason why it should be expected to reproduce the mortality of the clergy; nor in fact did it, according to what the author himself had stated. The Clergy Mutual Table, adjusted for increased longevity since the date of that experience, would probably have fitted the facts very much better, but, even assuming that, for the purpose of strengthening the valuation because of the small size of the Fund and possible fluctuations, the author had considered it well to employ a mortality table showing rather heavy rates, he (the speaker) would have thought that a table like Hewat's, which was based on the mortality of the clergy, would have been much more suitable. He considered it was very important, especially in the case of widows' Funds, to see that the incidence of mortality was correct. In such Funds the sum at risk varied with the age passed through, and the incidence of the mortality was surely a point that could not be overlooked. He had compared the rates of mortality by the Elementary Teachers' Table and Hewat's, and the results were very curious. From age 22 to 31, King's mortality was lower; from 32 to 45, Hewat's mortality was lower; from 46 to 60, King's was lower; and again from 61 to 70 Hewat's was lower.

He wished that the author had dealt with some of the special problems which arose out of the Fund. For example, there was an increase of pensions to orphans after the death of their mother. The author suggested adding a percentage margin. On what basis was that margin to be fixed? It would have been very interesting if the author had dealt with this problem scientifically. The value of the orphans' annuity consisted of two parts: the term annuity and an additional temporary deferred annuity commencing after the death of the mother. The value of that annuity depended upon the age of the mother, and he thought some very interesting facts might have been brought out on further investigation as to the relative ages of mothers and children. The problem of the widows' annuity ceasing on re-marriage had also not been dealt with. Re-marriage rates at the present time were of very great importance,

and if the author had been able to give any information on the subject it would have added to the store of information.

In conclusion, he must take exception to the author's statement that an approximation he had made in the valuation was in the direction of safety—"a direction always very popular with the actuarial mind." In his view an actuarial calculation should be exact: that is, as exact as it could be made. Of course margins were sometimes necessary to cover possible or even probable fluctuations, but he did not think actuaries ought to make a calculation which included an unnecessary loading and call it an actuarial calculation. In many cases an actuarial calculation had to determine definite liability as between two parties, and in such cases serious injustice might be done if any loading were added. It seemed advisable to emphasize the fact that an actuarial calculation was an exact calculation as far as the probabilities allowed actuaries to make it so.

The PRESIDENT, in moving a vote of thanks to the author, said that any actuary who valued his own reputation and that of the Institute must approach the problems arising out of widows' and similar Funds with extreme caution, on account of the paucity of the data usually available in connection with such Funds, a paucity which created doubt not only as to the value of the figures derived from the experience under immediate consideration, but also as to the value of any comparisons which might be made with other results derived from similar limited material. Caution was also necessary on account of the changes that took place in the conditions—economic or financial—under which such Funds were conducted—changes to which Mr. Bacon had referred. Changes of that kind might come very suddenly and seriously impair a Fund in the interval between two valuations. Other changes had also to be taken into account, or at any rate their possibility, namely, changes coming from pressure of the members interested in the Fund, pressure which it was often very difficult to resist successfully, because, as he had pointed out on another occasion, the constitutions of Funds were often of a very democratic nature, and allowed the possibility of damaging alterations in the rules.

DR. HOWELL, in reply, said, with reference to Mr. Tinner's remarks, that by using the Carlisle Joint Life Annuity Tables and rating down the wives' ages by three years, prospective annuities to existing wives could be valued with great celerity. That, he thought, was the only merit possessed by a prospective valuation so made. As regards the method of finding the average value of an annuity to the widow of a husband dying at age  $x$ , he had referred in the paper to the fact that the unweighted ages which he had obtained agreed very closely with those which Mr. Marr had got out on the weighted plan. Mr. Tinner took exception to the method of dealing with the members regardless of whether they were bachelors, married men or widowers. The method might not be correct, but Mr. Tinner's alternative method of assuming that every man upon the Fund was a married man or a widower, and then correcting that

assumption by making deductions on account of bachelors, seemed to him much more open to criticism. Mr. Tinner also had submitted certain figures relating to the London Fire Brigade and had compared them with the corresponding figures for Church of Ireland clergymen. The cases were entirely different. A Church of Ireland clergyman belonged to the professional class and was generally a university man; he started on a small salary and it took him a long time to progress beyond it. Eventually he might settle down as a married man. The fireman was able to marry much earlier and had not to keep up an appearance. He thought that explained the difference.

He gathered that Mr. Bacon was a strong advocate of the prospective method. In theory it might be possible to justify the claim of the prospective method; but to urge objections against the collective method and to omit any reference to the far larger assumptions which it was necessary to make under the prospective method seemed unfair.

With reference to Mr. Besant's remarks he thought that even in the case of a young Fund, it was right to use the collective method. One found certain proportions of married men, bachelors, and widowers, and he thought one ought to proceed upon the assumption, until it was disproved, that those proportions were likely to prevail. Much had been said about the care that should be taken to deal with changing factors in the social state, but he would like to know what form that precaution should take, and whether it could be expressed in a formula. He had little doubt in his own mind that it should take the form of holding back a substantial portion of the surplus.

Mr. Thomas had really replied to the whole discussion. He was glad to find that Mr. Thomas was in favour of using the experience of the living. The only doubt in his (the author's) mind had been as to how far this was legitimate. It was an extraordinary thing that until Mr. Allin had had the courage to publish a paper based upon living experience there had been no pronouncement on the subject. He hoped the Institute now officially recognized the legitimacy of the method.

In conclusion, he expressed disappointment that his reference to Mr. Lidstone's ingenious plan of deducing marriage rates from a collective experience had not brought out any contribution to that very interesting subject.

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The following letter with reference to Dr. Howell's paper appeared in the *Insurance Record* of 2 January 1920:

*To the Editor of the Insurance Record.*

SIR,—I should have liked very much to have been present at the reading of Mr. Howell's paper at the Institute of Actuaries and to have taken part in the discussion, but, unfortunately, from the position in which I was placed, I found that impossible. Perhaps

you will allow me now to give in few words what I should probably have said had I been able to attend.

In the first place, I should like to thank Mr. Howell for all the kind references to myself which he made in his paper, and to corroborate his statements that I co-operated with him in the valuations of the Church of Ireland Widows' and Orphans' Fund and entirely concurred in the principles and methods which he adopted, and that I also endorsed the results which were brought out and his recommendations thereon.

It may be well to place on record the origin of my paper on "Family Annuities." Some of the speakers seem to have thought that it was the result of a roving scientific enquiry without any definite object having been in view. That, however, is not the case. About the year 1890 the late Right Hon. Joseph Chamberlain, M.P., and the late Dr. W. A. Hunter, M.P. for Aberdeen, were collaborating on the question of National Pension and other Funds, and were going into it very thoroughly. I had the privilege of knowing Dr. Hunter very well, and we had many meetings at which the matter was discussed, and he asked me to make a report to Mr. Chamberlain and himself in which I was to try to estimate the cost of the schemes which they had in their minds. I took up the matter very gladly, and after reporting to these gentlemen I thought that a paper might be submitted to the Institute of Actuaries which would be interesting and useful. Hence my paper on "Family Annuities", read before the Institute in December 1892, and it is a great satisfaction to me to find it referred to again after all these years. The method was devised for the purpose of estimating the cost of the various benefits which Messrs. Chamberlain and Hunter hoped to provide by legislation for the industrial population of the country, and it was applied practically accordingly. I have not been able to find a copy of my report, but the tables in the paper were, I think, included in it, and were certainly some of those used in my enquiry. So far as I know, this was the first occasion on which the method now known as the Collective Method was employed. It may be remarked that this collective method is not the same, and has not quite the same objects, as the method devised by the late Mr. R. P. Hardy. This becomes apparent when his writings on the subject are studied, and need not be elaborated here.

A curious event happened in 1908, when the Royal Commission on the Poor Laws and Relief of Distress was sitting. The Commission applied to three actuaries to report to them on the cost of various schemes which were being investigated. The actuaries were the late Mr. T. G. Ackland, Mr. F. G. P. Neison and myself, and none of us knew that the others had been consulted until our reports came out in print in the volume published by the Commission. These reports were therefore absolutely independent, and those who care to look into them will see the different lines the actuaries took. As a matter of fact, although Mr. Ackland's report and mine differ considerably in many respects, yet there is a great similarity. The benefits which we had to investigate were sickness and invalidity



in respect of males; illness, maternity and burial in respect of wives; illness and burial of children; and what have been called "family annuities" to widows and children; and, lastly, unemployment. I myself followed wherever practicable the collective method, and I obtained statistics from many sources. These reports seem to be very little known, and, so far as I am aware, nothing has been heard of them since. I still possess a few copies of a reprint of all the three reports in one pamphlet, and, should anyone care to see them, I shall be happy, so far as my stock goes, to supply a copy to anyone to whom they may be useful.

The principal statistics required are those relating to the relative ages of husbands and wives, and to the number, sexes and ages of the children of fathers, and these statistics are very difficult to find. As to the relative ages, there are tables given, based upon the Census of 1901, which supply the information as at that date, but unfortunately these have been omitted from the published documents of the 1911 Census. I tried to get them when I was preparing my first report on that Census to the Registrar-General, but they were not available.

Then, in this country, we have no public statistics whatever of the kind required as to children, although some actuaries have private information relating to particular funds. It would be well if the forms which will soon have to be settled for the Census of 1921 should include questions as to the number, ages and sexes of the living children of fathers who are enumerated. This might cause a good deal of additional work and might be difficult to adopt, but, nevertheless, it is very important. In New Zealand the information is collected on the death certificates of males, but so far I do not think it is asked for at the Censuses. If we cannot get it, therefore, in the British Census, the Registrar-General might be urged to have it supplied when deaths are registered.

It is earnestly to be hoped also that at the Census of 1921 particulars as to the relative ages of husbands and wives should be included in the tables that will be published.

I have always advocated that, as far as possible, the statistics of the fund itself should be employed in the valuations, but here it is necessary to emphasize the words, "as far as possible." In starting funds it is seldom that full particulars can be obtained, and in any case many years must elapse in the history of a fund before a sufficient mass accumulates, and the actuary must fall back upon such statistics as he can find relating to lives of a similar class to those included in the fund he has in hand. It is here that sound judgment and experience on the part of the actuary are essential.

I am, Sir,

Your obedient Servant,

GEORGE KING.

15, Walbrook, London, E.C.

*A New Method of Valuing Policies in Groups.* By H. I. TRACHTENBERG, B.A., A.I.A., *Actuarial Assistant in the Statistical Department of the Medical Research Committee.*

[Submitted to the Institute, 2 February 1920.]

# I.—DESCRIPTION OF METHOD.

THE underlying idea of this paper is the valuation of the sum assured ( $S$ ) in groups of  $t$  years by multiplying the total sum by a tabulated function  $\alpha$  and  $\sum_{x=-\frac{t-1}{2}}^{\frac{t-1}{2}} Sx$  by a tabulated  $\beta$ . A similar arrangement can be adopted for premiums, &c. The method may be exemplified by the following valuation of a group of 10 years of age:

Age	S	$x$	$Sx$
	£		
57	192,702	4.5	867,159
56	180,889	3.5	633,112
55	206,423	2.5	516,058
54	129,708	1.5	194,562
53	147,563	0.5	73,782
			$\Sigma + 2,284,673$
52	126,722	-0.5	63,361
51	134,623	-1.5	201,935
50	114,779	-2.5	286,948
49	115,348	-3.5	403,718
48	165,686	-4.5	745,587
			$\Sigma - 1,701,549$
Totals	1,514,443	...	+ 583,124

In this case  $\Sigma S = 1,514,443$  and  $\sum_{x=-\frac{t-1}{2}}^{\frac{t-1}{2}} Sx = +583,124$ . The values of  $\alpha$  and  $\beta$  for the age group 48-57 are .58804 and .01056 so that the value of the group

$$\begin{aligned}
 &= 1,514,443 \times .58804 + 583,124 \times .01056 \\
 &= 896,711,
 \end{aligned}$$



showing a deviation from the true value £896,755 of £44 or less than .1 per mille.

We may now consider the forms taken by  $\alpha$  and  $\beta$ . Let  $f(x)$  be the quantity to be valued, *i.e.*, sum assured, premium, reversionary bonus, &c., and  $u_x$  the function by which the valuation is made, *i.e.*, A,  $a$ , &c. Also let  $\Sigma(t)$  stand

for  $\sum_{-\frac{t-1}{2}}^{\frac{t-1}{2}}$ . Then we require to express  $\Sigma(t)f(x)u_x$  in the form

$$\alpha \Sigma(t)f(x) + \beta \Sigma(t)xf(x) \quad . \quad . \quad . \quad . \quad . \quad (I)$$

This may be done by taking

$$\alpha = \frac{\Sigma(t)u_x}{t}, \quad \beta = \frac{\sum_{\frac{1}{2}}^{\frac{t-1}{2}} u_x - \sum_{-\frac{t-1}{2}}^{-\frac{1}{2}} u_x}{\frac{t^2}{4}} \quad . \quad (II)$$

The result is the reproduction of the true value as far as second differences of  $u_x$  in cases where  $f(x)$  is linear.

For if  $u_0$  is the value of  $u$  for the central age of the group, and  $a_0$  and  $b_0$  are the first and second central differences of  $u_0$ , then as far as second differences  $u_x = u_0 + a_0x + \frac{b_0x^2}{2}$ .

Applying this formula to each value of  $u$  in  $\alpha$  and  $\beta$  and summing:

$$\begin{aligned} \alpha &= [u_{-\frac{t-1}{2}} + u_{-\frac{t-3}{2}} + \dots + u_{-\frac{1}{2}} + u_{\frac{1}{2}} + \dots + u_{\frac{t-3}{2}} + u_{\frac{t-1}{2}}] \div t \\ &= u_0 + \frac{t^2-1}{24} b_0 \end{aligned}$$

$$\begin{aligned} \beta &= [u_{\frac{t-1}{2}} + u_{\frac{t-3}{2}} + \dots + u_{\frac{1}{2}} - u_{-\frac{1}{2}} - \dots - u_{-\frac{t-3}{2}} - u_{-\frac{t-1}{2}}] \div \frac{t^2}{4} \\ &= a_0. \end{aligned}$$

Thus the formula

$$\begin{aligned} &\alpha \Sigma(t)f(x) + \beta \Sigma(t)xf(x) \\ &= \left(u_0 + \frac{t^2-1}{24} b_0\right) (\Sigma(t)f(x)) + a_0 \Sigma(t)xf(x). \end{aligned}$$

But (*cf.* G. J. Lidstone, *J.L.A.*, vol. xlv, p. 485) the true value  $\Sigma(t)f(x)u_x$  as far as second differences

$$\begin{aligned}
&= (\Sigma(t)f(x)) \left\{ u_w + \left( \frac{\Sigma(t)x^2f(x)}{\Sigma(t)f(x)} - w^2 \right) \frac{\delta^2 u_w}{2} \right\}, \text{ where } w = \frac{\Sigma(t)xf(x)}{f(x)} \\
&= (\Sigma(t)f(x)) \left\{ u_0 + a_0 w + b_0 \frac{w^2}{2} + \left( \frac{\Sigma(t)x^2f(x)}{\Sigma(t)f(x)} - w^2 \right) \frac{b_0}{2} \right\} \\
&= (\Sigma(t)f(x)) \left\{ u_0 + \frac{\Sigma(t)x^2f(x)}{\Sigma(t)f(x)} \cdot \frac{b_0}{2} + a_0 w \right\} \\
&= (\Sigma(t)f(x)) \left\{ u_0 + \frac{t^2-1}{24} b_0 \right\} + (\Sigma(t)xf(x)) \cdot a_0,
\end{aligned}$$

since  $w$  has the value shown above, and the coefficient of  $\frac{b_0}{2}$  reduces, when  $f(x)$  is linear, to  $\frac{t^2-1}{12}$ . (Compare A. E. King, *J.I.A.*, vol. xlviii, pp. 121 *et seq.* and H. E. Melville's remarks in discussion). This expression is identical with that given by the formula (I).

It has been assumed above that  $t$  is even. When  $t$  is odd forms of  $\alpha$  and  $\beta$  which likewise cause (I) to reproduce the true value as far as second differences of  $u_x$  if  $f(x)$  be linear, are :

$$\alpha = \frac{\Sigma(t)u_x}{t}, \quad \beta = \frac{\Sigma_1^{\frac{t-1}{2}} u_x - \Sigma_{-\frac{t-1}{2}}^{-1} u_x}{\frac{t^2-1}{4}}. \quad \text{(III)}$$

A. E. King, in his method of approximation described in the paper just mentioned, used the valuation factor

$$\left( u_w + \frac{t^2-1}{24} \delta^2 u_w \right)$$

as an approximation to

$$u_w + \left( \frac{\Sigma x^2 S}{\Sigma S} - w^2 \right) \frac{\delta^2 u_w}{2},$$

and in the discussion H. E. Melville pointed out that if the sum assured was linear the error was  $\frac{w^2 \delta^2}{2}$ .

A. E. King's method has the advantage of a single multiplication; mine requires no interpolation.

As has been seen the method has been established quite generally for a group of any size. It could therefore be used to value the whole of the particulars in a single group. This

would, however, require more work at the time of the valuation than would be necessary if groups of 10 were employed. For then the ingenious system of card and class-book entries proposed by A. E. King for use in connection with his own method could be used for the same purpose of preparing the particulars for valuation as far as possible prior to the date of valuation. These entries are the same for my method as for his.

It will be noticed from formulæ (I) and (II) that both  $\beta$  and the  $\Sigma$  by which it is multiplied may be either positive or negative. Now since  $\beta$  depends only on the valuation factor its sign is always known in advance. The sign of the  $\Sigma$  depends on the actual sums assured (or other quantity to be valued) in the particular group being valued.

## II.—THE CONSTRUCTION OF THE TABLES.

Tables of the two quantities  $\alpha$  and  $\beta$  are easily constructed. For instance, taking the case of the sums assured by whole-life assurances we have only to deal with  $A_x$  and work out

$$\sum_{-4.5}^{4.5} A_x \div 10 \text{ for } \alpha \text{ and } \frac{\sum_{-4.5}^{4.5} A_x - \sum_{-1.5}^{-.5} A_x}{25} \text{ for } \beta. \text{ Obviously the easiest}$$

way to form such a table is to sum in fives throughout.

It has been seen that as far as second differences for any group,  $\beta$  is independent of  $t$ , the size of the group, because it is equal to the first central difference of the central term ( $a_0$ ). Thus  $u_{\frac{1}{2}} - u_{-\frac{1}{2}}$  the value of  $\beta$  for the simplest case of  $t=2$  could theoretically be taken for the value of  $\beta$  for groups of 10. It is, however, preferable to use the full expression for  $t=10$  since irregularities in two values would be magnified by the larger multipliers in a group of ten terms. For a similar reason I have not worked on differences in constructing the tables.

Like those of A. E. King my tables are based on  $O^M$  3 per cent and supply the functions  $\alpha$  and  $\beta$  corresponding to the detailed valuation factors  $A_x$ ,  $a_x$ ,  $10^5 D_x^{-1}$  and  $\bar{a}_{M-\overline{n+1}:\overline{n}}$ .

The tables shown below are abridged forms of the full tables which are deposited for reference. The values of  $\alpha$  and  $\beta$  for every tenth decennial group and for one maturity age 60 are shown:

For groups of 10 years.

 $O^M$ 

3 per-cent.

Age Group $x - x + 9$	$A_x$	
	$a$	$+ \beta$
10-19	·27688	·00565
20-29	·34052	·00705
30-39	·41720	·00829
40-49	·50693	·00967
50-59	·60925	·01069
60-69	·71615	·01049
70-79	·81338	·00878
80-89	·88812	·00611
90-99	·93560	·00345

For groups of 10 years.

 $O^M$ 

3 per-cent.

Age Group $x - x + 9$	$a_x$	
	$a$	$- \beta$
10-19	23·827	·194
20-29	21·642	·242
30-39	19·009	·285
40-49	15·929	·332
50-59	12·416	·367
60-69	8·746	·360
70-79	5·407	·301
80-89	2·841	·210
90-99	1·211	·118

For groups of 10 years.

 $O^M$ 

3 per-cent.

Age Group $x - x + 9$	$10^5 D_x^{-1}$	
	$a$	$+ \beta$
15-24	1·851	·062
25-34	2·610	·092
35-44	3·773	·145
45-54	5·693	·250
55-64	9·426	·537

For groups of 10 years.

 $O^M$ 

3 per-cent.

Un- expired Term Group $n - n + 9$	$\bar{a}_{n0-n+1:n}$			
	$a$	Diff.	$+ \beta$	Diff.
0-9	3·878	15	·810	4
10-19	10·654	57	·566	3
20-29	15·587	83	·428	2
30-39	19·390	94	·336	...

## III.—INCOMPLETE GROUPS.

If the method be applied to decennial groups based on special class books as suggested by A. E. King there will be incomplete groups at the beginning and end of the particulars for valuation, and it is well to formulate a rule to be followed in these cases. As the expressions for  $\alpha$  and  $\beta$  were obtained for any value of  $t$  no further algebraic investigation is required, but it will be useful to see how tables worked out for decennial groups can be used for smaller groups.

Since

$$\alpha_t = u_0 + \frac{t^2 - 1}{2t} b_0$$

$$\beta_t = u_0$$

$\beta_t$  is the same whatever the value of  $t$  as long as the group has the same central age and is therefore obtained by taking

the  $\beta$  from the tables for that decennial group having the same central age.

For  $\alpha_t$  the procedure is to take the value of  $\alpha_{10}$  in the Tables for the group having the same central age as the group  $t$ , and use the expressions shown in the following table where  $h$  is the mean of the two values in the detailed table on each side of the central value of the group.

$t$	$\alpha_t$
2	$h$
4	$\alpha_{10} - \frac{7}{8}(\alpha_{10} - h)$
6	$\alpha_{10} - \frac{2}{3}(\alpha_{10} - h)$
8	$\alpha_{10} - \frac{3}{8}(\alpha_{10} - h)$

No values are shown for incomplete groups of  $t$  where  $t$  is odd, as it may be assumed that there is another term at the free end of the incomplete group with zero against it, and the group is then dealt with as an even group.

Alternatively of course  $\alpha_t$  can be easily calculated directly from the formula for group  $t$ .

#### IV.—CONVERSION TABLES.

As regards the use of conversion tables for obtaining the  $\alpha$  and  $\beta$  for an A factor from the  $\alpha$  and  $\beta$  for the corresponding  $a$  factor, just as A. E. King enters the conversion tables in exactly the same way as would be done for the ordinary factor of the detailed valuation, so, in applying my method the  $\alpha$  for the Assurance (A) factor is obtained by entering the conversion table with the  $\alpha$  for the Premium ( $a$ ) factor. The  $\beta$  for the A factor, however, cannot be obtained from that for the  $a$  factor by using the ordinary conversion table.

Since, however,  $\beta$  for the Assurance factor

$$= \frac{-d(a_{4.5} + a_{3.5} + a_{2.5} + a_{1.5} + a_{.5} - a_{-.5} - a_{-1.5} - a_{-2.5} - a_{-3.5} - a_{-4.5})}{25}$$

$$= -d \times (\beta \text{ for the annuity factor}),$$

the one function can be obtained from the other by a single multiplication.

## V.—A VALUATION BY THE METHOD.

In order to test the results of the application of the method to an actual valuation, consider the case of a valuation on 31 December 1919, of an office whose whole-life assurance business has a constitution at that date represented by Mr. George King's model office No. 1 of 50 years of age, and an endowment assurance business represented by Dr. Buchanan's model office of 35 years of age.

For the whole-life office I constructed the sums assured, compound reversionary bonus of 30s. per-cent (Buchanan's formula, *J.I.A.*, vol. xxxix, p. 257) and premiums, for each attained age.

It will be observed that there is a jump at every fifth age, 26, 31, 36 . . . This is due to the assumption made in the model office that the entries occur at central ages (20, 25 . . . 65), and not at each integral age.

This makes the material rougher since two jumps occur in each decennial group valued. The test is therefore more severe than would otherwise have been the case.

In the case of the endowment assurances I constructed the sums assured for each unexpired term, but had then to insert the premiums by assuming that for the unexpired term in each quinquennial group for which the totals were published, they were proportional to the corresponding sums assured. The minor differences from the true results produced by this assumption could not interfere with the test, for the same figures were used for making the detailed valuation for comparison.\*

The maturity ages are determined by Mr. King's methods. These are (1) when the Z method is used obtain an average Z for the group by dividing the total Z by the total sum assured; (2) use Mr. Elderton's single maturity age throughout. I have followed the first because the Z method was used by

\* Buchanan's first group refers to unexpired terms 1-4, and by applying the proportion of premiums to sums assured for this group to the sums assured for unexpired term zero I obtained the premiums for unexpired term zero. He omitted these, presumably because their valuation factor is zero, none being due after the date of valuation, though the sums assured are payable the instant after. I have included them as they form no exception to the application of my method. They happen to form a group by themselves and their value comes to -1 instead of zero. It is a type of error that can never exceed 2 in magnitude, being due to setting down the  $\alpha$  multiplier for premiums to the first place of decimals and the two products whose sum gives the value to the nearest integer. (For sums assured the  $\alpha$  multiplier is set down to the nearest integer. For both sums assured and premiums the  $\beta$  multiplier is set down to the nearest integer.)



Buchanan.\* Jumps are likewise noticeable in the numbers for certain unexpired terms in the model office for endowment assurances.

Tables showing the particulars and result of the valuation are annexed, and it will be observed from Tables V and VI that the method now suggested gives results that approximate closely to those of a detailed valuation.

TABLE I.

*Model Office No. 1 for Whole-Life Assurances (50 years).  
Particulars for each age attained at date of Valuation.*

Age attained	Sums Assured	Reversionary Bonuses	Net Premiums	Age attained	Sums Assured	Reversionary Bonuses	Net Premiums
21	8,364	...	109.2	59	53,810	15,871	1312.3
22	7,330	...	95.7	60	51,828	16,227	1261.9
23	6,633	...	86.6	61	52,293	16,632	1355.1
24	6,064	...	79.2	62	50,087	16,951	1292.0
25	5,644	...	73.7	63	47,864	17,175	1229.7
26	27,233	80	403.4	64	45,739	17,326	1172.1
27	25,030	153	370.4	65	43,608	17,383	1115.1
28	23,357	214	345.6	66	42,536	17,440	1138.8
29	22,108	274	327.0	67	40,378	17,444	1076.6
30	21,084	330	311.7	68	38,141	17,315	1013.2
31	46,195	626	763.9	69	35,948	17,098	952.4
32	43,622	900	720.2	70	33,774	16,797	892.5
33	41,437	1,147	683.6	71	30,314	15,224	816.9
34	39,772	1,388	655.5	72	28,269	14,862	759.8
35	38,315	1,616	631.1	73	26,164	14,389	699.7
36	59,869	2,149	1092.7	74	24,051	13,794	641.6
37	57,192	2,653	1041.1	75	21,900	13,076	582.9
38	54,852	3,116	997.1	76	16,783	9,483	479.5
39	52,879	3,556	960.0	77	14,943	8,796	426.8
40	51,262	3,979	929.9	78	13,163	8,055	376.0
41	66,857	4,684	1333.5	79	11,459	7,273	327.8
42	64,353	5,350	1279.6	80	9,870	6,489	282.8
43	61,988	5,966	1229.8	81	6,398	3,835	205.1
44	60,116	6,559	1190.9	82	5,369	3,340	172.6
45	58,409	7,118	1155.9	83	4,445	2,867	143.2
46	68,554	7,898	1481.3	84	3,609	2,409	116.7
47	66,175	8,623	1425.2	85	2,874	1,982	93.3
48	63,938	9,311	1372.9	86	1,618	1,007	60.1
49	62,002	9,960	1328.5	87	1,255	809	46.8
50	60,159	10,565	1286.8	88	956	636	35.9
51	66,156	11,325	1536.7	89	718	493	27.1
52	63,923	12,030	1479.0	90	535	380	20.2
53	61,710	12,674	1422.7	91	262	167	11.4
54	59,757	13,270	1374.6	92	184	123	8.0
55	57,822	13,795	1327.9	93	118	81	5.1
56	60,272	14,417	1484.4	94	67	48	2.9
57	58,014	14,974	1422.1	95	29	21	1.3
58	55,812	15,453	1363.3	96	6	4	0.3

\* I reconstructed the Z's from maturity ages derived from the published valuation ages. Those for unexpired terms 0 and 1 were not given, but I took for these the maturity age 58 for unexpired terms 2 and 3.

TABLE II.

*Model Office for Endowment Assurances (35 years).  
Particulars for each unexpired term at date of Valuation.*

Un- expired Term	Sums Assured	Net Premiums	%	Un- expired Term	Sums Assured	Net Premiums	%
0	132,947	5,292.9	167,513	20	103,958	2,918.7	136,185
1	136,427	5,431.5	171,898	21	106,834	2,999.5	139,953
2	139,707	5,562.1	176,031	22	110,350	3,098.2	144,559
3	143,006	5,693.4	180,188	23	114,833	3,224.0	150,431
4	146,790	5,844.0	186,423	24	121,264	3,404.6	157,643
5	152,863	6,021.3	194,136	25	61,424	1,486.7	86,239
6	156,818	6,177.1	200,727	26	63,467	1,536.1	89,108
7	160,698	6,329.9	205,693	27	66,108	1,600.1	92,088
8	164,563	6,482.2	210,641	28	69,604	1,684.7	96,958
9	168,583	6,640.5	215,786	29	74,926	1,813.5	103,548
10	169,540	6,528.5	218,707	30	22,109	457.4	34,291
11	173,539	6,682.5	223,865	31	22,973	475.3	35,631
12	177,780	6,845.8	229,336	32	24,118	499.0	37,407
13	182,382	7,023.0	235,273	33	25,674	531.1	39,820
14	187,843	7,233.3	242,317	34	28,144	582.2	43,342
15	156,473	5,329.9	203,415	35	4,630	82.6	7,820
16	160,520	5,467.7	208,676	36	4,847	86.5	8,187
17	165,212	5,627.6	214,776	37	5,140	91.7	8,681
18	170,956	5,823.2	222,243	38	5,554	99.1	9,381
19	178,716	6,087.5	232,331	39	6,237	111.2	10,453

TABLE III.

*Summary of Particulars. Whole-Life Assurances.*

Years of Birth	Age Groups	SUMS ASSURED		REVERSIONARY BONUSES		NET PREMIUMS	
		$\alpha$ -multiplier	$\beta$ -multiplier	$\alpha$ -multiplier	$\beta$ -multiplier	$\alpha$ -multiplier	$\beta$ -multiplier
1823-29	90-96	1,201	— 2,770	824	— 1,997	49.2	— 109
1830-39	80-89	37,112	— 75,124	23,867	— 47,509	1,183.6	— 2,194
1840-49	70-79	220,820	— 208,908	121,749	— 90,832	6,003.5	— 5,276
1850-59	60-69	448,422	— 154,322	170,991	7,969	11,606.9	— 3,293
1860-69	50-59	597,435	— 82,671	134,374	48,562	14,009.8	— 565
1870-79	40-49	623,654	51,664	69,448	54,469	12,727.5	2,656
1880-89	30-39	455,217	234,222	17,481	29,232	7,856.9	5,137
1890-98	21-29	131,763	226,902	721	322	1,890.8	3,458
...	...	2,515,624	...	539,455	...	55,328	...

TABLE IV.

*Summary of Particulars. Endowment Assurances.*

Years of Maturity	Average Maturity Ages	Unexpired Term Groups	SUMS ASSURED		NET PREMIUMS	
			$\alpha$ -multiplier	$\beta$ -multiplier	$\alpha$ -multiplier	$\beta$ -multiplier
1919	58.0	0	132,947*	- 66,474	5292.9†	- 2,647
1920-29	58.1	1-10	1,538,995	326,806	60,710.5	11,490
1930-39	58.4	11-20	1,657,379	-370,327	59,039.2	-24,930
1940-49	58.9	21-30	810,919	-700,032	21,304.8	-22,467
1950-58	60.9	31-39	127,317	-250,660	2,558.7	- 5,310
...	...	...	4,267,557	...	148,906	...

\* Payable the instant after valuation.

† None due after the valuation, but the total is taken straight from the class book, like that for any other group, and provides no exception to the application of the method. See note (\*) page 44.

TABLE V.

*Valuation of Whole-Life Assurances. O<sup>M</sup> 3 per-cent.  
 $\alpha$ ,  $\beta$  Method.*

Years of Birth	SUMS ASSURED		REFERRSORY BONUS		NET PREMIUMS		PERCENTAGE DEVIATION IN VALUE OF			
	Value	Deviation	Value	Deviation	Value	Deviation	S.A.	R.B.	N.P.	Net L.
1823-29	1,110	- 1	761	...	128	...	.09	...	...	.06
1830-39	32,501	+ 7	20,907	+5	5,007	- 6	.02	...	.12	.04
1840-49	177,777	+ 2	98,230	-3	40,052	- 4	...	...	.01	...
1850-59	319,518	- 3	122,539	-2	114,306	+ 8	...	...	.01	...
1860-69	363,103	- 2	82,386	-2	188,162	+ 5	...	...	...	...
1870-79	316,649	+12	35,732	...	214,582	- 7	...	...	...	.01
1880-89	191,859	+12	7,535	-1	155,745	- 9	.01	.01	.01	.05
1890-98	46,468	- 1	263	...	41,974	+ 2	...	...	...	.06
Totals ...	1,448,985	+26	368,353	-3	759,956	-11	...	...	...	...

Total net liability ... .. 1,057,382Deviation ... .. + 34

TABLE VI.

*Valuation of Endowment Assurances. O<sup>M</sup> 3 per-cent.  
α, β Method.*

Years of Maturity	SUMS ASSURED		NET PREMIUMS		PERCENTAGE DEVIATION IN VALUE OF		
	Value	Deviation	Value	Deviation	S.A.	N.P.	Net L.
1919	132,947	...	- 1	- 1	...	*	...
1920-29	1,315,885	- 30	301,077	- 95	...	·03	·01
1930-39	1,108,341	+ 198	664,906	- 257	·02	·04	·10
1940-49	433,388	- 12	338,807	- 50	...	·01	...
1950-58	55,854	- 9	49,219	+ 2	·02	...	·17
Totals ...	3,046,415	+ 147	1,354,008	- 401	...	·03	·03

Total net liability ... .. 1,692,407

Deviation ... .. + 548

\* See note (\*) page 44.

*Some further suggestions on the subject of Approximate Valuations. By ALFRED HENRY, F.I.A., of the Government Actuary's Department.*

[Submitted to the Institute, 2 February 1920.]

THE present Note deals with a modification of the method described in vol. li, pp. 118-126.

The method suggested in the original Note necessitated the finding of the values of the first, second and third continuous summations of the data to be valued. These summations in some cases were laborious, and the object of the present contribution is to demonstrate how the arithmetical work can be curtailed without material loss of accuracy, firstly, by reducing the number of continuous summations, and secondly, by summing in quinary groups instead of by individual terms.

The theoretical basis of the original method was that the valuation factor could be represented throughout by an expression of the form  $a + bx + cx^2$  and in practice this

assumption was found to apply with considerable accuracy to a variety of factors. Further investigation shows that results of almost equal accuracy can be obtained by splitting the data into two (or, in exceptional cases, three) parts and applying to each of these parts an expression of the form  $a + bx$ . For example, in the case of whole-life assurances the data are valued in two groups, one group consisting of ages up to 54 and the other for ages 55 and over; separate expressions are then calculated for the valuation factors in each group. Having expressed our valuation factor in the form  $a + bx$ , we require to know for our calculations only the values of the first and second sum of the data in each of the two groups of ages.

If the summations are made by individual ages, the errors involved arise solely from the differences between the true valuation factor and the value of the factor as derived from the assumed expression  $a + bx$ . If, however, the summations are made in quinary groups, two further errors are introduced:

- (a) The theoretical error involved by substituting the value of the factor at the central age of the group for the average value of the factor over the group.
- (b) The errors arising from unequal distribution of the data over the group and the consequent failure to secure a balance of error when the total of the group is valued by the factor for the central age of the group.

In practice these two errors are not to be distinguished, but they are different in principle; the error under (a) is present, whatever the distribution of the data, that under (b) depends solely upon the facts which are the subject of valuation.

The former error can be corrected in the following way. In the original Note it was pointed out that the best method of obtaining the values of the constants  $a$  and  $b$  in the expression  $a + bx$ , was by means of weighted equations for all ages which were subsequently combined in groups. If now, in any quinary group, ranging from age  $x-2$  to age  $x+2$ , the respective weights are  $w_{x-2}$ ,  $w_{x-1}$ ,  $\dots$ ,  $w_{x+2}$ , and the valuation factors are  $F_{x-2}$ ,  $F_{x-1}$ ,  $\dots$ ,  $F_{x+2}$ , the following equation results:

$$w_{x-2}F_{x-2} + w_{x-1}F_{x-1} + \dots + w_{x+2}F_{x+2} \\ = (w_{x-2} + w_{x-1} + \dots + w_{x+2})(a + bx)$$

since, in using the summations of quinary groups, we assume

the total weight to be concentrated at the central age of the group. By using weighted equations of this character, the theoretical error under (a) above, is eliminated.

In order to find the values of  $a$  and  $b$ , two equations only are necessary, and therefore the weighted equations in respect of each quinary group must be suitably combined in two groups of ages so as to form the final equation from which the values of  $a$  and  $b$  are calculated. Thus for whole-life assurances, in order to find the constants up to age 54, the groups of ages 20 to 39 and 40 to 54 were selected; for subsequent ages the constants were found from the groups 55 to 69 and 70 to 89.

The above system of weighting the values for individual ages is slightly more laborious than dealing with weighted values for age groups, but it gives greater accuracy in the final results. Where, however, the data to be valued commence and end with zero values, the theoretical error already referred to, tends to correct itself over the whole range of values, and weighted values for quinary groups only may be used. In a case such as endowment assurances grouped according to unexpired term, where the initial amounts of sums assured, &c., are not zero, the error introduced by dealing with quinary groups in place of individual ages might be important.

Having obtained values of  $a$  and  $b$ , the next point for consideration is the manner in which these values are to be adapted to meet the valuation by quinary groups. If the origin be taken at the lowest age of the initial group, and if  $U_1, U_2, U_3, \dots$  be the totals of the sums assured (or whatever other data are being valued) then the total value required, on the assumption that the amounts  $U_1$ , &c., are concentrated at the central age of the group is :

$$\begin{aligned} & U_1(a+2b) + U_2(a+7b) + U_3(a+12b) + \dots \\ &= (a-3b)(U_1 + U_2 + U_3 + \dots) + 5b(U_1 + 2U_2 + 3U_3 + \dots) \\ &= (a-3b)\Sigma U_1 + 5b\Sigma^2 U_1 \end{aligned}$$

where  $\Sigma U_1$  and  $\Sigma^2 U_1$  represent respectively the first and second sums of the quinary groups from the bottom upwards.

The following examples are based upon the same data as were used in the original note in vol. li. A comparison is thus afforded of the relative accuracy of the two methods. The data are given in the appendix in quinary groups. The



first example is worked in detail, so as to indicate the small amount of clerical labour involved.

EXAMPLE 1.—Office A. Whole-life. With Profits. Valuation of sums assured and bonuses.  $O^M$  3 per-cent.

The origin is taken at age 15 for ages 15 to 54 and at age 55 for higher ages. Any policies existing at ages under 15 must be included in the first age group, and summation made up to and including that group.

Age Group	Sums Assured and Bonuses	Sum from bottom upwards	
			$a-3b=217252$
			$5b=0449525$
15-19	22	11,514 = $\Sigma$	$(a-3b)\Sigma=2,501\cdot4$
20-24	162	11,492	$5b\Sigma^2=3,134\cdot4$
25-29	624	11,330	
30-34	1,305	10,706	Total, Ages 15-54 <u>5,635\cdot8</u>
35-39	2,022	9,401	
40-44	2,328	7,379	
45-49	2,398	5,051	
50-54	2,653	2,653	
		69,526 = $\Sigma^2$	$a-3b=590640$
			$5b=0491891$
55-59	2,363	7,070 = $\Sigma$	$(a-3b)\Sigma=4,175\cdot8$
60-64	1,683	4,707	$5b\Sigma^2=834\cdot5$
65-69	1,755	3,024	
70-74	733	1,269	Total, Ages 55 & over 5,010\cdot3
75-79	294	536	Add for Ages 15-54,
80-84	159	242	as above ... .. 5,635\cdot8
85-89	57	83	Total, all Ages... .. <u>10,646\cdot1</u>
90-94	17	26	
95-	9	9	
		16,966 = $\Sigma^2$	True Value = 10,646

EXAMPLE 2.—Office A. Whole-life. With Profits. Valuation of tabular premiums.  $O^M$  3 per-cent.

The limited payment premiums are valued separately.

*Premiums payable throughout life.*

Ages 15-54.

$$\begin{aligned}
 a-3b &= 2637456 & 5b &= -154338 \\
 \Sigma &= 254\cdot3 & \Sigma^2 &= 1572\cdot9 \\
 (a-3b)\Sigma &= 6,707\cdot05 \\
 5b\Sigma^2 &= -2,427\cdot58 \\
 \hline
 \text{Total for Ages 15-54} &= \underline{4,279\cdot47}
 \end{aligned}$$

Ages 55 and over.

$$a - 3b = 13.55482 \qquad 5b = -1.68884$$

$$\Sigma = 208.5 \qquad \Sigma^2 = 500.9$$

$$(a - 3b)\Sigma = 2,826.18$$

$$5b\Sigma^2 = -845.94$$

$$\text{Total for Ages 55 \& over} = 1,980.24$$

$$\text{Add ,, ,, ,, 15-54} = 4,279.47$$

$$\text{Total for all Ages} = \underline{\underline{6,259.71}}$$

*Limited payment Premiums.*Unexpired terms 1-15.

$$a - 3b = -.41835 \qquad 5b = 3.43005$$

$$\Sigma = 50.8 \qquad \Sigma^2 = 104.7$$

$$(a - 3b)\Sigma = -21.25$$

$$5b\Sigma^2 = 359.13$$

$$\text{Total for terms 1-15} = \underline{\underline{337.88}}$$

Unexpired terms 16 and over.

$$a - 3b = 10.71142 \qquad 5b = 2.03275$$

$$\Sigma = 15.8 \qquad \Sigma^2 = 20.8$$

$$(a - 3b)\Sigma = 169.24$$

$$5b\Sigma^2 = 42.48$$

$$\text{Total for terms 16 and over} = 211.72$$

$$\text{,, ,, ,, 1-15} = 337.88$$

$$\text{Total for all terms} = 549.60$$

Add total for premiums

payable throughout life ... 6,259.71

$$\underline{\underline{6,809.31}} \quad \text{True value} = 6,802$$

EXAMPLE 3.—Office B. Endowment Assurances. With Profits. Valuation of sums assured and bonuses. OM 3 per-cent.

The policies are grouped according to year of maturity, the valuation date being 31 December. The unexpired term is defined as "Year of maturity—Year of valuation."

Unexpired terms 1-15.

$$(a-3b)=1\cdot016270 \qquad 5b=-\cdot1014552$$

$$\Sigma=7609 \qquad \Sigma^2=18235$$

$$(a-3b)\Sigma=7,732\cdot8$$

$$5b\Sigma^2=-1,850\cdot0$$

$$\text{Total for terms 1-15} = \underline{\underline{5,882\cdot8}}$$

Unexpired terms 16 and over.

$$(a-3b)=\cdot688749 \qquad 5b=-\cdot0612515$$

$$\Sigma=15532 \qquad \Sigma^2=33711$$

$$(a-3b)\Sigma=10,697\cdot7$$

$$5b\Sigma^2=-2,064\cdot9$$

$$\text{Total for terms 16 \& over} = 8,632\cdot8$$

$$\text{Add ,, ,, ,, 1-15 ...} = \underline{\underline{5,882\cdot8}}$$

$$\text{Total for all terms} = \underline{\underline{14,515\cdot6}} \quad \text{True value} = 14,499$$

EXAMPLE 4.—Office B. Endowment Assurances. With Profits. Valuation of tabular premiums.  $O^M$  3 per-cent.

Unexpired terms 1-15.

$$(a-3b)=-1\cdot057587 \qquad 5b=3\cdot4826$$

$$\Sigma=366\cdot5 \qquad \Sigma^2=860\cdot2$$

$$(a-3b)\Sigma=-387\cdot6$$

$$5b\Sigma^2=2,995\cdot7$$

$$\text{Total for terms 1-15} = \underline{\underline{2,608\cdot1}}$$

Unexpired terms 16 and over.

$$(a-3b)=10\cdot18614 \qquad 5b=2\cdot10313$$

$$\Sigma=560\cdot2 \qquad \Sigma^2=1116\cdot3$$

$$(a-3b)\Sigma=5,706\cdot3$$

$$5b\Sigma^2=2,347\cdot7$$

$$\text{Total for terms 16 \& over} = 8,054\cdot0$$

$$\text{Add ,, ,, ,, 1-15 ...} = \underline{\underline{2,608\cdot1}}$$

$$\text{Total for all terms} = \underline{\underline{10,662\cdot1}} \quad \text{True value} = 10,677$$

EXAMPLE 5.—Office C. Valuation of annuities on female lives.  $O^{[w]}$  3 per-cent.

Owing to the special formation of the curve of annuities it is desirable to deal with the data in three sections, namely, ages under 50, 50-79, and 80 and over. The origin is taken at age 25, and therefore any annuities in respect of ages below 25 should be included in the age group 25-29.

Ages under 50.

$$a-3b=23\cdot25486 \qquad 5b=-1\cdot288874$$

$$\Sigma=1239 \qquad \Sigma^2=5263$$

$$(a-3b)\Sigma=28,812\cdot8$$

$$5b\Sigma^2=-6,783\cdot3$$

$$\text{Total for Ages under 50} = \underline{22,029\cdot5}$$

Ages 50-79.

$$a-3b=17\cdot29445 \qquad 5b=-1\cdot809912$$

$$\Sigma=10670 \qquad \Sigma^2=44375$$

$$(a-3b)\Sigma=184,531\cdot8$$

$$5b\Sigma^2=-80,314\cdot9$$

$$\text{Total for Ages 50-79} = \underline{104,216\cdot9}$$

Ages 80 and over.

$$a-3b=6\cdot18746 \qquad 5b=-1\cdot132853$$

$$\Sigma=1199 \qquad \Sigma^2=1477$$

$$(a-3b)\Sigma=7,418\cdot8$$

$$5b\Sigma^2=-1,673\cdot2$$

$$\text{Total for Ages 80 \& over} = \underline{5,745\cdot6}$$

$$\text{Add ,, ,, ,, 50-79} = 104,216\cdot9$$

$$\text{,, ,, ,, under 50} = \underline{22,029\cdot5}$$

$$\text{Total for all Ages} = \underline{131,992\cdot0} \quad \text{True value} = 130,946$$

N.B.—Two-thirds of the error arises from one large contract.

The above examples show that, given the appropriate constants for any mortality table, a check valuation can be

made with very little labour and will give results in close agreement with the actual facts. The method has the advantages that it is independent of individual values of the valuation factors, that it differs entirely in its processes from those of the valuation itself, and that therefore errors of calculation and transcription will not be perpetuated.

Seeing that it gives effect to the actual distribution of the business of any office, it has obvious advantages as compared with the use of a model office in the estimation of liabilities under different valuation bases. Different constants can be applied to one set of summations of the valuation data, and the results for different bases can be obtained very rapidly. Since the errors in the method arise largely from irregularities in the data, it follows that the *differences* between the values on different bases will be very accurate, since the effect of such irregularities is included in both valuations and consequently disappears when the differences are taken.

On a net premium valuation, however, a difficulty arises from the fact that a change in valuation basis involves a change in the net premiums. Inspection shows, however, that the differences between the net premiums on different bases change very slowly from age to age; it is therefore permissible to take the average rate of net premium for any quinary group, and thus obtain an average entry age, for which the new net premium can be found. This method should yield good results, since at the youngest valuation ages where the value of premiums is greatest, the possible range of entry ages is small, whereas at the older ages where the possible range of entry ages is larger, the value of net premiums is smaller.

*Office A.*

Age Group	Sums Assured and Bonuses	Tabular Premiums	Age Group	Sums Assured and Bonuses	Tabular Premiums
15-19	22	·4	60-64	1,683	52·0
20-24	162	2·7	65-69	1,755	61·7
25-29	624	11·5	70-74	733	21·4
30-34	1,805	26·1	75-79	294	7·4
35-39	2,022	42·4	80-84	159	2·9
40-44	2,328	50·8	85-89	57	·7
45-49	2,398	51·8	90-94	17	·3
50-54	2,653	68·6	95-99	9	·3
55-59	2,363	61·8	...	...	...

*Premiums payable for limited periods.*

Unexpired Term	Tabular Premiums	Unexpired Term	Tabular Premiums	Unexpired Term	Tabular Premiums
1-5	14.0	16-20	12.2	31-35	.2
6-10	19.7	21-25	2.4	...	...
11-15	17.1	26-30	1.0	...	...

*Office B.*

<i>n</i> *	Sums Assured and Bonuses	Tabular Premiums	<i>n</i> *	Sums Assured and Bonuses	Tabular Premiums
1-5	1,091	60.2	26-30	3,268	101.9
6-10	2,410	118.9	31-35	1,540	42.7
11-15	4,108	187.4	36-40	500	12.4
16-20	5,521	236.0	41-45	70	1.6
21-25	4,625	165.4	46-50	8	.2

\* *n* = Year of maturity—Year of valuation. Valuation date is 31 December.

*Office C.*

Age Group	Amount of Annuity	Age Group	Amount of Annuity	Age Group	Amount of Annuity
25-29	27	50-54	754	75-79	3,412
30-34	11	55-59	1,544	80-84	921
35-39	89	60-64	1,651	85-89	278
40-44	613	65-69	1,437	...	...
45-49	499	70-74	1,872	...	...

## ABSTRACT OF THE DISCUSSION.

MR. R. E. UNDERWOOD, referring first to Mr. Henry's method, said that if the required constants were not already available, or if it should be thought desirable to use constants calculated from similar data to those being valued, the previous year's detailed valuation provided a very ready basis for the purpose. It had to be borne in mind, however, that the constants were produced by using the valuation data as weights, and were not therefore theoretically applicable to any set of data. He thought that some limitation must be placed on the general applicability of constants calculated from one set of facts to the valuation of another set. That was the more evident if the applicability of the method to a select valuation was considered. He thought the course to follow in that case would be to deal with the valuation in 11 parts, namely, year of assurance



0, year of assurance 1, and so on up to years of assurance 10 and over. Each of the first 10 parts would consist of a single year's new business, which would be much more erratic in its incidence over the valuation ages than would be the total business of all durations. In one instance that had come under his notice the new amounts assured at two of the ages in a quinary group were £57,000 and £100 respectively. In such a case the errors arising from unequal distribution of the data over the group must be appreciable, especially if the constants had been calculated on an entirely different distribution. It seemed to him a weak point in the method that it did not admit of any limits being placed on the amount of the error involved in a particular case. It might be of interest to place on record the results of a short investigation of the practical application of the method which he had made with the help of Mr. R. D. Anderson. They first extracted from the class books one year's new business of a particular class, using the sums assured only, and valued them at individual ages by the  $O^M$  3 per-cent functions. They then used the values thus obtained to calculate the constants  $a$  and  $b$  and made a valuation by Mr. Henry's new method. Owing to the very short time at their disposal, they confined their investigations to the older group suggested by Mr. Henry, and, working with age 55 as the origin, they obtained a total value of £34,199 against an actual value of £34,257. The error of £58 amounted to one-sixth of 1 per-cent of the true value. The data on which they worked were very unevenly distributed, the sums assured being £18,700 at age 61, £100 at age 65, and £750 at age 69. As an experiment they then transposed the sums assured at ages 61 and 69 and, having adjusted the true values, made a valuation by the new method, using the constants previously calculated. As was to be expected, the error resulting was much larger, being £524 on a true value of £35,670, or nearly  $1\frac{1}{2}$  per-cent of the true value. Had they dealt similarly with the premiums, the error in the net reserve would no doubt have been much smaller, but the percentage error would presumably have remained about the same. As similar errors of opposite sign were likely to occur in a valuation of the different classes of business, the amount did not appear to be serious. Apart from the calculation of the constants, the only work involved in obtaining the value of the data in any given case was that of tabulating them in two sets of quinary groups and obtaining the first and second summations of each set from the bottom upwards—the second summation being, of course, only a cast. Those summations were then used with the constants in evaluating the valuation equations. No such simple method of valuation had ever been submitted to the Institute. The simplicity of the work was equal to that of the valuation methods to which actuaries were accustomed, and, judging by the examples given, remarkably accurate results were obtained in all normal cases.

With regard to Mr. Trachtenberg's method, before the multiplication by the  $\beta$  factor could be carried out, it was necessary to complete a series of small multiplications, such as those shown

in the first tabulated statement in the paper, for the calculation of *S<sub>r</sub>*. It was there, he thought, that the method lost some of its practical value, although, of course, it was that process which gave it its power and accuracy. He did not wish to exaggerate the arithmetical difficulties of multiplying by such quantities as  $4\frac{1}{2}$ ,  $3\frac{1}{2}$ , &c., but the fact remained that it had to be done for each group of sums assured, of bonuses and of premiums, and it could not fail to form a considerable charge on the time-saving elsewhere effected. The errors introduced in the application of the method appeared from the examples to be of no importance whatever, but again there was the fact that it was impossible to determine within what limits they would lie. In that connection it was interesting to recall the words of Mr. Lidstone in the discussion on Mr. A. E. King's paper. In justifying discussions of unimportant details, he said: "They" (*i.e.*, such discussions) "might lead those who had to consider the question to see, first, that a greater degree of accuracy could be obtained if it were wanted, and, secondly and eventually, that it was not wanted at all." It was certain that no greater degree of accuracy than that obtained by Mr. Trachtenberg in his examples need be sought, and he was heartily to be congratulated on the success he had achieved.

As to the uses to which these approximate methods might be put, he gathered from the remarks of Mr. A. E. King in the paper just referred to that at least one large Assurance Company valued its liabilities by an approximate method: but he felt doubtful whether large Assurance Companies generally were likely to adopt such methods. The largest part of the work of a valuation was that of writing up the classification books, and the work on the valuation sheets was comparatively light. In the application of Mr. Trachtenberg's method it was to be noted that the special classification books suggested by Mr. A. E. King might be used, and that was a point in its favour. It seemed clear, however, that that system of books could not be used with Mr. Henry's method, owing to the fact that the same age groups were retained at each successive valuation. In his opinion actuaries had become accustomed to face the work of an ordinary aggregate valuation, and would not abandon that unless some greater attraction could be held out. But the possibility of using an approximate method for a select valuation seemed to him worthy of further consideration. It was difficult to conceive a more unsuitable time for valuing on an aggregate basis than the present, when one considered that Offices which had done a rapidly increasing new business up to the middle of 1914 suddenly reverted for over four years to a new business of from one-half to two-thirds of their ordinary total, and were now closing their books for 1919 with more new business than they had ever done before in one year since they were established. Whatever might be said in favour of an aggregate valuation, he thought actuaries would feel some doubts as to the soundness of their valuation of liabilities on such a basis at the present time. Further, what interest could they take in the surplus account

prepared on such a basis? To suggest a full select valuation at a time when everyone was working at such high pressure would be useless, as indeed it had been in the past. Possibly, however, the new labour-saving methods would prove ultimately to be the solution. He understood that Offices doing an extensive foreign business, and having to make valuation returns under various local laws, found these approximate methods a great help. In conclusion, he would call attention to Mr. Henry's remarks as to the facilities provided by his method for valuations of the same data on several different bases. That was a great advantage at times, and one which would seem to be peculiar to the method in question.

MR. R. D. ANDERSON said that when two methods of dealing with the same problem were brought into such close juxtaposition as they had been that evening, it was natural to compare them with a view to seeing where they agreed and where they differed. Both Mr. Henry and Mr. Trachtenberg had made use of a straight line, Mr. Henry to represent the valuation function and Mr. Trachtenberg to represent the sums assured. Both methods resulted in a valuation divided into two parts, the first part,  $a$  or  $a_0$ , applied to the total sums assured and the second part,  $b$  or  $\beta$ , applied to the second summation of the sums assured, for the  $\Sigma S_x$  employed by Mr. Trachtenberg could, of course, be expressed in that form. These points of similarity suggested that there must be some close connection between Mr. Henry's  $a$  and  $b$  and Mr. Trachtenberg's  $a$  and  $\beta$ , and in fact if the sums assured in Mr. Henry's method were assumed equal at all ages his  $a$  and  $b$  could be expressed, like Mr. Trachtenberg's  $a$  and  $\beta$ , in terms of the mean of the valuation factor for the first quinary group, and  $a_0$  and  $b_0$ , the first and second central differences of the valuation factor. The essential difference between the two methods, therefore, seemed to be that Mr. Henry used the actual sums assured as weights in finding his  $a$  and  $b$ , while Mr. Trachtenberg assumed that the sums assured were linear. It was to be expected therefore that, if Mr. Henry's method was used to give a check valuation of the same data from which his  $a$  and  $b$  were derived, it would give very accurate results, but if used for future valuations, with the same  $a$  and  $b$ , the correctness of the result would depend on how closely the distribution of the quantities to be valued reproduced the distribution from which  $a$  and  $b$  were derived. If the distribution materially affected the values of  $a$  and  $b$ , it would appear rather dangerous to apply the  $a$  and  $b$  derived from a distribution of 5, 10 or more years ago. On the other hand, if the distribution did not much affect  $a$  and  $b$ , a considerable amount of labour would be saved by omitting the weighting with the sum assured, and the values of  $a$  and  $b$  could, like Mr. Trachtenberg's  $a$  and  $\beta$ , be obtained direct from the Table. He had made a valuation of Mr. Henry's data by finding two straight lines to represent the mean value of  $A_x$  for groups of 5 years, *i.e.*, he had applied Mr. Henry's method, but left out the weighting with the sums assured, the only difference being that he took the origin of his straight line for the first half of the Table at age 55 instead of 15 and got a

negative  $b$ , and he had obtained a total value of £10,652 as compared with the correct value of £10,646—an error of £6 only. That might be a happy accident, but on the face of it there did not seem much to be gained by weighting with the sums assured.

Turning to Mr. Trachtenberg's paper again, it was remarkable how close a value he obtained in his example, although the sums assured were by no means linear, and it occurred to him that in assuming that his sums assured were linear Mr. Trachtenberg had unnecessarily restricted himself. If  $f(x)$  could be expressed as a parabolic function of  $x$ , it was clear, since  $\sum x^2 f(x)$  was taken symmetrically about a central origin, that all terms in  $f(x)$  involving odd powers of  $x$  would disappear on summation, so that  $f(x)$  need not be linear, provided no even powers of  $x$  appeared in  $f(x)$ . Having come to that conclusion, he endeavoured to find how much error would be introduced by assuming that the sums assured could be represented by a parabola and was rather surprised to find that the error was negligible in the particular case he tried. He came to the conclusion that Mr. Trachtenberg's method would give very good results if the sums assured were at all regular in their progression, but he would be glad to hear if anyone else could confirm this conclusion.

Mr. R. C. SIMMONDS said that he had applied Mr. Henry's method to the valuation of a large block of whole-life business, and the particulars might be of some interest to Mr. Henry and to the members generally. The sums assured in the business in question amounted to nearly £3,000,000 and, owing to lack of time, he was obliged to confine his investigation to the sum assured, and therefore any consideration of true reserves had to be omitted. The business was all of fairly recent date, so that the distribution was not equivalent to that in a 50-year model office. The result of applying the method with constants derived from the business itself was to produce for the value of the sums assured, on a total of over £1,600,000, an error of under £1,000, an error of rather more than one-half per 1,000. A curious feature was that, on applying the constants given in Mr. Henry's paper to the same figures, the already trivial error was reduced very substantially. He would like to mention that the business in question was net, so that any disturbance caused by unduly large cases could be taken as having been eliminated. He would very much like Mr. Henry to give some clearer idea of the exact method of weighting which he proposed to adopt. The statement in his paper that "given the appropriate constants for any mortality table, a check valuation can be made with very little labour and will give results in close agreement with the actual facts, &c.", would seem to suggest that it was not strictly necessary to weight with particular regard to the business under consideration. The point was of some practical importance since the determination of the constants on each occasion would involve a considerable amount of labour.

SIR ALFRED WATSON did not think that Mr. Henry would claim for his method that it supplied more than a check valuation ;



there was no suggestion in his paper that the method was one by which the ordinary systems of valuation could be superseded. In certain valuations that were now being made, where there were no variations in the benefits assured or in the premiums, but everything was on a uniform basis, the method lent itself admirably to checking the results by calculation of the reserves, thus substituting one process for a minimum of two if the operation were performed on contributions and benefits separately. The value of the method clearly depended upon the facility with which the constants could be found, and in the case referred to the constants had been found by a standard distribution of the lives over the several ages from 16 to 70 and when calculated on this general basis were applied to each particular case in which a check of an individual valuation made by the ordinary method was required. The results were extremely satisfactory and that went some way to answer the query raised by the last speaker as to whether it was necessary in every case to re-calculate the constants from the distribution of the particular business the valuation of which it was desired to check. In the case of which he spoke, the constants calculated at the outset were to be applied, after careful consideration of course, to about 10,000 or 12,000 different valuations in which the age distribution of the assured varied widely. He thought it was worth having that on record, because, although papers of the kind were and always must be of theoretical interest, it was useful to know of any circumstances in practical working in which actual use was being made of theoretical methods. Apart from the practical value of a method of the kind as supplying a much-needed check in times of great stress, when, with all the precautions that could be devised, something might go wrong in the ordinary process of valuation, the study of such methods, their mastery and their application, formed an extraordinarily stimulating mental exercise, the cultivation of which was bound to make men more efficient in the performance of their actuarial functions. It gave a mastery over data and working processes that could not be obtained by confining oneself to the orthodox and routine methods of the text-books. For his own part, he welcomed very gladly the method of valuation which Mr. Henry had devised. He had not had an opportunity of studying in practical use Mr. Trachtenberg's alternative method, but he was quite sure that in that case also the study of the method could not fail to be of benefit.

MR. W. P. ELDERTON believed that the earliest attempt at approximate valuation was the well-known method of Woolhouse, who grouped the ages for which the central values of  $A$  were  $\cdot 3$ ,  $\cdot 4$ , &c., and multiplied these central values of  $A$ , by the corresponding total sums assured. This method sometimes gave poor results, and he thought the reason was that it did not take sufficient account of the general run of the quantities to be valued. The methods that were being discussed that night were not open to any such objection. Going back to the beginning of this sort of approximate work, A. E. King's method used a mean age in certain groups, with a modified assurance value or annuity value which made some

allowance to counteract the use of a large group, but even with the suggested auxiliary Tables, a certain amount of interpolation was necessary. In Trachtenberg's paper interpolation of that kind was avoided, but it was necessary, on the other hand, to have two multipliers  $a$  and  $\beta$ . As between the two methods it was a choice between interpolation and two multiplications. Of course, if Trachtenberg's method were to be used to any great extent, a fairly large number of tables would have to be prepared, but that would be no real objection. If any method were found to be generally applicable, tables would be prepared just as they were at present for  $D_x$  or  $N_x$  or other moderately useful functions. Both methods assumed certain forms for the functions to be valued or by which the valuation was to be made, but they did so not by adopting a form to begin with, but by proceeding by some order of differences. Henry's method seemed to be different from the others in that respect: he actually re-graduated the functions by which he was valuing. In the original version of his method Henry suggested that the valuation function should be put in the form  $a + bx + cx^2$ , and by that process and three summations the result of the valuation would be found. He now suggested that the data should be split into two parts and a straight line be used for valuation, but a different straight line for each part. That had the advantage of shortening the process by cutting out one summation, and it could be further shortened by the use of quinary groups. The actual work of valuation might possibly be further shortened by using auxiliary columns something like those suggested by Mr. A. E. King in his paper.

He had had in his mind for some years an approximate method which would, if it could be used, save still more work. In the case of a whole-life assurance the sum assured or net premium or office premium could be described roughly by a frequency curve of Type I, with a range of from age 5 to age 105. Henry had already shown that  $A_x$  could, for the purpose of valuation, be expressed in the form  $a + bx + cx^2$ . The quantity to be valued and the valuation factor might therefore be represented approximately by two algebraical expressions, and the product of those expressions would be the sum of the area and the first and second moments of the Type I curve multiplied by certain constants. Hence, by appropriate tabulation the valuation could be reached with a single multiplication. As it is possible to fit a curve roughly by using broad areas with the help of certain auxiliary tables, the final table for use in the valuation would take the following form: The first column would be the proportion which the sums assured from age 5 to age 40 bore to the whole sum assured that was being valued; the next column the proportion that the sum assured from age 40 to age 70 bore to the whole that was being valued. That roughly fixed the curve. A third column would give the amount by which the total sum assured or net premium had to be multiplied to produce the final result. The broad groups would entail no additional work because it was no more trouble to add 90 items in three sets of 30 each than in one set of 90.



He believed that some actuaries had the feeling that such approximate work as Henry's, Trachtenberg's and A. E. King's was not very useful in practice. Personally he did not agree with that view. If it were possible to reduce 240 multiplications to 10, it was worth doing, even if there was a large amount of balancing work to be done in addition to the multiplications. But there was another defence of such approximate methods, and he thought a far better defence, that it was along such lines that future developments of actuarial science might come in ways that could not at present be foreseen.

The PRESIDENT, in proposing a vote of thanks to the authors of the papers, said that he would like to see a more extensive application of the methods to actual facts—by which he meant not only the application of the same method to different data, but also the application of the different methods to the same data. He believed that such approximations were, and were likely to be, very powerful instruments in practical work, and he thought they were likely to be more valuable owing to the pressure which would fall on all actuaries in the near future on account of the changed conditions which were certain to arise as the result of the war. They would need easy means of making check valuations or check calculations, and therefore it was very important to be able to rely thoroughly on approximate methods which would detect not only errors of theory, but also errors which were likely to arise merely owing to pressure of work, such as Mr. Henry referred to when he said: "The method has the advantages that it is independent of individual values of the valuation factors, that it differs entirely in its processes from those of the valuation itself, and that therefore errors of calculation and transcription will not be perpetuated." He agreed entirely with Sir Alfred Watson and Mr. Elderton that the study of such problems as approximate valuation was a very useful exercise of judgment and a stimulation to original thought. As he had said before in the Hall, he thought the methods of education and the type of mind thereby evolved should be very useful to the community. The mind evolved by such education would be extremely useful in applying its powers to practical problems involving imagination, and, of all the practical problems which were coming before the actuary, those which required imagination were going to be the most important.

Mr. ALFRED HENRY agreed with Mr. Anderson that the two methods were not widely different in principle. Mr. Trachtenberg took a series of ten-year groups, whereas he divided the data into two groups for the whole series of ages; Mr. Trachtenberg using the ten-year groups found it unnecessary to weight his quantities, whereas he (the speaker) found the weighting was of considerable importance. Then Mr. Trachtenberg used functions which involved the values  $x$  and  $f(x)$ , and he himself used consecutive summations which in effect gave the same results. Therefore he did not think there was very much between them in principle. With regard to Mr. Underwood's remarks, he was much interested

in hearing of a practical application of the method. He had not had access to office records in order to test the value of the method from the point of view of Life Office work, but had used the Board of Trade returns. It was satisfactory to learn from Mr. Underwood and also from Mr. Simmonds that in working from actual figures it was possible to get accurate results. With regard to the question of the irregularity of individual values being likely to throw out the result, if one had a very large group of data at any particular age and valued the re-insurances by the same method, the errors were bound to be reduced. Apart from that, errors of distribution as between the ages in a particular group did not relate to that group entirely; regard must be had to the whole series of data from the first group to the last. If the grouping was in five-year age groups unequal weighting in one group might be counterbalanced in another group. With regard to the question raised by Mr. Simmonds, he was convinced personally that to get the best results from the method it was necessary to adopt some method of weighting. The errors in the method when using quinary groups might arise in three ways. There was the question as to whether there was proper distribution over the quinary group, *i.e.*, whether the values assumed by the data at individual ages were such as to give a correct result when valued at a central age; there was the question as to whether the approximate curve did in fact fit the values of the function to which it had been applied; and thirdly there was the question as to whether the distribution of ages was approximately in accordance with the weighting. The object of the weighting was to make certain that, even if the approximate curve did not fit the curve of the valuation factor, on a normal distribution there would still be a correct result. Without some system of weighting that could not be obtained. It followed that, if there was a distribution somewhat dissimilar from the actual weights used in the function, such an accurate result might not be obtained. It all depended on the balance of error, and, although by weighting one secured the balance of error in a normal case, one might secure the balance of error in other ways, and he thought that accounted for Mr. Simmonds's results in applying his particular constants. One did not get a very different result from the constants obtained by his system of weighting. It was a well-established fact that if the system of weights was a reasonable one the values of the actual weights themselves did not interfere very much with the final functions obtained. He would like to endorse what Sir Alfred Watson had said as to the method not being intended as anything more than a check valuation. As things were at present it would be rather difficult to put forward a valuation made on his lines as one complying with the Assurance Companies Act. For that reason he had never considered the method as affording more than a check valuation, but the value of it as a check valuation was that it did not reproduce particular errors. If the value of, say, a reversion had been entered incorrectly, that error was not reproduced; the valuation was made *ab initio*. On the question

of auxiliary columns, he had always recognised that it was possible to use auxiliary columns, but it was, he thought, a good feature of his method that auxiliary columns had not to be used. It seemed to him that the use of auxiliary columns through all the classification books and alteration books entailed a lot of extra work.

MR. TRACHTENBERG said that Mr. Underwood had referred to the necessity by his method of multiplying by  $3\frac{1}{2}$ ,  $4\frac{1}{2}$ , &c. If done in that way it would take two columns in the class books, but his intention was to use Mr. King's classification, multiply by 0, 1, 2, up to 9, and then subtract 4.5 times the total sum. Mr. Anderson, in comparing his method with Mr. Henry's, said that whereas he assumed that the sum assured was a linear function, Mr. Henry took the sums assured as weights, and he suggested that that was the sole difference between the methods. But his method took account of second differences, whereas Mr. Henry's took account of first differences only. He would like to emphasize the fact that he (the speaker) put forward his method as an exact method and not as a check method. The operations were too elaborate for a check, whereas they would be simple enough for a method which replaced a detailed valuation, and that was what he proposed it should do. He agreed that it was very desirable to test the various approximate methods by applying them to the same data. An approximate method of valuation might seem strange at first, yet after repeated application be found as simple as the ordinary detailed method. No method should be criticized merely on the ground that it was unfamiliar. He admitted the simplicity of Mr. Henry's method and thought it was wonderfully fitted for the purpose for which he had put it forward—as a sufficient check.

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Mr. A. E. King writes :

“ . . . Mr. Trachtenberg gives at the outset a table of particulars of Sums Assured for Valuation, the particulars being identical with those given in §12 of my paper. It is necessary in working by Mr. Trachtenberg's method to multiply  $\Sigma S$  by  $a$  and  $\Sigma Sx$  by  $\beta$ , and advisable that  $S$  for each age should be multiplied by  $x$  without any material cutting down of figures. In working by my method it is necessary to find the weighted age by dividing  $\Sigma Sx$  by  $\Sigma S$  and then the sums assured are multiplied by  $A_w$  where  $w$  is the weighted age found as above; and it is to be noted that  $S$  may be cut down from six figures to four figures or even three figures, if we choose to do so, without any sensible loss of accuracy. . . . The advantage of Mr. Trachtenberg's method is that extended tables of assurance and annuity values are dispensed with, but as against this there are the following disadvantages :

- (1) Two factors have to be used instead of one.
- (2) The figures should not be materially cut down. In practice it is my rule to use but two figures of the sum assured in

finding the weighted age and this effects a great saving of time and labour.

- (3) The bonuses also require to be weighted afresh at every valuation; there is no corresponding device put forward to that given in my paper which allows the weighted age obtained from the sum assured to be employed in valuing bonuses and premiums.
- (4) In working by my method with a weighted age the actuary can watch the progression of the weighted ages for each 10-year group from year to year and thus has some check on the work of his subordinates. Mr. Trachtenberg's method does not carry this advantage.

It should be borne in mind that my extended tables when once calculated are ready for use for all time, whereas the trouble of multiplying sums assured, bonuses and premiums by  $x$  for each age to the last figure is a task which is inflicted afresh at every valuation.

. . . . . Mr. Trachtenberg states that the system of class books described in §§30-35 in my paper will equally well serve the purpose of his method. His tables, however, would have to be extended to ten times the present size, for a given 10-year group comprising ages 20-29 will next year comprise ages 21-30 and so on. Mr. Trachtenberg also would require weights for bonuses and they would need to be changed after every distribution of surplus.

It may interest members to know that although it is my practice to make a valuation on the usual lines in respect of each individual year of birth and maturity for the purposes of Board of Trade returns the only check employed on the calculations is a second valuation of extended grouping, employing weighted ages."

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Mr. Henry sends, at the request of several members, the following example of the calculation of the constants  $a$  and  $b$ :

This example represents the calculation of the constants for the O<sup>M</sup> 3 per-cent annuity.

In this case the weighting and the calculations relate to quinquennial groups, but in certain circumstances it may be desirable to weight the values for individual ages (see remarks on p. 50); the principles of the calculation are, however, the same.

If  $a_x$  be the annuity value at age  $x$  and  $w_x$  be the corresponding "weight", then the following equation holds good:

$$w_x a_x = w_x (a + bx).$$

The data are divided into two groups for which separate constants are required, namely, ages under 55 and ages 55

and over. In the former group, for convenience, the origin is taken at age 15, and in the latter group at age 55. The following table can therefore be constructed for central quinary ages :

Age	Weight	Weighted Annuity	Weighted value of the function $a + bx$
17	0	0	...
22	14	311	$14a + 98b$
27	51	1073	$51a + 612b$
32	88	1736	$88a + 1496b$
37	113	2068	$113a + 2486b$
42	124	2079	$124a + 3348b$
47	127	1918	$127a + 4064b$
52	121	1614	$121a + 4477b$
57	109	1253	$109a + 218b$
62	93	897	$93a + 651b$
67	74	580	$74a + 888b$
72	50	308	$50a + 850b$
77	25	116	$25a + 550b$
82	9	30	$9a + 243b$
87	2	5	$2a + 64b$
92	0	0	...

Taking now the group relating to ages under 55, only two constants are required, and therefore only two equations are necessary. The weighted equations are therefore combined, somewhat arbitrarily, into two groups, ages 17-37 and 42-52 respectively, thus giving the following equations :

$$266a + 4692b = 5188$$

$$372a + 11889b = 5611$$

Whence  $a = 24.94853$   $b = -.308676$

Similarly for ages over 55, the following equations result :

$$276a + 1757b = 2730$$

$$86a + 1707b = 459$$

Whence  $a = 12.041515$   $b = -.337768.$

In the above example the weights used are based on the model office ; in actual practice, the weights should be chosen so as to reflect the circumstances of the particular calculation in view.



*Valuation of Redeemable Securities with allowance  
for Tax.*

IN view of the prospect of a high rate of income-tax being maintained for many years, it has become the usual practice—at any rate among actuaries—to estimate yields on a net basis after allowance for tax at the current rate. It may be useful, therefore, to discuss two or three special types of redeemable security that arise in investment work. In dealing with such securities it is not as a rule worth while to obtain an explicit formula for the approximate yield. It is sufficient to value at two rates and make a first-difference interpolation. Or in these times when quotations are often wide and business a matter of negotiation, the investor may have decided on the net yield at which he would regard a particular security or class of security as worth consideration, and may want a value to compare with current quotations. In either case the question becomes one of valuation to give a specified net yield—rather than of direct determination of the yield.

The rates of dividend and tax will be denoted by  $g$  (per interval) and  $t$ , and the net rate of interest by  $i$  (per interval).

This type of security presents no difficulty, as all that is necessary in order to allow for income-tax is to replace  $g$  by  $g(1-t)$  in the usual formulas.

Loan repayable  
by fixed  
instalments.

If repayments take place by drawings at par the value of a unit of the loan, regarded as an average sample of the whole loan, to yield the net rate  $i$  will be  $K + \frac{g(1-t)}{i} (1-K)$ , where  $K$  is the present value at rate  $i$  of the remaining instalments per unit of the outstanding amount of the loan.

If the repayments may be made by purchase when the loan is below par, then the value of a unit of the loan to yield the net rate  $i (> g-gt)$ , on the usual assumption that the unit in question will be held until the end of the term of the loan and will then be repaid at par, is  $1 - (i - g\overline{1-t})a_{\overline{n}|i}$ , where the final instalment of the loan is repayable at the end of  $n$  intervals. The amounts of the instalments do not enter into the formula, but it is obvious that towards the end of the term the demand for repayment account must greatly exceed the normal supply and that opportunities may consequently arise for the realization of a higher rate than  $i$  by sale.

An example of a loan repayable by fixed instalments by



drawings or purchase is afforded by the Belgian 3 per-cent Sterling Loan of 1914. On the basis of 6s. tax throughout the value as at 5 February 1920, to yield  $3\frac{1}{4}$  per-cent net half-yearly was  $100 - 0.22a_{38}^{3\frac{1}{4}}$  which  $= 52.37$ .

Loan repayable by annuity. A loan repayable by an annuity comes under the last heading, but admits of special treatment.

The remaining term of the loan being  $n$  intervals the successive instalments per unit of the amount outstanding will be  $1/s_n^g$ ,  $(1+g)/s_n^g$ , &c., and  $K$  will be

$$[v + v^2(1+g) + \dots + v^n(1+g)^{n-1}]/s_n^g,$$

which (in the usual case of  $i$  exceeding  $g$ )  $= a_n^{i'}/(1+g)s_n^g$  where  $i' = (i-g)/(1+g)$ , or more conveniently  $(a_n^-/a_n^g - g/i)/(1-g/i)$ . Hence the value of a unit of the loan to yield the net rate  $i$  is

$$g(1-t)/i + [1-g(1-t)/i](a_n^-/a_n^g - g/i)(1-g/i).$$

For interpolation purposes this expression may be written in the form  $(1+t/\lambda)a_n^-/a_n^g - t/\lambda$ , where  $i = (1+\lambda)g$ .

The Indian Railway Annuities are virtually loans of this type, but as they are quoted per £1 of annuity the expression for the value will take the form  $(1+t/\lambda)a_n^- - ta_n^g/\lambda$  where  $n$  is the remaining term. Thus, in the case of Great Indian Peninsula Railway A annuity (of which the schedule for income-tax purposes is calculated on the basis of interest at 1.425 per-cent half-yearly), if we neglect the final fraction of the annuity for the period from 1 July to 17 August 1948, and also the small deduction for expenses and management, the value at 1 July 1920, to yield, say, 3 per-cent net half-yearly on the basis of 6s. tax will be  $\frac{1}{2}[1.271a_{56}^{3\frac{1}{2}} - .271a_{56}^{1.425\%}]$ , which  $= 11.9$ . The B annuity presents a different problem. If the income of the existing sinking fund accumulations together with future sinking-fund contributions could be invested in the purchase of the B annuity at prices to yield 3 per-cent half-yearly, then the value of the annuity at 1 July 1920, to yield 3 per-cent net half-yearly on the basis of 6s. tax would be  $11.9$  (as above)  $+ p + qa_{56}^{3\frac{1}{2}}$ , where  $p$  and  $q$  (which could be estimated from the published list of sinking-fund investments) represent respectively the present value of the existing sinking-fund per unit of annuity and the half-yearly net interest thereon. Clearly the supposed sinking-fund

procedure could not be followed throughout the term of the annuity. Nevertheless, if consistently followed as long as possible it must (subject to possible further depreciation of  $p$  and conversion of A annuity into B on too favourable terms) have the effect of enabling an investor at the price above-stated to realize at least the required yield by sale.

Loan repayable by C S. F. Suppose the amount per interval applied to the service of the loan to be  $z+g$  per unit of the outstanding amount of the loan.

If repayments take place by drawings at par at the end of each interval the average value of a unit of the loan to yield the net rate  $i$  will be  $(1+t/\lambda)a_n/a_n^g - t/\lambda$ , where  $i = (1+\lambda)g$  and  $(z+g)a_n^g = 1$ . When the drawings take place less frequently than interest is payable this can be allowed for, if desired, by a modification of  $t$ . For the general formula becomes  $K + \frac{gs_i^{(m)}(1-t)}{i}(1-K)$ , and all that is necessary is to put  $(1-t)s_i^{(m)} = 1-t'$ . Thus, if drawings take place yearly and interest is payable half-yearly, the value to yield the net yearly rate  $i$  will be  $(1+t'/\lambda)a_n/a_n^g - t'/\lambda$ , where  $t' = t - \frac{1}{4}i(1-t)$ .

The Copenhagen 4 per-cent Loan of 1901, is repayable by annual drawings at par in approximately 47 years from 15 November 1901, interest being payable on 15 May and November. Hence the value on 15 November 1920, to yield the net yearly rate of 5 per-cent on the basis of 6s. tax will be  $2.165a_{\frac{5\%}{28}}/a_{\frac{4\%}{28}} - 1.165$ , which = 77 without regard to any profit on exchange.

If repayments may be made by purchase under par and  $i$  is  $> g$ , it must be assumed, as usual, that the purchases for repayment account will be made at prices to yield the net rate  $i$ . If  $n$  be the number of intervals in which the loan will be repaid,  $S_r$  the proportion of the loan (per unit of the existing amount) outstanding at the beginning of the  $(r+1)$ th interval, and  $X_r$  the price per unit at the beginning of the  $(r+1)$ th interval—so that  $S_0=1$ ,  $S_n=0$ ,  $X_0$ =the present value per unit and  $X_n=1$ —then

$$z+g = gS_{r-1} + X_r(S_{r-1} - S_r)$$

and

$$X_r = 1 - [i - (1-t)g]a_{n-r}^-$$

Hence, beginning with  $z+g = (1+g)S_{n-1}$ , a schedule of the future operation of the loan can be constructed backwards

until the value  $S_0 = 1$  is reached. The process is shown by the following extract from a schedule of the Argentine (Port of the Capital) 5 per-cent Loan,  $z + g$  being taken at the date of valuation as  $\cdot 04674$  on the amount outstanding,  $t$  as  $\cdot 3$ ,  $i$  as  $\cdot 03$  and therefore  $i - (1 - t)g$  as  $\cdot 0125$ :

$n - r$	$X_r$	$\log(z + g + X_r S_r)$	$\log(g + X_r)$	$(3) - (4)$	$S_{r-1}$	$S_{r-1} X_{r-1}$	$\log X_r$
0	1	$\cdot 6697$	$\cdot 0107$	$\cdot 6590$	$\cdot 0456$	$\cdot 0450$	...
1	$\cdot 9879$	$\cdot 9628$	$\cdot 0056$	$\cdot 9572$	$\cdot 0906$	$\cdot 0884$	$\cdot 9947$
2	$\cdot 9761$	$\cdot 1310$	$\cdot 0005$	$\cdot 1305$	$\cdot 1351$	$\cdot 1303$	$\cdot 9895$
...	...	...	...	...	...	...	...
25	$\cdot 7823$	$\cdot 8945$	$\cdot 9970$	$\cdot 9875$	$\cdot 9716$	$\cdot 7544$	$\cdot 8934$
26	$\cdot 7765$	$\cdot 9037$	$\cdot 9039$	$\cdot 9998$	$\cdot 9995$	$\cdot 7705$	$\cdot 8901$

so that the term of the loan to yield 3 per-cent net half-yearly should be  $13\frac{1}{2}$  years and the price 77. When an investment is made in a loan of this kind, it is very desirable that a complete schedule should be constructed and the market price compared at the end of each interval with the price shown in the schedule, because if purchases are made for repayment account at prices in excess of schedule prices the term of the loan may be lengthened and the investor will not realize the intended yield unless he sells.

But it would obviously be inconvenient to construct a complete schedule merely for valuation, and an approximate value may be obtained by some such method as the following. Since

$$X = (z + g)a_n^- - tg(vS_0 + v^2S_1 + \dots + v^nS_{n-1})$$

$$= (z + g)a_n^- - tg(va)_n^- \text{ say,}$$

and 
$$X = 1 - [i - g(1 - t)]a_n^-$$

it follows that

$$X = [z + g - tg(va)_n^- / a_n^-] / [z + i + tg(va)_n^- / a_n^-]$$

If  $S$  decreased from 1 to 0 in A.P. (which will be roughly the case),  $(va)_n^- / a_n^-$  would  $= (1/a_n^- - 1/n)/i$ , and this expression does not differ greatly, for a wide range of  $n$  and  $i$ , from  $\cdot 6$ .

Hence  $X = \frac{z + g - \cdot 6tg}{z + i + \cdot 4tg}$  approximately.

In the example given above this would make the 6 per cent net value 77·2. It gives also the convenient direct approximation to the yield  $i = (z + g - \cdot 6tg)/X - (z + \cdot 4tg)$ .

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\*.\* With reference to the article in the October 1919 number of the *Journal* on "The Valuation of Victory Bonds held to pay Estate Duty", Mr. Lidstone draws attention to the fact that the formula given in the footnote on p. 363 is applicable not only at date of issue, but also after  $t$  years (when  $t$  is integral) if  $z$  be taken as the proportion per unit of the outstanding amount of the loan to be redeemed at the next drawing, or in other words the ratio of the Sinking Fund including interest on bonds previously drawn to the outstanding amount of the loan; but the particular result  $1\cdot125a - \cdot125a'$  applies only at the date of issue.

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## UNEMPLOYMENT INSURANCE BILL.

### REPORT BY THE GOVERNMENT ACTUARY ON THE FINANCIAL PROVISIONS OF THE BILL.

To THE RT. HON. SIR ROBERT S. HORNE, K.B.E., K.C., M.P.,  
MINISTER OF LABOUR.

Government Actuary's Department,  
Wellington House,  
Buckingham Gate,  
London, S.W. 1.

SIR,

I have investigated the effect of the financial provisions of the Unemployment Insurance Bill, including the proposed conditions as to contributions and benefits, and submit the following report thereon.

#### NUMBER OF INSURED PERSONS.

1. The scheme applies to all employed persons under contract of service in Great Britain between 16 and 70 years of age with certain exceptions as set out in Part II of the First Schedule. The principal of these exceptions are the groups of persons employed in agriculture, in domestic service, in the teaching profession, in the naval, military or air service of the Crown, in the permanent civil service (in an established capacity), and in the service of local or public authorities, or in that of railway companies or other public

companies engaged in statutory undertakings, where the conditions of service are certified by the Minister to be such as to render insurance against unemployment unnecessary. Persons employed otherwise than by way of manual labour and at a rate of remuneration exceeding £250 a year are also excepted.

In the case of Ireland the persons included in the scope of the Bill are those employed in the "Insured Trades" under the National Insurance (Unemployment) Acts, 1911 to 1919.

The estimated numbers of employed persons, classified according to Industries, are given in the Appendix. These numbers for the most part relate to October 1919, and are based upon statistics obtained in the course of periodical investigations made by the Board of Trade as to the state of employment in the United Kingdom, certain modifications being made to allow of the risks of the class of General Labourers being separately treated in the course of the calculations. The data do not distinguish between non-manual workers remunerated at rates under and over £250 respectively, or separately indicate the numbers of employed persons who are over the age of 70. In this respect the figures shown represent a somewhat greater number of employed people than would be affected by the Bill at the present time, and some reduction should be made. There are, however, counter-balancing factors. In the first place, the Bill does not come into operation till October 1920, and the natural increase of the insurable population during the intervening 12 months must be allowed for. Secondly, there were in October 1919, considerable numbers in receipt of out-of-work donation, or still in the Forces and awaiting demobilization, and although the re-absorption of these groups in industry may chiefly tend to fill up, directly and indirectly, the numbers of the two large industrial sections which are excluded from insurance, namely, agriculture and domestic service, a substantial residue may be expected to add to the ranks of the insured population. After allowing for these factors it would appear reasonable to assume that the figures given above supply a sufficiently close approximation to the numbers to whom the scheme will apply on 1 October 1920, and they have been adopted accordingly.

The total numbers thus estimated to come under the scheme are as follows :

Men ... ..	8,332,900
Women ... ..	3,429,800

Of these numbers, 628,300 men and 455,900 women are estimated to be between 16 and 18 years of age, and therefore subject to reduced rates of benefits and contributions.

2. The scheme provides for insurance through a central Unemployment Fund (Clause 12), but by Clause 16 provision is made for insurance through separate schemes in cases where that method appears to the Minister to be the more appropriate, facilities being given for the establishment of schemes made by Joint Industrial Councils or associations of employers and employees, or by the



Minister himself. The effect of these provisions needs careful consideration from the financial point of view, since with a uniform rate of contribution the immediate effect of segregation under separate schemes, where that is resorted to in the case of industries with less than the general average amount of unemployment, must be to reduce the income of the central Unemployment Fund to a greater extent than the liabilities of the Fund are diminished. The provisions of Clause 16 (5) under which the State grant is fixed as a maximum of 30 per-cent of the normal rate (*i.e.*, one-tenth of the contributions of employers and employees instead of one-third) in cases where separate schemes are established emphasize this point, since segregation will not, as a rule, be practicable unless the rate of unemployment in an industry contemplating separate insurance is at least 15 per-cent below the general average. While it is impossible to say to what extent segregation may be resorted to, these considerations make it imperative that a full estimate of its effect should be included. I am instructed that the conditions of the industry may be expected to facilitate the establishment of separate schemes in the case of certain large industrial groups, including, in the aggregate, rather more than 3,000,000 employed persons, the average of whose estimated rates of unemployment\* is under 50 per-cent of the average of the rates applicable to all industrial groups within the Bill. I have accordingly assumed that these groups will be outside the central Unemployment Fund, while, for somewhat different reasons to those which govern the ordinary case, the Mercantile Marine is expected to require separate treatment (Clause 18) and is also excluded. It may further be assumed that in some other cases where industrial groups can be distinguished so far as to render possible the joint arrangements which Clause 16 contemplates, resort will be had to the powers given by this clause, and I have accordingly added to the numbers above stated 514,000 persons with a very low rate of unemployment as likely to go under special schemes. I thus arrive at the conclusion that of the employed persons brought under the Act, 7,995,000 will be insured through the central Unemployment Fund, and 3,767,700 through special schemes set up under the powers conferred by Clause 16. The distribution of the number assumed to be insured through the Central Fund, with reference to age and sex, is as follows :

Men aged over 18	...	...	...	5,146,700
„ „ 16 to 18	...	...	...	416,100
Women aged over 18	...	...	...	2,097,500
„ „ 16 to 18	...	...	...	334,700
Total				<u>7,995,000</u>

#### THE INCOME OF THE UNEMPLOYMENT FUND.

3. The joint contribution of employer and workman is fixed by the Third Schedule at *6d.* a week for men and *5d.* a week for women,

\* See paragraph 4 as to rates of unemployment.



these rates being reduced to 4*d.* and 3*d.* respectively in the case of persons under 18 years of age. The State grant is fixed at one-third of the contribution of employers and workpeople, thus bringing up the total contribution to 8*d.* a week for men and 6½*d.* a week for women. The estimated average number of contributions payable yearly, after allowing for periods of unemployment, sickness and holidays, is 46. The total annual income of the Unemployment Fund from the contributions paid in respect of the 7,995,000 employed people estimated to be insured through this Fund is accordingly £11,246,000, of which £2,812,000 will be contributed by the Exchequer. In arriving at these sums no deduction has been made in respect of the persons exempted from insurance under Clause 3, and in whose case employers' contributions only, with the accompanying State grant, are payable. The experience of Health Insurance in respect of corresponding provisions shows that the number of exempted persons is, relatively to the total, negligible, and reductions both of income and expenditure due to exemptions have accordingly been disregarded.

The Income of the Fund also includes the interest accruing from the investments of the Fund accumulated under the Acts of 1911 to 1919, amounting at the present time to approximately £19,000,000. The receipts from this source, which are not subject to Income Tax, amount approximately to £900,000 a year.

#### THE RATE OF UNEMPLOYMENT.

4. The assumed rate of unemployment, *i.e.*, the average proportion of persons who are unemployed at any time, is shown in respect of each Industrial group in the last column of the Appendix. As regards the "Old Insured Trades" the rate adopted in framing the financial basis of the Act of 1911 was 8·6 per-cent, the method adopted in arriving at this rate being described in the Actuarial Report issued when the Bill was before the House of Commons. As explained in my report on the National Insurance (Unemployment) Bill, 1919 (Cmd. 439), I am satisfied from the statistics which have subsequently become available that the risk was over-valued in 1911, and I estimate that the appropriate rate for this group of trades is 6·5 per-cent.\*

In the case of the other industrial groups the rates shown have been estimated by the Ministry of Labour. Wherever possible these rates have been based on the mean annual percentages of Trade Union members unemployed as supplied to the Board of Trade over a prolonged period, but in some cases adjustments have been con-

\* It appears from the Actuarial Report on the Bill of 1911, that in the absence of data as to the rates of unemployment in the several branches of the Building Trade the rate for the whole trade was taken as double the rate appropriate to one branch of the trade, namely: Carpenters and Plumbers. The experience of the years 1913-14, which may be taken on the whole as sufficient to indicate the relations between one branch of the Trade and another, suggests that the proportion should have been taken as 4 to 3, and this proportion has now been adopted.

sidered necessary to provide for the higher rates of unemployment believed to prevail among workers outside the Trade Union membership, and for disturbing factors introduced by recent changes in industrial conditions. In view of the high rate of unemployment among general labourers and the migratory character of this class, an estimate has been made of the probable number of general labourers employed in each of the groups, and these have been deducted and treated as a separate group subject to an unemployment rate of 10 per-cent.

In certain groups of industries no information as to the experience of unemployment appears to be available, and in these cases, namely: Transport (excluding Docks and Railways), Clerks and Shop Assistants and Commercial and Public Utility Services, the rates submitted to me have no statistical basis, though it is evident that they have been very carefully considered.

It is clear that, while in order to obtain a financial basis for the scheme some estimate of the general rate of unemployment is necessary, the material available for the purpose is far from satisfactory. That is inevitable in the circumstances, and the defect is only to be cured by the institution of such a scheme of insurance as is now proposed and in connection with which adequate data for the study of the subject will gradually be accumulated. In the present uncertainty I regard some adjustment of the data supplied to me by the Ministry of Labour as essential, and I have accordingly added 10 per-cent to the unemployment rates of all the industrial groups except those representing the trades insured under the Acts of 1911 and 1916. In the result I obtain average rates of unemployment in respect of persons insured through the central Unemployment Fund of 5·32 per-cent,\* representing nearly 17 working days per annum. It should be clearly understood that this is the yearly average over a series of years, *i.e.*, a trade cycle, and that no inferences as to its validity can be drawn from the experience of single years, or even of several years, if these do not represent a complete cyclical period.

It is stated above that the rate of unemployment adopted in the case of the "Old Insured Trades" is 6·5 per-cent, and the general average being 5·32 per-cent, it follows that as a group the old insured trades are subject to a rate of unemployment about 22 per-cent above the general average of all trades; while further analysis shows that this rate is 37 per-cent above the average of all trades exclusive of their own group. In view of the pronounced excess in the liability thus brought by the old insured trades it becomes of importance to consider how far the contribution income of the Fund from this group is supplemented out of the interest earned by the present Fund. It is found on investigation of the accounts that of the total Fund, now amounting to not less than £19,000,000,

\* It has been considered advisable to assume a somewhat higher rate for persons under the age of 18, in view of the time probably lost by many of those who, as this age is approached, are excluded from juvenile occupations, and pass into adult employment.

the sum accumulated from the contributions paid in respect of the old insured trades is not less than £16,000,000, producing an interest income of £750,000. This sum represents an annual payment, for about 2,600,000 persons, of nearly 6s. each, equivalent to an addition of about 20 per-cent to the contributions payable under the Bill. It may be assumed therefore that so long as the existing Fund is maintained intact its interest earnings viewed as a contribution made to the common stock by the old insured trades will put those trades, in relation to the general body, in a not less favourable position than will be several important groups now brought into insurance for the first time.

#### THE LIMITING PROVISIONS AS TO BENEFITS.

5. The Second Schedule provides that benefit shall be payable, at the rate of 15s. for men and 12s. for women, for each week in a "spell" of unemployment following the first week, subject to certain limitations of which the most important are (i) that benefit shall not be paid for more than 15 weeks in any insurance year and (ii) that no person shall receive more unemployment benefit than is represented by the proportion of one week's benefit for every six contributions paid by him.

The effect of the waiting week and the limitation of benefit to 15 weeks in any period of 12 months\* was fully investigated in the course of the actuarial enquiry made in connection with the Bill of 1911. The opinion was arrived at that these limitations would reduce the "benefit days" to 71 per-cent of the days of unemployment and the further limitation under the "one-in-five" rule (for which it is now proposed to substitute "one-in-six") was estimated, apparently, to reduce this proportion to about 64 per-cent. I have myself made an independent investigation of the subject from such facts as are available from the working of the Act of 1911 prior to the War, and have arrived at the conclusion that the several limitations proposed will restrict the benefit days to a figure between 60 per-cent, in periods of good trade, when spells of unemployment are relatively short as well as relatively few, and 70 per-cent in periods of bad trade when the contrary conditions prevail. Material for more precise calculation is not at present available, and for the purpose of this report I have concluded, therefore, that it will be sufficient if payment of benefit in 66½† per-cent of the estimated number of days of unemployment is assumed, and have framed my estimates accordingly.

#### THE EXPENDITURE ON BENEFITS.

6. I thus estimate that the expenditure on benefits from the Unemployment Fund will be £10,059,000 per annum in respect of

\* The Act was subsequently amended to substitute "the insurance year" for "any period of twelve months", thus increasing the liability to some extent.

† A somewhat larger percentage is assumed in the case of persons under the age of 18, in correspondence with the assumption of a higher unemployment rate at these ages.

the 7,995,000 persons assumed to be insured through the Fund. This sum represents nearly  $22\frac{1}{4}$  million weeks of unemployment, in respect of which benefit will be payable for  $14\frac{3}{4}$  million weeks.

The estimated expenditure in respect of the cost of administration, taken at 10 per-cent of the income of the Fund (Clause 9), is £1,214,000.

Provision is made (Clause 28) for payment towards expenses of insured persons in travelling to obtain employment. Judging by past expenditure under this head a sum of £100,000 a year appears to represent a reasonable estimate of this liability, and I have adopted it accordingly.

#### SUMMARY OF INCOME AND EXPENDITURE.

7. The following is a summary of the estimated annual income and the average expenditure of the Fund on the basis of a fixed number of 7,995,000 persons insured through the Fund.

<i>Receipts.</i>				£	£
From contributions of Employers	...	...	...	4,217,000	
"    "    " Workers...	...	...	...	4,217,000	
"    "    " the Exchequer	...	...	...	2,812,000	
From Interest on Investments	...	...	...	900,000	
				<hr/>	12,146,000
<i>Expenditure (average).</i>					
On Unemployment Benefit	...	...	...	10,059,000	
" Travelling Expenses of Insured					
Persons	...	...	...	100,000	
" Costs of Administration	...	...	...	1,214,000	
				<hr/>	11,373,000
Surplus (average)	...	...	...	...	<hr/> £773,000

The scheme is thus estimated to show, on the average of a series of years, a surplus of £773,000 a year. This amount appears large, if considered without reference to the transactions to which it relates, but on being examined with due regard to those transactions it is seen to represent no more than a prudent provision for unforeseen contingencies. It is equal to something over  $7\frac{1}{2}$  per-cent of the claims and would be fully absorbed by a permanent rise in the level of the claims to this comparatively slight extent, having in view the fact that an increase in the claims would of necessity be accompanied by a reduction in the receipts from contributions.

8. The estimated surplus is subject to one other qualification. If the claims did not vary, but year by year were in agreement with the average assumed, the surplus would be realized with regularity. But in fact, the claims will fluctuate, and even though the average, taken over a series of years, may accord with the expectation, the

ultimate surplus of a cyclical period may be considerably affected by the course which unemployment has taken from year to year. In the early part of the period, if the cycle should begin with employment at its highest point, and claims at a minimum, the existing Fund will be increased and further interest income will be secured towards meeting the strain when unemployment is at its maximum. If, on the other hand, the maximum rate of unemployment is reached quickly, the Fund will be reduced for a time, its interest income will fall, and the estimated surplus will be diminished by this loss of interest. Experimental calculations indicate that if the maximum rate of unemployment is deferred until the middle of a trade cycle, while the average rate over the cycle agrees with the present estimates, the surplus shown above will be fully secured, but that if the maximum rate of unemployment is reached in the early part of the cycle, the ultimate surplus may be 30 to 40 per-cent below the theoretical expectation. It is obviously impossible to predict the course of a trade cycle or to measure the extremes in the rate of unemployment which will be touched in its progress; in this respect the Fund is subject to contingencies, such as do not operate in any other sphere of insurance, and while for practical purposes the average conditions alone can be investigated, it is clear that the resulting "surplus" is not of the same dependable character as the surplus found on the valuation of a life assurance company or even of a friendly society.

#### THE EXCHEQUER GRANT.

9. The charge to the Exchequer in respect of the central Unemployment Fund is, as shown above, £2,812,000, but to this there falls to be added the payment from public funds to special schemes established by Joint Industrial Councils or associations of employers and employees, or by the Minister, under Clause 16. The estimated maximum expenditure under this head is £410,000, bringing up the total charge upon public funds, under the conditions assumed, to £3,222,000. It is, however, to be observed that if the degree of segregation proves to be lower than assumed in the estimates the charge upon the Exchequer will be increased. If, for instance, the number of employed persons brought under special schemes should be no more than two millions the cost to the Exchequer will, it is estimated, be £3,670,000 per annum; in this event the estimates as to the position of the Central Fund would be favourably affected, since the unemployment risks of all the industries assumed to segregate are, as indicated in para. 2, lower than the general average, and are materially lower therefore than the average of the classes assumed as insured through the Central Fund.

I am, Sir,

Your obedient Servant,

(Signed) ALFRED W. WATSON.

23 December 1919.



## APPENDIX.

*Estimated Number of Employed Persons within the scope of the Bill,  
classified according to Industries.*

Trade Group	MEN		WOMEN		Total	Rate of Unem- ploy- ment
	Over Age 18	Ages 16-18	Over Age 18	Ages 16-18		
Trades insured under 1911 Act ... ..	2,233,000	153,700	208,300	14,600	2,609,600	6·5
Trades insured under 1916 Act ... ..	620,600	55,100	266,900	45,300	987,900	4·0
Boot and Shoe .. ..	78,000	9,000	49,400	9,000	145,400	3·0
China, Earthenware, Glass and Mis- cellaneous ... ..	72,000	8,900	44,900	6,500	132,300	6·0
Clerks, Shop Assistants and Commercial ...	1,020,200	129,800	860,500	143,000	2,153,500	2·8
Clothing ... ..	99,500	9,500	338,600	54,400	502,000	6·0
Docks ... ..	98,000	2,000	...	...	100,000	10·0
Food, Drink and Tobacco ... ..	266,300	17,700	181,700	33,300	499,000	4·0
Furniture ... ..	133,900	13,600	27,600	9,900	185,000	6·0
General Labourers ...	190,000	10,000	...	...	200,000	10·0
Hotels, Boarding Houses, Restaurants and Places of Enter- tainment ... ..	80,000	9,000	237,000	24,000	350,000	2·5
Mercantile Marine ...	241,000	7,000	1,000	...	249,000	3·0
Mining ... ..	1,047,400	92,600	8,000	1,500	1,149,500	1·5
Paper and Printing ...	184,400	18,300	94,800	25,200	322,700	4·8
Public Utility Services	160,300	2,000	5,800	...	168,100	1·0
Railways ... ..	550,700	38,300	34,900	4,000	627,900	1·0
Textiles—						
Cotton ... ..	146,300	17,000	286,900	37,900	488,100	3·0
Wool and Worsted...	102,900	6,900	140,200	16,800	266,800	3·0
Other Textiles ...	124,400	14,600	159,000	29,000	327,000	3·0
Transport (not Docks or Railways) ... ..	255,700	13,300	28,400	1,500	298,900	6·0
Total ... ..	7,704,600	628,300	2,973,900	455,900	11,762,700	...



## NATIONAL HEALTH INSURANCE BILL, 1920.

REPORT BY THE GOVERNMENT ACTUARY UPON THE  
FINANCIAL PROPOSALS OF THE BILL.

TO THE RT. HON. CHRISTOPHER ADDISON, M.D., M.P., CHAIRMAN  
OF THE NATIONAL HEALTH INSURANCE JOINT COMMITTEE.

Government Actuary's Department,  
Wellington House,  
Buckingham Gate,  
London, S.W. 1.

SIR,

In compliance with your request, I submit the following Report explanatory of the financial provisions of the National Health Insurance Bill, 1920.

## BENEFITS AND CONTRIBUTIONS.

1. Comparing the benefits and contributions proposed in the Bill with the existing benefits and contributions, the increase in the case of each item is set out in the following table :

	MEN			WOMEN		
	At present	As proposed	Increase	At present	As proposed	Increase
Benefits :	<i>s. d.</i>	<i>s. d.</i>	<i>s. d.</i>	<i>s. d.</i>	<i>s. d.</i>	<i>s. d.</i>
Sickness, per week ...	10 0	15 0	5 0	7 6	12 0	4 6
Disablement... ..	5 0	7 6	2 6	5 0	7 6	2 6
Maternity ... ..	30 0	40 0	10 0	30 0	40 0	10 0
Annual charge upon Insurance Funds towards the cost of Medical Benefit ...	6 6	9 6	3 0	6 6	9 6	3 0
Sanatorium Benefit	0 9	Nil	*	0 9	Nil	*
Weekly contribution† :						
Employer ... ..	0 3	0 5	0 2	0 3	0 5	0 2
Insured Person ...	0 4	0 5	0 1	0 3	0 4	0 1
Total ... ..	0 7	0 10	0 3	0 6	0 9	0 3

\* The withdrawal of the sanatorium benefit from the Act is equivalent to the reduction of liabilities by 9*d.* per annum for each insured person.

† See paragraph 11 (a) as to the reduction of these rates in Ireland.

2. It is also proposed to increase the yearly allowances to insurance committees from 3*d.* to 4*d.* per insured person, and with

reference to the administration expenses of approved societies, I am informed that an amendment of the Regulations under section 35 of the Act of 1911 is intended, and that the normal allowance for this purpose will be increased, as from 1st January 1920, from 3*s.* 5*d.* a year to 4*s.* 5*d.* The additional contributions for which the Bill provides have regard to the liability in respect of these increases.

#### THE RELATIVE VALUES OF BENEFITS AND CONTRIBUTIONS.

3. The Bill provides for a State contribution of two-ninths of the cost of benefits (including the expenses of administration of approved societies and insurance committees), and seven-ninths of the expenditure therefore falls to be provided from the contributions of insured persons and their employers. The values of the liabilities and of the contributions applicable thereto have been actuarially calculated, in respect of persons entering into insurance at the age of 16, on the basis prescribed for the valuations of approved societies by the National Health Insurance (Valuation) Regulations, 1919 (S. R. & O. 1919, No. 1119), and the contribution appropriate to each item of the liabilities is found to be as follows :

	Men	Women
Weekly Contribution for $\frac{2}{9}$ Benefits :	<i>d.</i>	<i>d.</i>
Sickness Benefit ... ..	3·02	2·68
Disablement Benefit ... ..	1·11	·92
Maternity Benefit ... ..	·68	·49
Medical Benefit ... ..	1·92	2·01
Expenses of Administration ...	·94	·98
Total ... ..	7 $\frac{2}{3}$ <i>d.</i> = 7·67	7 $\frac{1}{12}$ <i>d.</i> = 7·08

The contributions payable in respect of women provide for insurance until marriage in the case of all women, for the continuance of insurance in the case of those women who remain in employment after marriage, and for special benefits (section 22 of the National Health Insurance Act, 1918, as extended by the Bill\*) to which women who give up employment on marriage are entitled for a limited period after the payment of contributions in respect of their insurance has ceased.

4. By section 55 of the Act of 1911, as amended by section 1 (1) of the Act of 1918, the Insurance Commissioners were required to retain from each contribution 1 $\frac{5}{8}$ *d.* in the case of a man and 1 $\frac{1}{8}$ *d.* in the case of a woman, for the redemption of reserve values, and by section 1 (2) of the Act of 1918 it was provided that from these

\* The weekly rate of sickness benefit payable in these cases is increased by the Bill from 5*s.* to 7*s.* 6*d.* ; maternity benefit is similarly increased from 30*s.* to 40*s.*

amounts there should be carried, in assigned proportions,  $\frac{4}{3}d.$  in the case of a man, and  $\frac{2}{3}d.$  in the case of a woman to the Contingencies Funds of societies and the Central Fund created by that Act. Under the altered conditions set up by the Bill it is necessary to provide larger contributions for these funds and also for the redemption of reserve values, the increase corresponding approximately, in each case, with the addition made to the money benefits. The Bill accordingly provides that the retained sums shall be increased respectively from  $1\frac{5}{6}d.$  to  $2\frac{1}{2}d.$  and from  $1\frac{1}{6}d.$  to  $1\frac{1}{2}d.$  The apportionment of the full contribution of  $10d.$  (men) and  $9d.$  (women) is thus as follows :

	Men	Women
To Benefit Fund (including administration) ...	$7\frac{2}{3}d.$	$7\frac{1}{2}d.$
„ Contingencies Fund and Central Fund ...	$1\frac{1}{6}d.$	$1\frac{1}{2}d.$
„ Redemption of Reserve Values... ..	$1\frac{1}{6}d.$	$1\frac{1}{2}d.$
Total ... ..	$10d.$	$9d.$

It is to be observed in respect of these contributions that, taking the insured population as a whole, they contain no margin other than that provided by the Contingencies Funds, the purpose of which is to protect the Benefits Funds of societies against the consequences of deficiency. Any addition to the benefits or other charges on the funds of approved societies in excess of the present proposals would therefore require a corresponding addition to the income of the Insurance Funds, as proposed by the Bill, or a corresponding modification of the benefits or charges in other directions.

5. The contributions shown above as payable to the Benefit Fund (namely,  $7\frac{2}{3}d.$  men,  $7\frac{1}{2}d.$  women) provide, as stated, for seven-ninths of the normal benefits in the case of an entrant at age 16, and the full value of the normal benefits to which such a person is entitled is therefore, as nearly as possible,  $10d.$  in the case of a man and  $9d.$  in the case of a woman. The normal benefits in the case of a person entering at age 16 are thus of equal value with the contributions, and it follows that the State grant of two-ninths of the benefits counterbalances the charge made upon new entrants in respect of both the Contingencies Fund and the redemption of reserve values. But since seven-eighths, at least, of the contributions paid to the Contingencies Fund remains with societies as an additional provision for expenditure on benefits, to be applied either in the reduction of a deficiency which otherwise would involve a diminution of benefits or, if not required for this purpose, in the provision of additional benefits, this contribution, with the accompanying State grant, must be regarded as available for the persons in respect of whom it is paid. Further, in the case of women, the Act of 1918

provides for a special State grant now estimated to amount, at the maximum, to £280,000 per annum through the medium of the Women's Equalization Fund. In view of the additions to be made to the benefits, it is proposed by the Bill to increase this grant by 25 per-cent, *i.e.*, to an estimated maximum (on existing numbers) of £350,000, a sum which, allowing for the State grant of two-ninths of the expenditure to which it is related, represents an effective grant of £450,000, or of about one halfpenny per insured woman per week. The full value of the benefits provided to an entrant of 16 years of age, apart from such provision as may be made by the Central Fund in cases of exceptional deficiency on valuation, is therefore the equivalent, in the case of a man, of 10·6*d.* per week, as against a contribution of 10*d.* and in the case of a woman of 10·1*d.* as against a contribution of 9*d.*

The foregoing figures relate, as stated, to the position of an entrant at age 16. In the case of persons at present in insurance the value of the benefits is considerably higher, the additional liability being provided for by the reserves already accumulated (including with these the reserve values credited under Section 55 of the Act of 1911) and by the new reserve values to be credited after the passing of the Bill (*see* paragraph 10 below).

#### THE NUMBER OF INSURED PERSONS.

6. In view of the changes in the industrial population due to the war it is a matter of some difficulty to estimate at the present time the number of persons who will be within the scope of the National Health Insurance Acts on 5 July 1920, but according to the best calculation of which the case admits it would appear that the number of such persons will be 15,850,000, of whom 10,850,000 will be men and 5,000,000 women. The distribution of these numbers between the several parts of the United Kingdom, and between approved societies and the class of deposit contributors, is as follows :

Country	MEN		WOMEN	
	Approved Societies	Deposit Contributors	Approved Societies	Deposit Contributors
England ... ..	8,300,000*	250,000	3,920,000	125,000
Scotland ... ..	1,130,000	30,000	530,000	15,000
Ireland ... ..	490,000	10,000	260,000	5,000
Wales ... ..	630,000	10,000	140,000	5,000
United Kingdom	10,550,000	300,000	4,850,000	150,000

\* Including persons entitled to benefits out of the Navy and Army Insurance Fund.

## CONTRIBUTIONS FROM EMPLOYERS AND INSURED PERSONS.

7. The estimated amount of contributions payable under the present Acts in respect of the number of insured persons above shown is £20,525,000 per annum, and the addition of 3*d.* per week in the joint contribution of employer and employee (in Ireland 2½*d.*) proposed by the Bill increases this sum to £29,750,000. The apportionment of these sums between the several parts of the United Kingdom and between employers and insured persons is as follows :

*Contributions.*

Country	AT PRESENT		AS PROPOSED		INCREASE	
	Employers	Insured Persons	Employers	Insured Persons	Employers	Insured Persons
	£	£	£	£	£	£
England	7,388,000	9,082,000	12,314,000	11,545,000	4,926,000	2,463,000
Scotland	1,002,000	1,232,000	1,670,000	1,566,000	668,000	334,000
Ireland	373,000	397,000	596,000	546,000	223,000	149,000
Wales	462,000	589,000	770,000	743,000	308,000	154,000
United Kingdom	9,225,000	11,300,000	15,350,000	14,400,000	6,125,000	3,100,000

NOTES.—(1) The normal division of the contributions is varied in respect of low-wage earners, the employers paying more and the insured persons less, with, in some cases, a State grant of 1*d.* a week. In the absence of information as to the number of persons falling within this category, which in any event must in present circumstances be very small, no account has been taken of this provision in estimating the respective totals of contributions payable.

(2) For the purpose of this Table the number of contributors is taken as 350,000 fewer than the number of insured persons, this difference representing persons who, having left insurable employment, remain entitled to benefits for the ensuing twelve months. Women who have left employment on marriage constitute a large proportion of this number.

(3) In view of the very small number of voluntary contributors it has been assumed in the preparation of this table that all insured persons are employed contributors.

## EXPENDITURE ON BENEFITS.

8. In proceeding to estimate the present expenditure on benefits for which the Bill provides, certain assumptions have been made. These are as follows :

- (i) That the present distribution of the membership of approved societies with reference to age accords with the distribution



obtained by tabulating the claims made by societies for Reserve Values in respect of entrants into insurance in 1912.

[The age distribution existing at 31 December 1918 will be ascertained when the valuations of approved societies, now proceeding, are completed. It is not expected that this will differ sufficiently from the figures relating to 1912 to affect materially such an estimate as is here presented.]

- (ii) That the claims upon approved societies in each part of the United Kingdom will be in accordance with the tables adopted as the basis of valuation, due regard being had to the ages of insured persons in each such part, and, in the case of women, to the relative number of unmarried and married persons and to the additional provision made for benefit expenditure out of the Women's Equalization Fund.
- (iii) That the claims of deposit contributors for sickness, disablement and maternity benefits will be in accordance with the past experience in respect of this class.

Subject to these conditions the expenditure in the year beginning 5 July 1920, for which provision is made, is estimated to amount under the present Acts, to £20,357,000, and under the provisions of the Bill to £28,653,000. The distribution of these amounts between several the parts of the United Kingdom and between the membership of approved societies (distinguishing, in this connection, between men and women) and deposit contributors is shown in the statement given on page 9. It should be explained that the figures of expenditure given in this paragraph refer only to payments derived from the contributions (with the reserve values) and the related State grant. The additional payments made by the Exchequer by way of grants-in-aid in respect of medical benefit and analogous services, as referred to in paragraph 9 below, are not included. As indicated in paragraph 1 above, it is proposed by the Bill that the expenditure on medical benefit (apart from these special grants-in-aid), for which 6s. 6d. a year for each insured person was provided in the finance of the Act of 1911, should be statutorily fixed at 9s. 6d.

#### EXCHEQUER GRANTS.

9. The statement given on page 10 shows the estimated liabilities of the Exchequer under the present and the proposed conditions, in respect of the first complete year over which the new scale of benefits will operate, the calculations being subject to the assumptions named in paragraph 8 above.

Under the Act of 1911 the State proportion of benefits (including expenses of administration) is fixed at two-ninths in the case of men and one-quarter in the case of women. A uniform grant, in future, of two-ninths (22·2 per-cent), applicable to both men and women, is proposed by the Bill (*see* paragraph 3 above), and in the case of



women the effect of adding the Women's Equalization Fund, which was created by the Act of 1918 and is now to be augmented, will be to make the State grant 26·5 per-cent of the benefits.

The total amount of the charge under present conditions is £5,170,000, and under the new conditions proposed by the Bill £6,942,000.

These sums include the normal State grant in respect of medical benefit, but not the further sums in respect of medical benefit which have been voted as grants-in-aid under the authority of section 1 of the Act of 1913. The amount of these grants in 1919 was about £3,100,000, and the corresponding charge in the immediate future will depend upon the issue of the arbitration between the Government and the medical profession now pending.

Separate figures are given for the State grant in respect of each part of the United Kingdom, save in regard to the Women's Equalization Fund and the Central Fund, which are common to the whole country and are under the control of the National Health Insurance Joint Committee.

#### REDEMPTION OF RESERVE VALUES.

10. To meet the increase of liabilities made by the Bill in respect of their existing members, approved societies will require supplementary reserve values which, it is estimated, will amount in all to about £45,000,000. At 5 July 1920, the date at which it is proposed to bring the amending Act into operation, the existing reserve values then unredeemed will amount to about £65,000,000. The total sum to be redeemed will thus be about £110,000,000, and it is estimated that the weekly contributions proposed to be applied to the Sinking Fund (namely, 1 $\frac{2}{3}$ d. men, 1 $\frac{5}{2}$ d. women) will produce a sufficient income to complete the redemption in about 35 years. The aggregate income available for this purpose in the first complete year under the new conditions will amount to about £4,800,000, of which £3,300,000 will be required for interest, leaving £1,500,000 to be applied towards the reduction of the outstanding liability.

In this connection it should be observed that the redemption period is dependent to some extent upon the purposes to which any surplus accumulating in the Reserve Suspense Fund set up under section 15 of the National Health Insurance Act, 1918, may be applied. If surplus balances in this Fund are appropriated to the redemption of reserve values which, having regard to the constitution of the Fund, would appear to be their proper use, the period of the Sinking Fund may be materially shortened.

#### MISCELLANEOUS QUESTIONS.

11. The changes made in the Bill in respect of the following classes require only brief explanation :

(a) *Persons resident in Ireland.*—Medical benefit is not provided in Ireland, but the Bill provides for the retention of sanatorium

benefit in that country. Setting off the annual charge for this benefit, namely, 1s. 3d. per insured person (including domiciliary treatment), against the relief of liability in respect of medical benefit, there is a net annual saving of 8s. 3d. per insured person, which allows of a reduction of contributions by 2d. a week. The appropriate rate of contribution in Ireland is accordingly 8d. for a man, or 7d. for a woman, and the Bill provides for the payment of these amounts, the contributions of employer and employee being respectively 1d. a week less than in the other parts of the United Kingdom.

(b) *Insured Persons serving in H.M. Forces.*—While an insured person is serving with the colours, contributions are required in respect of him firstly to provide for the cost of maternity benefit and his share of his society's expenses of administration while he is serving, and secondly to build up a reserve of such an amount as to secure that when he resumes full insurance on discharge, his society will be able to meet its liabilities to him without incurring loss. The present rate of contribution in respect of serving men is 3d. a week, but, having regard to the increased benefits conferred under the Bill, it is found to be necessary to increase the contribution to 4½d. a week. The Bill provides that no part of this contribution shall be deducted from the man's pay.

(c) *Seamen employed on Foreign-going Ships.*—The Act of 1911 makes special provision, rendered necessary by the circumstances of the employment, as to the contributions and benefits of certain classes of the mercantile marine. Under conditions defined by section 48 of the Act, as amended in subsequent statutes, the right to sickness, disablement and medical benefits is limited, broadly, by the period during which the insured person is resident in the United Kingdom, and while fewer than the normal contributions are required to be paid to maintain full benefits the rate of contribution is also reduced by 1d. a week. It is found that under the proposed new conditions this reduction should be 2d. a week, and provision is made by the Bill to this effect.

I am, Sir,

Your obedient Servant,

(Signed) ALFRED W. WATSON.

1 March 1920.

[Statements of estimated Expenditure on Benefits and of Exchequer Grants.  
See pp. 89 and 90.]

*Estimated Expenditure on Benefits (July 1920-July 1921).*

	AT PRESENT				AS PROPOSED				INCREASE			
	Approved Societies		Deposit Contrib- utors	Total	Approved Societies		Deposit Contrib- utors	Total	Approved Societies		Deposit Contrib- utors	Total
	Men	Women			Men	Women			Men	Women		
			£	£			£	£			£	£
England	11,356,000	4,804,000	224,000	16,384,000	15,967,000	6,786,000	294,000	23,047,000	4,611,000	1,982,000	70,000	6,663,000
Scotland	1,556,000	617,000	27,000	2,200,000	2,190,000	876,000	34,000	3,100,000	634,000	259,000	7,000	900,000
Ireland	510,000	235,000	5,000	750,000	721,000	341,000	6,000	1,068,000	211,000	106,000	1,000	318,000
Wales	856,000	157,000	10,000	1,023,000	1,202,000	224,000	12,000	1,438,000	346,000	67,000	2,000	415,000
United Kingdom	14,278,000	5,813,000	266,000	20,357,000	20,080,000	8,227,000	346,000	28,653,000	5,802,000	2,414,000	80,000	8,296,000

*Estimated Exchequer Grants. (Exclusive of Special Grants to Medical Services.)*

	AT PRESENT				AS PROPOSED				INCREASE			
	Approved Societies			Deposit Contri- butors	Total	Approved Societies			Deposit Contri- butors	Total		
	Men	Women				Men	Women					
		£	£	£	£		£	£	£	£		
England	2,523,000	1,201,000	51,000	3,775,000	3,549,000	1,509,000	66,000	5,124,000	1,026,000	308,000	15,000	1,349,000
Scotland	346,000	154,000	6,000	506,000	487,000	195,000	8,000	690,000	141,000	41,000	2,000	184,000
Ireland	147,000	79,000	2,000	228,000	206,000	100,000	3,000	309,000	59,000	21,000	1,000	81,000
Wales	190,000	39,000	2,000	231,000	267,000	49,000	3,000	319,000	77,000	10,000	1,000	88,000
United Kingdom	£3,206,000	1,473,000	61,000	4,740,000	4,509,000	1,853,000	80,000	6,442,000	1,303,000	380,000	19,000	1,702,000
<i>Supplementary Grants :—</i>												
Women's Equalization Fund	...			280,000				350,000	...	70,000	...	70,000
Central Fund	...	...	...	150,000				150,000	...	...	...	...
Total	...	...	...	£5,170,000			£	6,942,000	1,303,000	450,000	19,000	1,772,000

NOTES.—(1) The State grant in respect of Ireland is apparently in excess of two-ninths of the expenditure on benefits shown in the preceding Table. This is due to the fact that medical benefit is not provided in Ireland. The contributions are reduced by the equivalent of the full cost of medical benefit, and the Exchequer pays to the Insurance Funds the proper proportion of the amount represented by the reduction, this being treated as an addition to the State charge in respect of benefits.

(2) The State grant does not include the liability arising on the special grant of 1d. a week towards the contributions of low-wage earners. In present circumstances this grant is insignificant; see note to Table in paragraph 7.

## LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law.*

Presumption  
of death.

THE case of *Greig v. The Trustees of the Widows' Fund of the Company of Merchants of the City of Edinburgh*, reported (1919) W.C. and Ins. Rep. 170, was an action by Mrs. Agnes Douglas or Greig against the trustees of the Widows' Fund of the Company of Merchants of the City of Edinburgh (1) for declarator that her husband, David Greig, Junior, a contributor to the defenders' Widows' Fund, must be presumed to have died prior to 31 December 1900, and that the pursuer, as his widow, was entitled to an annuity of £40 out of the said fund, and (2) for decree ordaining the defenders to make payment to the pursuer of the annuity of £40 as from Whitsunday 1901.

The facts are briefly as follows: The pursuer's husband was born in 1857, and the marriage took place in 1879. The marriage proved unhappy, and in 1896 the pursuer obtained a decree of judicial separation from her husband on account of his cruelty, and a decree for payment of aliment. He was never able to make any payment to the pursuer under the said last mentioned decree either for aliment or for expenses. He left Edinburgh shortly after the decree and lived a vagrant life, and was never heard of after 1900. The pursuer believed and averred that he was dead and that he probably died about the year 1900 as a pauper without his identity having been known or discovered. The register of deaths had been searched for the years from 1896 to 1914 in Scotland, and from 1898 to 1917 in England and Wales, but no entry was found relating to the pursuer's husband. The defenders pleaded that the pursuer's averments were irrelevant and insufficient to support the summons, which should be dismissed, and that the presumption of life at common law had not been displaced.

The Lord Ordinary (Blackburn) in dismissing the action, said: "Had this been an application under the Presumption of Life Limitation (Scotland) Act, 1891, the mere disappearance of the pursuer's husband for 18 years would have entitled the Court to presume that he had died at least as far back as the year 1907. But this case appears to be one of those which are expressly excluded from the provisions of the Act by the terms of Section 11 thereof,

“ which provides that anyone claiming under a policy of  
“ assurance on the life of a person who has disappeared must  
“ in a question with the insurers prove the death of the person  
“ who is insured in the same manner as if the Act had not  
“ been passed. Be that as it may, the question in this case is  
“ raised in a common law action, and the presumption at  
“ common law is that a person who has disappeared is  
“ presumed to continue in life until he would have reached  
“ the ordinary limit of human age, unless sufficient evidence  
“ is adduced to raise a presumption that he has died within  
“ that period. It is not sufficient to prove that the person  
“ disappeared at a certain date, and that there is a ‘ general  
“ probability ’ that he is dead, but his disappearance must be  
“ associated with some particular incident which might involve  
“ a more than ordinary risk to life, and which coupled with  
“ unaccountable silence on his part since the date of the  
“ incident, raises a presumption that he died on or about that  
“ date. . . . The pursuer’s averments amount to nothing  
“ more than a statement that her husband has disappeared,  
“ coupled with a ‘ general probability ’ that he is dead. I do  
“ not think that they are strengthened by the somewhat vague  
“ averment that her husband’s friends and relations are  
“ satisfied that he is dead. It appears to me that these  
“ averments fall far short of what is requisite to elide the  
“ common law presumption, and that something more specific  
“ is required to enable the Court to presume that he is dead.  
“ The pursuer’s inference that her husband has probably died  
“ as an unidentified pauper appears to me to be less plausible  
“ than the suggestion of the defenders, that he had good  
“ reasons for concealing himself from his wife when he found  
“ that he could get nothing more out of her. It is not  
“ impossible that he may have married again, which would  
“ provide the best of reasons for concealing himself from her  
“ and his former friends and associates. Under these  
“ circumstances it appears to me useless to allow the pursuer  
“ a proof on the record as it stands.”

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The case of *Turton v. O'Brien*, 7 Tax Cases 170, deals with a claim for allowance for Income Tax purposes in respect of a lump sum paid in commutation of a portion of the future annual premiums payable under a policy of life assurance.

Amount paid in  
commutation of  
future premiums  
not allowable  
as a deduction  
for Income Tax  
purposes.



The case was stated by the Commissioners for the general purposes of the Income Tax Acts for the opinion of the High Court, and the facts are as follows :

The appellant had effected an insurance in 1896 with the Clergy Mutual Assurance Society for an annual premium of £19. 7s. 6d. Certain bonuses were applied in reduction of the premium with the result that in 1917 it stood at £13. 10s. 10d. On 28 November 1917 the Society offered to commute that annual premium and all future annual premiums for four annual payments of £33. 16s. 7d. each. The appellant did not accept this offer, but on 4 January 1918 it was arranged that he should pay to the Society the sum of £31. 5s. 10d., namely, the sum of £20 as a commutation payment in lieu of £2. 5s., part of the annual premium of £13. 10s. 10d., as from 1 December 1917, and the sum of £11. 5s. 10d., being the reduced annual premium payable on 1 December 1917 in respect of the said policy.

The appellant contended that under Section 54 of the Income Tax Act, 1853, and Section 17 (1) of the Finance Act, 1915, interpreted, as he submitted, by the decided case of *Hunter v. The King* (1904) A.C.161, 5 Tax Cases 13, he was entitled to allowance on any sum which in any year he had actually paid out of his own pocket to the Insurance Company, whether termed premium for that year or by whatever other name designated, so long as it did not in each year exceed one-sixth of his net income and since 1915, seven per-cent of the sum assured, and that if he paid more one year he naturally paid less the next.

The respondent, on behalf of the Crown, contended (*inter alia*) that the words "annual premium paid" or "the annual sum paid" under Section 54 of the Income Tax Act, 1853, were not altered by the words used in Section 17 (1) of the Finance Act, 1915, and that the latter Act limited the allowance granted under the Act of 1853, and that the appellant was entitled only to allowance of the amount paid as annual premium.

The Commissioners decided to refuse the claim for allowance of the payment of £31. 5s. 10d., but allowed £13. 10s. 10d., being £11. 5s. 10d. reduced premium and the £2. 5s., the part of the previous premium commuted.

Rowlatt, J., in giving judgment in favour of the Inland Revenue, said : "The case turns upon Section 54 of the Act

“ of 1853, and that says that premiums on policies, and  
 “ annual sums paid or deducted to secure deferred  
 “ annuities and so on, shall be deductible for Income Tax  
 “ purposes. . . . The point of the whole thing is that  
 “ it must be a payment of an annual character, because the  
 “ whole basis of the commercial transaction which this Act is  
 “ contemplating is that there should be a payment of an  
 “ annual character. The position is that a man has an  
 “ uncertain time to live, but that the probability of his life  
 “ can be estimated if you take a sufficient number of cases.  
 “ Therefore it becomes commercially possible to agree  
 “ future benefits in respect of a series of payments which need  
 “ not be the same every year, in amount, but it must  
 “ essentially be a scheme of payments to be spread over a  
 “ certain number of years, because the probability of those  
 “ years being lived out or not is only one of the elements  
 “ which go to the calculation of the amount. Now Section 17  
 “ of the Finance Act, 1915, says: ‘in respect of any premiums  
 “ or other payments payable on a policy for securing a capital  
 “ sum on death.’ This Section does not add a new payment  
 “ to the category in Section 54 of the Act of 1853, but it is  
 “ dealing only with payments which, apart from this Act of  
 “ 1915, would be within that Section 54. Therefore it is only  
 “ dealing with payments which are annual payments upon the  
 “ actuarial basis, an actuarial basis such as I have been  
 “ endeavouring to describe. If the payment is a commutation  
 “ payment and therefore is nothing in the nature of an annual  
 “ premium or payment—not an insurance payment at all, but  
 “ a commutation in order to bring the insurance adventure to  
 “ an end so far as the liability of the assured is concerned—  
 “ then it is not a payment to which this new section has any  
 “ relation at all. I have no alternative but to dismiss the  
 “ appeal with costs.”

Whether  
 premiums are  
 due on specified  
 dates or at end  
 of days of grace.

The conditions in a policy of life assurance governing the payment of premiums were considered in the case of *McKenna v. City Life Assurance Company*. The case is reported (1919) K.B. 491.

The facts are as follows: A policy of life assurance provided that the premiums were to be “payable on or before the last day of January, April, July and October” in each

year. By the conditions attached it was provided that "Thirty days of grace without liability to fine are allowed for the payment of each renewal premium", and that "any policy which has acquired a surrender value will not immediately lapse if a renewal premium be not paid within the days of grace, but will be kept in force for twelve calendar months from the date upon which the last premium became due", subject to payment of the arrear premiums and interest within that period. The premium payable on 30 April 1915 was paid. At that date the policy had acquired a surrender value. The premium payable on 31 July 1915, and all subsequent premiums were unpaid. On 7 August 1916 the plaintiff, who was assignee of the policy, with the object of keeping the policy on foot, offered to pay to the insurers the premiums then in arrear, contending that the twelve months mentioned in the conditions ran from the last of the days of grace, but the insurers refused to accept the premiums.

Scrutton, L.J., held that the last premium "became due" on the date specified in the policy as that on or before which it was payable—namely, 31 July 1915; that the offer on 7 August 1916 to pay the premiums in arrear was consequently too late, and that the policy had lapsed.

Company subject  
to British and  
Colonial  
Income Tax.

What is proper  
deduction from  
dividend  
on Preference  
Shares.

In the case of *Rover v. South African Breweries, Limited*, recently reported in these Notes, *J.I.A.*, vol. li, p. 272, Astbury, J., held that a company which has paid British income tax and also Colonial income tax, and has obtained from the British authorities repayment of the Colonial income tax can only deduct from the dividend due to the preference shareholders the amount of British tax less the Colonial tax recovered.

The same point came before the Second Division of the Court of Session (Scotland) and the three Lords Justices held that in paying the dividend on the preference shares the company was entitled to deduct the full amount of the British income tax, and was not bound to take into account the repayment it had received in respect of the Colonial income tax which it had paid.

The case is *New Zealand and Australian Land Company*,

*Limited, v. Scottish Union and National Insurance Company*, reported 57 S.L.R. 15. This was a special case to determine the amount of income tax deductible by the Land Company from a dividend of 4 per-cent on preference shares held by the Insurance Company in the Land Company's undertaking. The Land Company having paid United Kingdom income tax for the year in question at the rate of 5s. per £1 on its profits satisfied the Special Commissioners that Colonial income tax at the rate of at least 1s. 6d. per £1 had been paid in respect of part of the income earned in the Colonies. Accordingly in terms of section 43 of the Finance Act, 1916, they had claimed repayment, in respect of the profits earned in the Colonies, of part of the United Kingdom income tax equal to the difference between the amount so paid, namely, 5s. per £1, and the amount it would have paid if the United Kingdom income tax had been charged at the rate of 3s. 6d. per £1, *i.e.*, a difference of 1s. 6d. per £1. The Commissioners of Inland Revenue admitted the claim and repaid the sum of £28,435. 17s. 6d. The profits earned by the Land Company for the year in question were more than sufficient to pay the dividend of 4 per-cent to the preference stockholders without taking into account the said repayment. For the same year the Land Company paid to its ordinary stockholders a dividend of 10 per-cent per annum and a bonus of 5 per-cent free of income tax.

The Lord Justice Clerk said: "The fact that the Colonial Government had exacted at least 1s. 6d. per £1 of income tax from the Company's profits earned in the Colonies did not in any way affect the preference shareholders' right to a dividend of 4 per-cent. They were entitled to be paid their full 4 per-cent, subject always to the British income tax, altogether irrespective of what had been paid in the Colonies as Colonial income tax. They had been so paid. I cannot see, therefore, how the Insurance Company can be said to have paid in any way anything by way of, or in respect of, Colonial income tax. But the Land Company were not entitled to pay a dividend of 4 per-cent plus the whole or any part of the 5s. income tax chargeable and charged in Great Britain. If the Company were to pay 4 per-cent, less income tax at the rate of 3s. 6d. per £, they would in my opinion be paying a larger dividend than 4 per-cent, and so contravening the company's articles of associa-

“ tion. The Insurance Company were, in my opinion, not within  
 “ the class the hardships on whom section 43 was intended  
 “ to alleviate in respect that they did not pay double income  
 “ tax. It seems to me that the contention of the Insurance  
 “ Company would result in this, that at any rate as between  
 “ them and the company of which they are stockholders they  
 “ are to be dealt with, so far as payment of dividend is  
 “ concerned, as only liable to pay income tax at the rate of  
 “ 3s. 6d. per £1. In my opinion there is not sufficient warrant  
 “ in the statute to support this view . . . . . It may be  
 “ that other considerations would have to be dealt with if the  
 “ Insurance Company had not received their 4 per cent. for  
 “ the year in question. But on the facts of this case that  
 “ point does not arise.”

Lord Dundas in assenting said “. . . . I am of course  
 “ aware that we are differing from, and in effect overruling,  
 “ the decision of Astbury, J., in the case of *Rover v. South*  
 “ *African Breweries*. The point at which I am constrained  
 “ to dissent from the carefully reasoned opinion of that  
 “ learned Judge seems to arise at the conclusion of the first  
 “ paragraph thereof, where his Lordship says that ‘ although  
 “ ‘ the words of the section ’ (54 of the Act of 1842) ‘ refer to  
 “ ‘ the allowance of duty charged, it clearly, in my judgment,  
 “ ‘ means charged in the sense of paid or payable.’ The  
 “ point lies at the very root of the judgment in *Rover’s* case  
 “ and in this case. I have endeavoured to demonstrate from  
 “ the statutory words used that the duty ‘ charged ’ upon the  
 “ first parties is 5s. and no lesser rate.”

Courts  
(Emergency  
Powers) Act.  
Enforcing the  
lapse of a policy  
of insurance.

Section 1 (1) (b) of the Courts (Emergency Powers)

Act, 1914, provides as follows: “ No person shall

“ . . . . . enforce the lapse of any policy of  
 “ insurance to which this sub-section applies, for  
 “ the purpose of enforcing the payment or recovery of any  
 “ such sum of money, except after such application to such  
 “ Court and such notice, as may be provided for by rules or  
 “ directions under this Act.”

This sub-section applies to life or endowment policies for  
 an amount not exceeding £25, or payments equivalent thereto,  
 the premiums in respect of which are payable at not longer  
 than monthly intervals, and have been paid for at least the



two years preceding 4 August 1914. It has been considered and acted upon that this provision operated to prevent an assurance company from lapsing a policy to which the sub-section applies while the Act remains in force, notwithstanding that payment of premium has ceased. The Divisional Court in a recent decision, however, refused to accept this view and held that the setting up of the defence that the policy had lapsed owing to a failure to pay the premiums was not an attempt to "enforce the lapse" of a policy within the meaning of Section 1, sub-section 1 (*b*) of the Act of 1914.

The case is *Barker v. Phillips & Others*, and is reported 122 L.T.R. 123. The facts of the case are as follows: On 1 July 1912, William Barker became a member of the defendant lodge, and on the same date took out with the lodge a policy of life assurance for the sum of £10, the premiums on which were paid up to 7 January 1918, after which date no premiums were paid. The said William Barker died on 11 November 1918 while on active service in Egypt, but the lodge, although requested so to do refused to pay over the policy moneys to the plaintiff, the widow and administratrix of the said William Barker. On 11 January 1919 the lodge was requested in writing by the plaintiff to submit the matter to Arbitration in accordance with the rules of the lodge, and on 25 January 1919 both the lodge and the Central Body of the Order declined to entertain the plaintiff's claim. No application under the Courts (Emergency Powers) Acts 1914-1917 was made by the lodge to the Court, and the plaintiff claimed the £10 insured, less 9s. 3d. due from the deceased for arrears of premiums between 7 January 1918 and 11 November 1918.

The County Court Judge before whom the case came in the first instance held that the provisions of the Emergency Acts had not been complied with and gave judgment for the plaintiff for the balance of £9. 10s. 9d. The defendants obtained leave to appeal, and on 28 October 1918 the matter was brought before Lush and Sankey, JJ., sitting as a Divisional Court. The grounds of appeal were:

- (1) That the learned judge was wrong in law in deciding that the defendants were enforcing the lapse of any policy of insurance to which the Courts (Emergency Powers) Acts, 1914-1917, applied.
- (2) That the learned judge misdirected himself in holding



that any notice was necessary before the suspension of the said William Barker from funeral benefits under Rule 11 of the lodge became effective.

- (3) That no application was necessary under the Courts (Emergency Powers) Acts, 1914-1917, before the said suspension from funeral benefits became effective as a defence to any claim in respect thereof.

In giving judgment Lush, J., said: "In this case the administratrix of one W. Barker sued the defendants to recover the sum of £10 said to be due under a policy of assurance, less the sum of 9s. 3d. for which the plaintiff gave credit, for arrears of premiums. The defendants set up the defence that they did not owe this or any other sum to the plaintiff, because by the terms of contract with the deceased the policy moneys were not to be payable if the premiums were in arrears for more than twenty-six weeks, and they were so in arrear. So far as the contract is concerned it is plain that the defendants were not indebted, and it is conceded that the contract was at an end. But it is contended that by the provisions of the Courts (Emergency Powers) Acts, 1914-1917, the defendants are debarred from setting up this defence that they are not indebted to the plaintiff. . . . I will first of all deal with the question what is the true construction of the Courts (Emergency Powers) Act of 1914 and of the amending Act of 1916, and see whether the provisions of these Acts apply to a case of this nature. . . . Turning then to Section 1(1)b of the Act of 1914, we find it provides as follows: 'No person shall . . . levy any distress, take, resume or enter into possession of any property, exercise any right of re-entry, foreclose, realize any security . . . forfeit any deposit, or enforce the lapse of any policy of insurance to which this sub-section applies, for the purpose of enforcing the payment or recovery of any such sum of money, except after such application to such court and such notice as may be provided for by rules or directions under this Act.' Now it is said that when an insurance company sets up by way of defence to a claim the answer that by the terms of the contract the plaintiff cannot recover because the premiums have not been paid for a certain specified time, the insurance company are endeavouring to 'enforce a lapse of a policy of insurance'

“ within the meaning of the section. It is said, and I assume  
“ for the purpose of this action, that this is a policy of  
“ insurance to which that section of the Act applies, and we  
“ have to decide whether the words of the section which I  
“ have read are applicable to the defence which the company  
“ are seeking to raise in this case. In my opinion this sub-  
“ section has no application to a case of this sort. . . . The  
“ section means this, that a creditor such as a landlord shall  
“ not levy distress, and shall not exercise his right to enter,  
“ and take possession, because the tenant has made default  
“ in payment of rent, except by leave of the Court. In other  
“ words, the plain object of the section is to protect people  
“ who cannot pay what they owe from having a distress levied  
“ or from other harsh measures that a creditor might take, and  
“ so give them the opportunity of setting up as defence a set  
“ of facts which would enable the Court to protect them against  
“ the consequences of their inability to pay owing to the war.  
“ If the respondent’s interpretation of this section is right, it  
“ would have exactly the opposite effect to that intended. It  
“ would enable a person who had no claim to any sum of money  
“ at all, to come to the court and, by seeking to take the benefit  
“ of an Act which was passed for entirely different purposes,  
“ compel a defendant to pay money which he did not owe. With  
“ all respect to the learned County Court Judge that  
“ interpretation must be wrong. The section when it speaks  
“ of a person enforcing a lapse of a policy of insurance is  
“ dealing with the case of a company which may have a right  
“ to obtain money from some policyholder which they cannot  
“ enforce until they have lapsed the policy. In other words,  
“ the section, when dealing with this class of case, is restricting  
“ the right of insurance companies to enforce the payment of  
“ some sum of money, and is putting them in precisely the  
“ same position as the other creditors to which the section  
“ refers. The defendants in this action are not enforcing a  
“ lapse of a policy at all, they are not taking any steps to  
“ obtain money from the plaintiff, they are asking for no  
“ relief, but are merely seeking to set up the defence that  
“ this money is not due and owing under the contract.  
“ The plaintiff’s argument if acceded to, would have the very  
“ remarkable result that if a large number of policyholders  
“ who had failed to pay their premiums were suddenly to  
“ bring actions against an insurance company to compel them

“ to pay the policy moneys, in every one of those cases the  
“ insurance company would be prevented from saying that  
“ they did not owe the money. It would amount to this, that  
“ a company would have to admit a claim which they did not  
“ owe, and which had ceased to be payable months before the  
“ action was brought. In my view the section has nothing to  
“ do with a case like the present one . . . . . One  
“ other consequence would follow if the respondent’s contention  
“ is right—namely, that although the company had been  
“ without the premiums for several months, and had had no  
“ opportunity of investing that money and providing the fund  
“ for the payment of the policy money, yet, on the arrears of  
“ premiums being handed over to them, they would be bound  
“ to pay the policy money; I do not see how, under those  
“ circumstances, an insurance company could carry on its  
“ business. In my opinion, therefore the learned County Court  
“ Judge was wrong. This section was never meant to apply to  
“ a case like the present, and the appeal must be allowed.”

Sankey, J., expressing agreement, said . . . . .  
“ I think that the scheme of the Act is that where someone  
“ wants to start any process of Court himself, he must obtain  
“ the leave of the Court. I do not think the Act applies  
“ to where he is not anxious to start anything, but merely  
“ wishes to put in an appearance and defend a process started  
“ by another party. In this case if anyone lapsed the policy  
“ it was the plaintiff, who did so by failing to pay the premium.  
“ What meaning, then, can be given to the words ‘enforcing  
“ ‘the lapse of a policy of insurance.’ I agree that some  
“ meaning must be given to them as Parliament cannot be  
“ assumed not to have meant anything by those words. I  
“ hesitate to give a meaning to them as it is not necessary for  
“ our decision, but it may be that the inclusion of these words  
“ might prevent an insurance company suing for some penalty  
“ due on a lapse, unless they had applied to the Court for  
“ leave.”

The remarks of the learned judges are certainly interesting, although some doubts have been thrown upon the soundness of the decision. So far as I know, no companies have altered their practice or endeavoured to act upon the decision, and probably any attempt to do so would be restrained by further legislation.

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*An elementary demonstration of Stirling's approximate formula for the value of factorial  $n$ .*

1. IN the higher branches of the Theory of Probabilities and in many other mathematical investigations, we are concerned with the ratios between the factorials of large numbers, and for this purpose Stirling's approximate formula is largely used. The following elementary demonstration of that formula is based partly on an investigation given by Bertrand in his "Calcul des Probabilités", pp. 72-3, and partly on one by the Italian mathematician, Cesàro\*; but Cesàro's simple process—which involves only a well-known logarithmic series and an elementary application of the principle of limits—is here extended so as to produce limits of error much narrower than his, and much narrower also than those produced by a more difficult algebraical investigation by Chrystal—*Algebra*, Part II, pp. 369-71. This extension is believed to be new.

2. If we put  $n! = u_n$ , we have

$$u_{n+1}/u_n = n+1 = \frac{(n+1)^{n+1}}{n^n} \left(1 + 1/n\right)^n$$

But  $(1 + 1/n)^n \rightarrow e$  as  $n \rightarrow \infty$ .† If therefore we put

$$n! = u_n = n^n e^{-n} \phi(n),$$

we shall have

$$\phi(n+1)/\phi(n) = e/(1 + 1/n)^n, \text{ which } \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Now  $\log [e/(1 + 1/n)^n] = 1 - n \log (1 + 1/n) = \frac{1}{2n} - \frac{1}{3n^2} \dots$

and  $\log [(n+1)/n]^{\frac{1}{2}} = \frac{1}{2} \log (1 + 1/n) = \frac{1}{2n} - \frac{1}{4n^2} \dots$

Thus  $\phi(n)$  will change in approximately the same ratio as  $n^{\frac{1}{2}}$  changes, if  $n$  is at all considerable. We are thus led to put  $\phi(n) = n^{\frac{1}{2}} \psi(n)$ , or  $n! = n^{n+\frac{1}{2}} e^{-n} \psi(n)$ . We proceed to find limits for  $\psi(n)$ . From the last formula we have

$$\psi(n)/\psi(n+1) = [(n+1)/n]^{n+\frac{1}{2}} e^{-1};$$

or, taking logs.,  $-\Delta \log \psi(n) = \log \{ [(n+1)/n]^{n+\frac{1}{2}} e^{-1} \}.$

\* "Corso di analica algebrica", Torino, 1884, pp. 270 and 480. (See "The Mathematical Theory of Probability", vol. I, by Arne Fisher. The Macmillan Company, N.Y., 1915, pp. 93-5).

† The useful symbol  $\rightarrow$  stands for "tends to the value."—See *J.I.A.*, vol. li, p. 135.



ratio  $\rightarrow 1$ , since  $t(n) \rightarrow 0$  as  $n \rightarrow \infty$ . They must therefore tend as  $n \rightarrow \infty$  to a common limit, say  $C > \psi(n)e^{-1/12n}$  and  $< \psi(n)e^{-1/12[n+t(n)]}$ .<sup>\*</sup> Thus

$$\psi(n)e^{-1/12n} < C < \psi(n)e^{-1/12[n+t(n)]}$$

or

$$\begin{aligned}\psi(n) &> Ce^{1/12[n+t(n)]} \\ &< Ce^{1/12n}\end{aligned}$$

Substituting in the expression  $n! = n^{n+\frac{1}{2}}e^{-n}\psi(n)$ , we have

$$n! > Cn^{n+\frac{1}{2}}e^{-n} \cdot e^{1/12[n+t(n)]} \quad . \quad . \quad . \quad . \quad (2)$$

$$< Cn^{n+\frac{1}{2}}e^{-n} \cdot e^{1/12n} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

5. The foregoing argument may be expressed in the following alternative way. Since

$$\Delta \log \psi(n) > 1/12(n+1) - 1/12n$$

$$\text{and} \quad < 1/12[n+t(n+1)+1] - 1/12[n+t(n)],$$

we have by addition of differences,

$$\begin{aligned}\log \psi(N+m) - \log \psi(N) &> 1/12(N+m) - 1/12N \\ &< 1/12[N+m+t(N+m)] - 1/12[N+t(N)]\end{aligned}$$

But as  $N \rightarrow \infty$  the fractions  $\rightarrow 0$  and therefore  $\log \psi(N+m) - \log \psi(N) \rightarrow 0$ , i.e.,  $\log \psi(N)$  tends to a limit, say  $\log C$ . Hence making  $N \rightarrow \infty$  in the inequalities

$$\begin{aligned}\log \psi(N) - \log \psi(n) &> 1/12N - 1/12n \\ &< 1/12[N+t(N)] - 1/12[n+t(n)]\end{aligned}$$

$$\text{we get} \quad \log C - \log \psi(n) > -1/12n$$

$$< -1/12[n+t(n)]$$

or

$$\begin{aligned}\psi(n) &> Ce^{1/12[n+t(n)]} \\ &< Ce^{1/12n}\end{aligned}$$

which is equivalent to our former results.

<sup>\*</sup> Geometrically  $\psi(n)e^{-1/12n}$  and  $\psi(n)e^{-1/12[n+t(n)]}$  are both asymptotic to, but lie on opposite sides of, a line parallel to the axis of  $n$ .



6. To find the value of  $1/12[n+t(n)]$  we have, putting  $t(n)=1/20n$ , see § 3,

$$\begin{aligned} 1/12[n+t(n)] &= \frac{1}{12(n+1/20n)} = \frac{1}{12n} - \frac{1/20n^2}{12(n+1/20n)} \\ &> \frac{1}{12n} - \frac{1}{240n^3} \end{aligned}$$

Thus finally

$$n! > Cn^{n+\frac{1}{2}}e^{-n} \cdot e^{1/12n-1/240n^3} \quad . \quad . \quad . \quad (4)$$

$$< Cn^{n+\frac{1}{2}}e^{-n} \cdot e^{1/12n} \quad . \quad . \quad . \quad (5)$$

7. A value of the constant C sufficiently accurate for all practical purposes may be obtained by equating the approximate values, found from the formula

$$n! = Cn^{n+\frac{1}{2}} \cdot e^{-n} \cdot e^{-1/12n},$$

to the tabulated value of  $n!$  for a large value of  $n$ . The logs of the factorials to six decimal places up to  $n=1600$  are tabulated in De Morgan's *Treatise "The Theory of Probabilities"* in the *Encyclopædia Metropolitana*, vol. ii, pp. 486-490, and the logs to 7 places up to  $n=1000$  are given in Karl Pearson's *"Tables for Statisticians"*, pp. 98-101. Logs to 18 places up to  $n=120$  are given by Robbins, *Transactions*, A.S.E., vol. lii, p. 171.

If, for example, we take  $n=100$  so that  $\log_{10} e^{1/240n^3} = .000,000,002$ , the value of  $\log_{10} C$  may be found correct to 8 decimal places.\*

The following is the calculation to 9 decimal places:

$$\begin{aligned} \text{Log}_{10} 100^{100.5} &= 201.000,000,000 \\ \text{Log}_{10} e^{-100} &= -100 \times \text{mod.} = -43.429,448,190 \\ &\quad \underline{157.570,551,810} \\ \text{Log}_{10} e^{1/1200} &= \text{mod.}/1200 = \quad .000,361,912 \\ &\quad \underline{157.570,913,722} \\ \text{Tabular log}_{10} 100! &= 157.970,003,655 \\ &\quad \underline{\text{Log}_{10} C = 0.399,089,933} \end{aligned}$$

which shows an error of  $-1$  in the last place.

\* If we take  $n$  so low as 10,  $e^{1/240n^3} = .0000018$ , so that we may get a value of  $\log C$  correct to 5 places.

8. By higher methods it may be shown that the constant  $C = \sqrt{2\pi}$  and that the true value of  $n!$  lies between  $\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \cdot e^{1/12n}$  and the same multiplied by  $e^{-1/360n^3}$ , as against our factor  $e^{-1/240n^3}$ .\* Thus the proportional (or logarithmic) range of error found by the above simple process is only  $1\frac{1}{2}$  times that found by much more advanced work. Cesàro does not give the inequality (B), but works on (A) and  $\Delta\psi(n) < 0$ , producing the much wider limits  $Cn^{n+\frac{1}{2}} \cdot e^{-n}$  and this quantity multiplied by  $e^{-1/12n}$ .

G. J. L.

\* Since the Note was set up the writer has found that  $1/12(n^2 + n + 1)$  is  $> 1/12n - t(n) - [1/12(n+1) - t(n+1)]$ , where  $t(n) = 1/360(n^3 - n)$ . Hence the factor  $e^{-1/240n^3}$  in the lower limit may be replaced by  $e^{-1/360(n^3 - n)}$ , which is both nearer the true value and in closer agreement with that found by higher methods.

## REVIEW.

*Pulmonary Tuberculosis: Mortality after Sanatorium Treatment.* By NOEL D. BARDSWELL, M.V.O., M.D., F.R.C.P., and J. H. R. THOMPSON, F.I.A.

[Pp. 112. Medical Research Committee. Special Report Series, No. 33. 1919.]

THIS report, issued by the Medical Research Committee (National Health Insurance) is of great interest, and further is indicative of the increasing tendency towards the application of actuarial and sound statistical methods in the solution of problems beyond the normal sphere of actuarial activity.

The report deals with the mortality experienced to the anniversary of discharge in 1916 by patients discharged from the King Edward VII Sanatorium at Midhurst during the eight years 1907-14, 1,707 cases in all, of which 1,053 were males and 654 females.

The method of tabulation of the data is fully set out in an appendix, and is by the "select" method, an exposed to risk having been obtained in respect of grouped ages at discharge for each year elapsed since discharge. The expected deaths by the English Life No. 8 Table are compared with the actual deaths in the form of a ratio. This is the basis of working throughout, results being given according to condition on admission in three groups, Incipient, Moderately Advanced, and Far Advanced, and for each of these groups according to conditions on discharge Arrested, Much Improved, Improved, and Worse.

Further, the mortality after treatment with tuberculin, according to type of onset of disease, in respect of the presence or absence of tubercle bacilli in the sputum, among cases in which there existed tuberculosis of the larynx, and among cases who gave a family history of tuberculosis are also made subjects of inquiry.

There are also various tests as to homogeneity of data in the way

of subsidiary statistical tables. Indeed, so far as the data allow the investigation is most thorough, and the results are looked at from various points of view.

In studying the ratios in the various tables an eye should be kept on the expected and actual deaths on which they are based, as occasionally the data are lamentably small. For example, in comparing the Midhurst ratios with those for the Adirondack experience (Tables VIII and X) it is pointed out that for all durations in Group 3 Midhurst shows the better results, but as not one of the Adirondack expected deaths is so great as unity and four out of six range from .25 to .04 a comparison is not of much avail. The same applies to Table XIII as regards females discharged worse—it is remarked that for group 1 these show a ratio lower than might be anticipated (pp. 33-4), but the facts are two deaths on an expected of .188.

As regards the experience according to condition on admission. the size of the ratios should give much food for thought. Even amongst the Incipient cases they are 6.4 and 4.7 for males and females respectively, the corresponding figures for the Moderately Advanced being 15.6 and 19.0, and the Far Advanced 39.2 and 37.5. The ratios are much higher for the younger ages, but as is pointed out, this does not mean that the rate of mortality is decreasing with advanced age. Taking all stages of the disease at admission together the rate of mortality does not appear to vary sensibly between age groups. In order to carry conviction it would have been more satisfactory had this been dealt with for the three groups according to condition on admission. An examination of the figures, however, confirms the same feature in each separate group. The consistency of the results between males and females and with a previous investigation into American data adds great weight to the conclusions reached.

Tables XI, XII and XIII with reference to the experience according to condition of discharge are extremely useful in so far as they emphasize the greater chances a patient has of life if admitted in an early stage of the disease. 55.7 per-cent of the males and 71.2 per-cent of the females admitted in the Incipient stage are discharged with the disease arrested. These percentages drop to 19.2 and 17.9 respectively in the Moderately Advanced and are practically negligible in the Far Advanced stage. While the condition on discharge is important from the prognosis standpoint, it should be clearly borne in mind that the condition on admission is of far higher importance as regards chances of practical recovery. The observations made at the end of this chapter deal largely with prognosis, and to these might have been added one urging treatment without delay immediately tuberculosis is diagnosed.

In the light of statistics one must sadly admit the inadequacy of sanatorium treatment. In some individual cases no doubt it is effective, but as a whole one finds little satisfaction in the results. One reads occasionally in the Press of "medical orthopaedics" and possibly the solution lies in this direction, in the training of a convalescent to earn his living at such work and in such surroundings

as will assist and maintain his recovery. Statistics relating to the mortality experience of such cases would be interesting indeed.

To pass to the chapter on tuberculin, the principle adopted for the comparison of cases treated with and without this factor was to compare the general results for three years in which it was not, and three years in which it was used. During the latter period 67.6 per-cent of the patients discharged had been treated with tuberculin, and the argument was, justly, that if tuberculin were of service its value would be reflected in an improvement in the total results. This method seems undoubtedly the best, as it avoids any fallacy of selection. Various tables are inserted to prove the homogeneity of the data, and dealing with the immediate results of treatment, *i.e.*, improvement or otherwise in condition during treatment. These and the after results, however, are disappointing, no marked difference being apparent. The investigation was, moreover, carried further and an examination made of the experience of patients who actually received tuberculin treatment, comparison being made with a group of cases, as similar as possible, selected by a third party. Still the results were quite inconclusive, and the conclusion is reached that tuberculin injections had no appreciable effect either for good or ill.

A further chapter deals with the mortality according to type of onset of the disease, how it first manifests itself, whether by hæmoptysis, with an attack of pleurisy, or insidiously, *i.e.*, by gradual loss of strength accompanied by a cough. Here some well-defined features are brought to light. Hæmoptysis on the whole leads to an earlier diagnosis of the disease than in the other types of onset, and patients whose chief initial symptom was hæmoptysis experienced lighter rates of mortality. These conclusions of the authors are acceptable, but it is noticeable that the third one is only a suggestion—that an initial hæmoptysis may be associated with a relatively less severe type of disease. The percentages of patients with type of onset hæmoptysis in the groups at admission Incipient, Moderately Advanced and Far Advanced are for males 30.8, 25.1 and 18.3 respectively, and for females 18.2, 15.6 and 14.2. As regards the males especially this decreasing percentage rather indicates not only earlier diagnosis but a general shifting of the weight of such cases towards the less severe limit of a stage of disease at admission, which would be consistent with the lower ratios brought out. As regards females, this shifting tendency is practically non-existent and curiously the ratios for the females do not show to the same advantage over the cases with insidious type of onset, so one cannot regard the association of an initial symptom of hæmoptysis with a less severe type of disease (as opposed to a less severe stage at diagnosis) as proved, without further investigation.

No elaborate comments are necessary on the chapters dealing with the presence or absence of tubercle bacilli in the sputum or with cases in which there existed tuberculosis of the larynx. Both positive conditions have an adverse effect on the mortality. On the other hand the mortality of patients who gave a family history of

tuberculosis gave no indication of differing essentially from the normal. This, of course, does not have any bearing on the chances of a person who has a family history of tuberculosis contracting the disease, but once having contracted it he does not appear to be worse off than another person. An Appendix is added, dealing with Messrs. Rusher and Kenchington's paper, *J.I.A.*, vol. xlvii, p. 433, but this, of course, is familiar to actuarial readers.

There is one further point that perhaps may be mentioned. It is stated in the Report (p. 7) that an endeavour is made to admit to the Sanatorium only those who are suffering from the disease in its earlier stages. The endeavour, however, appears to have been singularly unsuccessful if one may judge by the following comparison with similar efforts made during the later admissions to the Adirondack Sanitarium :

	MALE			FEMALE		
	Incipient	Advanced	Far Advanced	Incipient	Advanced	Far Advanced
Adirondack, Dec. 1903, to July 1909 ...	222	298	8	206	247	5
Midhurst ...	289	541	223	171	307	176

One would imagine rather that only those who had little chance of recovery were refused and this, of course, is justifiable from a humanitarian point of view unless patients in the early stages were displaced by those in the later stages.

Generally speaking, the Report avoids serious criticism, not being dogmatic in its assertions ; in a true scientific spirit obviously not setting out to prove a case for or against sanatorium treatment, but investigating facts and reading only reasonable interpretations into the results.

S. J. P.

## CORRESPONDENCE.

### THE YIELD ON A REDEEMABLE BOND WHEN INCOME-TAX IS TAKEN INTO ACCOUNT.

*To the Editors of the Journal of the Institute of Actuaries.*

SIRS,—If your readers are not weary of the subject of the effect of income-tax on the yield on a redeemable security, they may possibly find the following formulæ of practical use. The expressions may be approximately derived from formula (6) given by Messrs.



Lidstone and Todhunter (*J.I.A.*, vol. xlix, p. 369), or more simply from first principles, but they are admittedly in the main empirical, and I am not able to give any theoretical reason why these particular approximations seem to suit present conditions any better than many others.

The true gross yield allowing for tax may be taken as

$$J - \frac{t}{1-t} \cdot \frac{K - \frac{1}{2}K^2}{n}, \text{ where } K \text{ is negative,}$$

or 
$$J - \frac{t}{1-t} \cdot \frac{K - \frac{6}{5}K^2}{n}, \text{ where } K \text{ is positive,}$$

$J$  being the gross yield without allowance for tax,  $1 + K$  the present price,  $n$  the outstanding term, and  $t$  the rate of tax per unit.

The advantage claimed for these formulæ is simplicity of application. The rate of dividend is not involved, and it is therefore possible to construct a short table of  $\frac{t}{1-t} \left( K - \frac{1}{2}K^2 \right)$  and  $\frac{t}{1-t} \left( K - \frac{6}{5}K^2 \right)$  for all prices of securities, for any value of  $t$  desired, and divide by  $n$  in any particular case required.

As regards accuracy, single examples prove little and are not worth setting out. I have, however, tested the formulæ in about 100 cases for terms from 2 to 40 years, rates of dividend from 3 to 7 per-cent, rates of net yield from  $3\frac{1}{2}$  to 6 per-cent, and rates of tax from 4s. to 8s. in the £, and find that for terms of 20 years and under the errors are under 3*d.* per-cent in 77 per-cent of the cases, between 3*d.* and 6*d.* per-cent in 15 per-cent, and between 6*d.* and 1s. 6*d.* per-cent in the remainder, which are nearly all cases at the highest rate of tax. For terms over 20 years the errors range up to as much as 3s. per-cent for the longest term and highest rate of tax, 71 per-cent being under 1s. per-cent.

I have compared the errors given by Messrs. Lidstone's and Todhunter's formula in about 30 of the above cases, and find that the latter is somewhat more accurate for terms under 10 years and less accurate for terms over 10 years.

This degree of accuracy seems amply sufficient in dealing with so uncertain a factor as the future rate of income-tax.

Yours faithfully,

O. F. DIVER.

15, *St. James's Square, S.W. 1.*

1 *March* 1920.



## THE INSTITUTE OF ACTUARIES.

## ADMISSION OF WOMEN TO MEMBERSHIP.

At a Special General Meeting, held at Staple Inn Hall, on 24 November 1919, the President moved the following resolution on behalf of the Council:

“That women be admitted to the Institute on the same conditions as men, and that the masculine shall include the “feminine in all the Regulations of the Institute.”

The PRESIDENT said that the reasons which had influenced the Council in deciding to bring forward the resolution might be summed up in the words that it was in accordance with the spirit of the times. There was evidence of that in resolutions which had been passed by other institutions, in the fact that legislation was contemplated which would have the effect of requiring such institutions to admit women on the same terms as men, and in the example of their sister institution, the Faculty of Actuaries in Scotland, which had already passed a resolution in terms similar to the one before the meeting. He had been informed shortly before the meeting that there was a feeling among some of the members that the proposed change should be postponed at least for the period during which the revised syllabus was to remain unaltered. That was a point which the Council had not considered, and consequently any opinion which he expressed upon it must be regarded as his personal opinion, and not as that of the Council. Although one object of the alteration of the syllabus was to make as easy as was compatible with efficiency the examination test to which the members who had been serving in the war were to be subjected, it was not the whole object of the Council in making the alteration. He thought he might say that the alteration had been made in accordance with a preconceived general policy, which was that the syllabus should aim at providing a test which would produce competent actuaries and at the same time save those who desired to specialize in a particular direction from having to overload themselves with work which might not be useful in connection with the particular subject in which they desired to specialize. He thought it was obvious that that was in the minds of the Council, because they had further developed their policy in the direction which he had indicated, by offering prizes for special research. Was there any real injustice to the members in the immediate admission of women? He submitted that there was not. Those members who had been on service in the war to the extent to which they had had an opportunity of studying professionally or of obtaining practical experience in offices before the war, had an advantage as compared with any women who might join the Institute. He admitted that on the point of practical experience some advantage had been lost by the fact that most of the women who would apply for admission to the Institute had had

some office experience during the war, but that did not apply to preliminary study. So far as new members were concerned, whether they were men or women, they started on an equal footing. It seemed to him perfectly fair that that should be so. After the resolution had been seconded the Council would be glad to hear any remarks that the members might wish to make on it.

Mr. S. G. WARNER, in seconding the resolution, said that he was very glad to have the opportunity of associating himself personally with the recommendation of the Council. The resolution had not been arrived at without very grave consideration and without the best attention which the Council were able to give to it. The arguments in its favour need hardly be elaborated at the moment; they could not be better summed up than they had been by the President. The Institute was, in fact, asked to follow the great current of general civilized public opinion. He presumed that any opposition or hesitation among the members must be due to some trepidation as to the results of such a profound and far-reaching change in the constitution of the Institute. With reference to that, he thought it might be taken as the testimony of experience that no profession, learned or otherwise, which during the past quarter of a century had, in obedience to that general feeling and movement, opened its doors to women, had, so far as he knew, had any occasion to regret it. Some of the members knew something about the struggle which took place before women obtained admission to the medical profession, and they knew now how women occupied an honoured place and almost an indispensable place there. In the actuarial profession he did not personally expect a very great incursion of women members for a good many years to come. The step they were proposing to take involved perhaps some courage and some liberal feeling and sympathy and good fellowship, but he thought it was a step which they as an Institute would honour themselves and in the end abundantly justify themselves by taking.

Mr. A. HENRY said he had been asked to express the feelings of the Committee of the Students' Society on the matter. The Committee were in touch with a considerable body of their members and the opinion which they held was the result of discussions amongst quite a large number of the younger members, including not merely the younger Fellows of the Institute but also members who were actually going in for the examinations and who had a real interest in the matter. As a Committee they did not oppose in any way the admission of women. As a matter of fact, in consonance with the attitude of most young people in the present days, they welcomed it; but at the same time they did view with some misgiving the effect of allowing women to enter under the present restricted syllabus—not because they were women or because there were any doubts as to their capacity to deal with a more difficult syllabus, but because it was an aggravation of the situation which had been brought about by the policy of cutting down the range of the examination syllabus. The President had said that, although the syllabus was the result of an attempt to meet war conditions,

yet it was part and parcel of a rather larger policy involving the ultimate qualifications of the practising actuary. In so far as the present syllabus was to be regarded as an attempt to meet the special conditions arising out of the war, he did not think anybody would criticize it. They all recognized that men had lost valuable years which would have been used for qualifying themselves in their profession, and that some measures must be taken to set that right. But the question of the final qualification of the actuary was a very different matter. The general scope of the examination was too wide a subject to discuss at the present meeting. At the same time, he thought there were some things which must be deemed essential as part of the equipment of the practising actuary. In his view the splitting of the final examination into two alternative parts might result in very unfortunate consequences to the profession. He thought that, from the point of view of the standing of the profession in the opinion of the outside public, the qualifications of an actuary should cover all subjects which properly fall within his purview. If in the future an actuary were to apply to be put on the list of public valuers, the Registrar might consider it his duty to enquire what part he had taken in the Final—the Friendly Society part or the Life Office part. He did not think that would be satisfactory from the point of view of the profession or from the point of view of the applicant. In referring to the question of the syllabus he wished to indicate the objections that were felt not as regards women only, but as regards women in common with the young members who were coming forward for their examinations and who did not require the concessions that were naturally extended to the members who had suffered during the war. The Committee did not propose to take any steps to move an amendment or to suggest any other course, but they would be very grateful if, in taking the necessary steps in connection with the admission of women to the Institute on the same terms as men, the Council would consider the views which he had represented.

The PRESIDENT said there appeared to be nothing in Mr. Henry's presentation of the case of the students which was inconsistent with the passing of the resolution. It must be borne in mind that their examinations were not competitive examinations; it was merely a pass degree which the Institute conferred. The admission of women would make no difference to the possibility of a return to the old syllabus, and if in the judgment of the members as a body it was desirable later on that the syllabus should be restored to its pre-war standard, the syllabus so restored would apply just as much to women as to men students, and therefore the position would be that both sexes were submitted to the same test. He thought that the fear that any considerable number of unqualified women might attain to the Fellowship in the interval was quite without foundation. If there was any foundation for it, it was a fear which would apply as much to men students as to women students. He could see no difference in principle. At a later stage it would be quite within the competence of any member or body of

members of the Institute to address the Council with a view to an adequate consideration of the views put forward by Mr. Henry on behalf of the Students' Society, and he was quite sure that they would receive attention.

The motion was carried unanimously.

### EXAMINATIONS, DECEMBER 1919.

Examinations were held on the 15th, 16th and 17th December 1919, at London, Liverpool, Norwich, Edinburgh, Dublin, Melbourne, Sydney, Perth, Dunedin, Montreal, Toronto, Ottawa, Winnipeg, Bombay, and Calcutta, with the following results:

#### PART I.—SECTION A.

Ninety-seven candidates sent in their names, of whom eighty-three presented themselves, and thirty-seven passed, namely:

Allen, P. A.	Gibberd, J. A.	Polden, L. S.
Balmford, W. C.	Godfrey, G. H.	Rowan, H. R.
Blossom, John	Haynes, T. W.	Sanger, C. W.
Bourke, G. W.	Holmes, G. L.	Saunders, F. W.
Carr, A. I.	Hornsby, S. J.	Schwehr, W. E. P.
Clough, W. H.	Horsley, N. A.	Starke, L. G. K.
Creese, H. R.	McLachlin, J. L.	Taylor, J. A. G.
Cutler, D. G. S.	Mann, A. H.	Vaidyanathan, L. S.
Davis, Miss D. B.	Miller, I. A.	Vaidyanathan, M.
Dodwell, Miss M. E.	Murrell, R.	Williamson, W. H. R.
Dowsett, R. E.	Parker, R. O.	Yetton, J. L.
Finch, G.	Paul, H. C.	
Forster, W. A.	Pedoe, A.	

#### PART I.—SECTION B.

Eighty-seven candidates sent in their names, of whom seventy-six presented themselves, and thirty-four passed, namely:

Balmford, W. C.	Kingham, C. E.	Sanger, C. W.
Blossom, John	Legge, Miss P.	Southwell, M. G.
Bourke, G. W.	Lockwood, B.	Starke, A. W.
Chapman, H. V.	McLachlin, J. L.	Starke, L. G. K.
Charles, A. H.	Miller, I. A.	Taylor, J. A. G.
Elrick, W.	Milnes, H. L.	Tulloch, Miss M. M. N.
Finch, G.	Murrell, R.	Turner, G. C.
Forster, W. A.	Nisbet, J. D. P.	Vaidyanathan, L. S.
Freeman, H.	Patterson, J. R.	Walker, J. R.
Gardner, W. F.	Pedoe, A.	Walker, R. B.
Haynes, T. W.	Perks, W.	
Jones, H. G.	Reynolds, P. C.	

## PART II.

Thirty-eight candidates sent in their names, of whom thirty-two presented themselves, and twenty passed, namely :

Barnsley, J. C.	Housemann, D.	Taylor, F. G.
Brown, E. A. L.	Johnston, W. N.	Thomas, J. H.
Bryson, S. F. A.	Lochhead, R. K.	Waller, F. W.
Childe, E. R.	Miles, G.	Watson, D. J.
Freeman, H.	Millward, G. D.	Williams, E.
Garland, W. E.	Perryman, F. S.	Woodrow, G.
Hooker, P. F.	Rowell, A. H.	

## PART III.—SECTION A.

Twenty-nine candidates sent in their names, of whom twenty-eight presented themselves, and eight passed, namely :

Clegg, C.	Mabon, J. B.	Warren, C. F.
Knowles, M. B.	Maddex, G. H.	Woffindin, R. H.
Lafford, H. G.	Thompson, F. A.	

## PART III.—SECTION B.

Thirty-one candidates sent in their names, of whom twenty-nine presented themselves, and nineteen passed, namely :

Atkins, F. C.	Jones, R. McN.	Spurgeon, C. B.
Blake, W. T. C.	Lafford, H. G.	Thompson, F. A.
Brown, S. P.	Maddex, G. H.	Watson, A. D.
Carpmael, C.	Moore, W. R.	Weyer, D.
Davidson, A. R.	O'Brien, H.	Woffindin, R. H.
Denmark, R. J.	Reid, W. C.	
Gostelow, C.	Rutherford, C. D.	

## PART IV.—SECTION A.

Twenty candidates sent in their names, of whom nineteen presented themselves, and seven passed, namely :

†Brown, A. E.	†Harrington, E. W.	†Pollard, E. C.
†Chatham, E. F.	†Klagge, O. C. J.	
†Drake, C. C. H.	†Penn, C. S.	

Those marked (†) have now completed the Examination for the Class of Fellow.

By Order of the Council,

A. C. THORNE,

*Chairman of Board of Examiners.*

W. PALIN ELDERTON,

H. M. TROUNCER,

*Joint Honorary Secretaries.*



## LECTURES ON STATISTICS.

THE following is the Syllabus of a course of five lectures delivered during the current session before the members of the Institute, at Staple Inn Hall, by Miss Ethel M. Elderton, Research Fellow, University College, London :

SOME PRACTICAL EXAMPLES OF STATISTICAL WORK  
WITH SPECIAL REFERENCE TO CORRELATION.

1. Explanation of terms ; collection of data, &c. ; graphic work.
  2. Means and standard deviations and their numerical calculation and why they are wanted in measuring correlation.
  3. Simplest method of calculating correlation.
  4. Regression and the product-moment method.
  5. Interpretation of results and discussion of various points that arise.
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**Obituary.**

HARRY WATSON, Student of the Institute, 2nd Lieutenant, King's Liverpool Regiment.

*Killed in Action 12 August 1916.*

ARTHUR VERNON CLARE, Probationer of the Institute, 2nd Lieutenant, 1/21 Battalion London Regiment.

*Killed in Action 15 September 1916.*

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# JOURNAL

OF THE

## INSTITUTE OF ACTUARIES.

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*Newton's Interpolation Formulas. Further Notes by* DUNCAN  
C. FRASER, M.A., F.I.A.

THROUGH the kindness of Mr. Lidstone, I have had an opportunity of seeing a letter from an eminent mathematical authority who writes from Edinburgh, and in the course of a friendly reference to previous contributions on this subject expresses the view that in my enthusiasm for Newton I have done some injustice to Newton's great contemporary, James Gregory. He says:

"There are, I think, strong reasons for concluding that  
"Newton's formula

$$f(a+n) = f(a) + n\Delta f(a) + \frac{n(n-1)}{2} \Delta^2 f(a)$$

$$+ \frac{n(n-1)(n-2)}{3} \Delta^3 f(a) + \dots$$

"was really discovered first by Gregory. At any rate  
"there is no doubt that Gregory was the first person to  
"communicate the formula to any other human being,  
"namely, in the letter to Collins, of date 23 November 1670,  
"which is printed in Rigaud's Correspondence of Scientific  
"Men of the XVIIth Century, v. ii, p. 209. He gives a  
"worked out example, Rigaud, p. 211. Now we know

“ that Collins kept up a regular correspondence with both  
 “ Gregory and Newton, and that he transmitted to Newton  
 “ what he got from Gregory (he says so himself, Rigand,  
 “ v. ii, p. 335); and we know that Gregory in his letter  
 “ calls the formula ‘ my formula ’ and that neither he nor  
 “ Collins at the time suggested that Newton might know  
 “ of it, and that Newton did not mention it in any letter or  
 “ publication for some years after. The world owes  
 “ much to Collins who kept copies of these letters. His  
 “ introduction of himself in his first letter to Gregory is  
 “ delightful: ‘ Sir, it was once my good hap to meet with  
 “ you in an ale house.’ ”

The criticism comes from a source which is entitled to the highest respect. The evidence, however, for the priority of Newton is, I think, complete and convincing, and I am glad to have the opportunity of setting it out in full detail. But first of all it may be appropriate to give some account of the persons who come into the story and of their relations to one another.

James Gregory (1638–1675), one of the most eminent mathematicians and astronomers of his day, is now best remembered by his simple and familiar series for an angle in terms of the powers of its tangent, announced in the following terms in a letter dated 15 February 1671, to John Collins (Rigand, v. ii, p. 216) :

Sit Radius =  $r$

Arcus =  $a$

Tangens =  $t$

Secans =  $s$

$$\text{Erit } a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - , \text{ \&c.}$$

The series does not I believe appear in any of his published works, Gregory being satisfied that the announcement to Collins was as good as publication to the world. He was the inventor of the reflecting telescope, and it was when he came to London in 1664 or 1665 to get an instrument constructed that he made the acquaintance of Collins, who was able to introduce him to the best glass grinders. From London he went to the University of Padua where he spent the next three or four years and published his earliest works. From 1668–1674 he was Professor at St. Andrews, and was then

appointed the first exclusively mathematical Professor in the University of Edinburgh, but he died suddenly after occupying that position for only fifteen months. He was the greatest representative of one of the most remarkable scientific families this country has produced. From his day to ours, the intervals have been few and short in which a Gregory has not occupied a Professorship or Lectureship in some department of science. His lineal descendants in four successive generations supplied Professors of Medicine to the Universities either of Aberdeen or Edinburgh, and his great-great-grandson, Duncan Farquharson Gregory (1813-1844), was well known as the first editor of the *Cambridge Mathematical Journal*. Of his collateral descendants two nephews, David and James, may be specially mentioned. David Gregory (1661-1708) became Professor of Mathematics in the University of Edinburgh at the age of 23, and after occupying that chair for seven years went to London in 1691 and made the acquaintance of Newton, whose intimate friend he became. Newton introduced him to Flamsteed, and through the active support of these influential friends he was chosen Savilian Professor of Astronomy at Oxford. David's brother, James Gregory the younger, succeeded him as Professor of Mathematics at Edinburgh, and was also on friendly and familiar terms with Newton. Another collateral descendant, Thomas Reid, Professor of Philosophy at Aberdeen and at Glasgow, has preserved an anecdote of the younger James Gregory which I quote with the special aim of placating my Edinburgh friends. James Gregory, he tells us, being one day in familiar conversation with Sir Isaac Newton at London, Sir Isaac said: "Gregory, I believe you do not know that I am a Scotchman." "Pray, how is that", said Gregory. Sir Isaac Newton said he was informed that his grandfather (or great-grandfather) was a gentleman of East (or West) Lothian, that he went to London with King James I at his accession to the Crown of England, and that he attended the Court in expectation as many others did until he spent his fortune, by which means his family was reduced to low circumstances. "At the time this was told me", Reid remarks, "Mr. Gregory was dead; otherwise I should have had his own testimony, for he was my mother's brother."

John Collins (1625-1683) was an accountant in the City of London, expert in his profession and interested in the scientific side of his work. Although the science of life contingencies

had not yet appeared above the horizon he may be regarded as the nearest equivalent in his time to the modern actuary. He spent seven years of his life, between the ages of 17 and 24, during the period of the Civil War, in an English ship employed by the Venetians in defending Candia against the Turks. At sea he learned the elements of navigation and mathematics, and on returning to London in 1649, he got into communication with Foster, lecturer in astronomy at Gresham College, who supplied him with lists of mathematical books so that he might extend his studies. His first letter to James Gregory throws an interesting light on his occupations and interests. My Edinburgh critic has already quoted a few words from it and I extend the quotation. Sion College, on the Thames Embankment, mentioned at the beginning of the extract, was and is famous for its library, and before the great Fire of London was attractive to mathematicians on account of its mathematical books and MSS. The early mathematician Torporley (1564–1632), who had been Secretary to Francis Viëta of Fontenoy, the founder of modern algebra, left his MSS. to Sion College where he spent the last years of his life: but nearly all its treasures were destroyed in the Fire, and the library had to be reconstituted. The letter from which I quote was written to Gregory at Padua and the date, which is not given, would appear to be early in 1668 (Rigaud, v. ii, p. 178):

Sir, it was once my good hap to meet with you in an ale house or in Sion College, and though I have not been educated at Universities and so my attainments are mean, yet I have an ardent love to these studies, and endeavouring to raise a catalogue of mathematical books, and to procure scarce ones for the use of the Royal Society and my own delight, I crave your assistance in procuring what I mention in this letter or the enclosed paper. . . . . I can give you an account of many books printed lately at Paris, and some of them for private persons, which I wish I could have for my money and perchance you will desire to meet with some of them in your return. I am now in a troublesome employment that prevents my studies, to wit, Chief Accountant to the Commissioners for examining the accounts of the late war, and manage an accountantship in the Excise Office by a substitute; so that if you vouchsafe an answer, direct it to me as an Accountant at the Excise Office in Bloomsbury, to be left with Mr. Bourne at the posthouse.

This may be supplemented from a letter of Collins written to another correspondent in 1672 (Rigaud, v. i, p. 200):

I correspond with Dr. Wallis, Mr. Bernard, Mr. Newton, Mr. Towneley, Mr. Gregory, Pere Bertit, Slusius, Borellius, &c. . . . . I am about to turn stationer myself; and as I have been, so I believe I shall continue as eager as any man to get good books printed. I have been employed nearly  $2\frac{1}{2}$  years under Mr. Slingsby, Mint Master, as Secretary (and a Member) of the Council of Plantations, and have received but little more than a tithe of my pay. A pension I had of £50 per annum, half my salary, a little while of the King for the loss of my place as an accountant in the Excise Office, by reason of altering the administration thereof since the late Lord Treasurer's death, and that hath been stopped these twelve months, as is likewise my wife's pay as laundress of the table linen to the Queen; the King's debts and occasions for the war diverting the money. And now the Council of Plantations is likewise to be a Council of Trade; and Mr. Slingsby, conceiving the trouble will be great, and the pay as uncertain, leaves his Secretary's place, and advises me to leave that employment and to manage the Farthing Office, to deliver out all that are coined on Tuesdays, Thursdays and Saturdays in the mornings, in crown-papers ready tied up; the salary £50 per annum and a fair dwelling-house, which I think may be in or near Fenchurch Street, where having a convenient shop I intend, God willing, to set up a Stationer's trade (and have a promise of serving the Mint), and afterwards hope to fall into the printing of books, especially some of the copies of the members of the Royal Society, and some of my own, particularly one of the modern advancement of Mathematical Sciences, and an Account of the best authors of that kind, and some others which I intend, of which more hereafter. The last ages were too ignorant of mathematics . . . . .

I married the younger daughter of two only children of Mr. Wm. Austin, who being one of His Majesty's Cooks when Prince of Wales, was by Dr. Wilkins' means, made and continued master cook of Wadham College in Oxford during the late troubles, and is now master cook to His Majesty of the Lord's kitchen. I live at my said father-in-law's house in Petty France, Westminster, over against the Adam and Eve.

He had been elected in 1667 a member of the Royal Society, of which Dr. Wilkins (afterwards Bishop Wilkins) was one of the founders. His dreams of setting up as Bookseller and Publisher were never fulfilled, probably for want of capital. He was not a man of means, though he was lavish in his expenditure on books. Here is an extract from another letter in 1677 (Rigand, v. ii, p. 23):

It hath been my misfortune to be concerned in public employments, as in the Council of Plantations, &c., wherein I have not been paid, and have great arrears due to me, for want whereof I am almost ruined; and having a numerous family to maintain,



to wit, a wife and seven small children, I am forced to undertake such occasional business as offers and by consequence to neglect a correspondence with the learned, which though unworthy I much covet.

Such occasional and remunerative business as the disentangling of intricate accounts necessarily occupied much of his time. He had published in 1652 an introduction to *Merchants' Accounts*, of which he brought out a second edition in May 1665; but he writes in the following February that by reason of the contagious death of the bookseller (during the plague) it was not yet procurable; and a year later he mentions that this work, lately re-printed, underwent the fate, *i.e.*, the issue was destroyed in the great Fire of London.

His fame rests not on any original work of his own, but on his correspondence with the learned men of his time. He constituted himself the principal centre and medium for the circulation of information on new discoveries, and for the collection and distribution of new books, British and Foreign. The publication of the works of Barrow and Wallis and other authors was arranged by him, and he supervised them while they were passing through the press. He writes to Wallis (*Rigaud*, v. ii, p. 468):

Mr. Pitts, a stationer in Little Britain (of whom I can give no other than a good character), understanding from Mr. Branker, a late fellow of Exeter College, who now lives with Lord Brereton in Cheshire, that you had prepared sundry tracts for the press, is very desirous to undertake any of them and to treat with you for that purpose, promising that they shall come out on very good paper like Schooten's *Miscellanies*, and the cuts to be suitable thereto, which may be done by Mr. Marke, and indeed passing by morning and night, and at noon too, if need require, I should afford my endeavour to have it carefully corrected.

The position of Collins as a medium of communication was so generally accepted that mathematicians writing to one another sent their letters through Collins with the view of ensuring their general circulation; and in that age of mutual jealousies and suspicions, particularly between British and Foreign writers, there can be no doubt that in taking this course they were also influenced by the desire that the credit of their original work should be secured to them and placed upon record by his impartial and disinterested authority. Thus Wallis, writing in 1676, sends to Collins a long Latin which is meant for transmission abroad to Tschirnhaus,



and with it, in merciful consideration for Collins's weak Latinity, an English translation. Some remarks of Wallis in a P.S. as to the printing of letters probably refer to their publication in the "Philosophical Transactions" issued periodically by the Royal Society. He says (Rigaud, v. ii, p. 600):

Mr. Baker's (letter) I doubt is too long to transcribe. I had rather see it in print; and Mr. Newton's also. As to our correspondence with the French I like it best when it is done in print, they being apt to be disingenuous in claiming all for their own which they have from hence, without owning whence they have it.

Collins's collection of letters passed, 25 years after his death, into the hands of William Jones, to whom the publication of Newton's "Method of Differences" was due. William Jones was tutor to George Parker, afterwards second Earl of Macclesfield and President of the Royal Society, and lived for many years with him as a member of his family. In this way the Collins papers came into possession of the Earl of Macclesfield and are usually known as the Macclesfield Collection. They were edited in 1841 by S. J. Rigaud and published under the title of "Correspondence of Scientific Men of the 17th. Century." It should be mentioned, however, that the series of letters in the Macclesfield Collection and in Rigaud's volumes is not complete. After the death of James Gregory, the letters received from him were deposited with the Royal Society by Collins, who retained only copies and rough notes of some of them. The documents deposited with the Royal Society were used in the compilation of the *Commercium Epistolicum*, and from that source it is possible to supplement the information which can be gained from Rigaud's two volumes.

Collins was first brought into touch with Newton by Barrow, who, being solicitous that Newton should not lose the credit of his early discoveries, adopted the recognized method of securing publicity and wrote to Collins the following letters which are numbered I, II and III in the *Commercium Epistolicum*. That compilation is entirely in Latin, and I have translated the portions which are here quoted.

#### I.—20 JUNE 1669.

A friend of mine in residence here, who possesses uncommon ability in these matters, handed me the day before yesterday some papers in which he described methods of computing the

dimensions of magnitudes, similar to the method used by Mercator for the hyperbola but much more general; and these methods serve also for the solution of equations; I think they will please you and I will send them with my next letter.

### II.—31 JULY 1669.

I send you, as promised, my friend's papers, which, as I hope, will afford you no little pleasure. Send them back I beg, when you have read as much of them as you desire, for he made this stipulation when I first asked for permission to communicate them to you. I adjure you, therefore, let me know that you have received them; I am anxious because I have taken the risk of sending them by the public carrier so as to comply with your wishes as quickly as possible.

### III.—20 AUGUST 1669.

I am glad my friend's papers have pleased you. His name is Newton, a Fellow of our College, and a young man, being still only in the year following that in which he took the Degree of Master of Arts: he has made very great progress in this subject, in which he has uncommon ability. Communicate the papers if you wish to our noble friend, Viscount Brounker.\*

Collins immediately entered into a correspondence with Newton, and on the 19 January 1670, Newton writes to him as follows (Rigaud, v. ii, p. 281):

I received Dr. Wallis his *Mechanics*, which you sent to Mr. Barrow for me. I must needs acknowledge you more than ordinarily obliging and myself puzzled how I shall quit courtesies.

The problems you proposed to me I have considered, and send you herewith the best solutions of one of them that I can contrive; namely, how to find the aggregate of a series of fractions, whose numerators are the same and their denominators in arithmetical progression . . . .

He proceeds to discuss this question at length. Rigaud dates this letter 1669; but it is clear from the reference to Wallis's *Mechanics* that the year is wrongly given and should be 1670. In dealing with the correspondence of that period caution is always necessary in considering letters dated in the early part of any year on account of the confusion of the old and new styles. The work mentioned was seen through the press for Wallis by Collins, and Wallis acknowledges receipt of the earliest copies on the 11 January 1669-70 (Rigaud, v. ii, p. 520). There is the same confusion as to the date of the next letter from Newton to Collins, which is given by Rigaud as 6 February 1669, but should be 6 February 1670.

\* President of the Royal Society.

This letter (Rigaud, v. ii, p. 287), in which Newton gives an approximate solution of the problem of determining the rate of interest in an annuity has been quoted and discussed by De Morgan (*J.I.A.*, vol. viii, p. 64).

Then comes a letter from Newton to Collins, the date of which is clearly expressed as 18 February 1669–70 (Rigaud, v. ii, p. 296). I give the letter in full :

Sir, Two days since, I received yours and Mr. Dary's letter with the book, for which I thank Mr. Dary and have herein enclosed sent him my thoughts of what he desired. That solution of the annuity problem, if it will be of any use, you have my leave to insert it into the Philosophical Transactions, so it be without my name to it. For I see not what there is desirable in public esteem, were I able to acquire and maintain it. It would perhaps increase my acquaintance, the thing which I chiefly study to decline. Of that problem I could give exacter solutions, but that I have no leisure at present for computations. I now see a way, too, how the aggregate of the terms of musical progressions may be found (much after the same manner), by logarithms, but the calculation for finding out these rules would be still more troublesome and I shall rather stay till you have leisure to do me the favour of communicating what you have already composed on that subject.

De Morgan, who drew up the useful table of contents to Rigaud's volumes (Rigaud, v. ii, p. 611–644), remarks on this letter that it is "one of the earliest evidences of Newton's temperament on the publication of discoveries." Some years later, Newton writes to Collins in a letter dated 8 November 1676 (Rigaud, v. ii, p. 403) : "I have learned what is to my convenience, which is to let what I write lie by till I am out of the way" ; and there is no doubt of his morbid shrinking from publicity. Much of Newton's work would have been lost to the world and much would have been credited to other people but for the assiduous efforts of Collins, as well as of other friends, in gathering and disseminating information as to what he had done.

Collins had mentioned Newton's name to Gregory in a letter dated 25 November 1669, which does not appear in the Macclesfield correspondence, but is given as Item No. XIV in the *Commercium Epistolicum*, as follows :

Barrow has handed over his position as public lecturer to a young Cantab named Newton, who has been mentioned by him as a man of the finest ability in the preface to his Optical Lectures ; and who before the appearance of Mercator's

Logarithmotechnia had hit upon the same method and applied it in a general manner to all curves, and in a variety of ways to the circle.

At this point there is a gap in the correspondence, but what happened is clear from later letters. Gregory had expressed a strong desire for further information about Newton's work (see his letter of 23 November 1670, given later); and Collins writing on the 24 December 1670 (Comm. Epistolicum, XIX) recalls that:

When Mr. Dary published his *Miscellanea* (1669) he sent a copy to Newton, who in return gave the said Dary a series for the zone of a circle which I sent to you. Without any doubt this is a legitimate and excellent series. Through the good offices of Barrow I received a number of other series, derived from Newton's general method . . . .

Newton's acknowledgment of Dary's book has already been given above. Dary, who was a gauger in Bristol and afterwards in Newcastle, had considerable mathematical ability, and his interesting method of obtaining approximate numerical solutions of equations has recently been revived by Sir Ronald Ross. The series which Newton sent to Dary and which Collins sent along with other series to Gregory is recorded by Gregory in a letter dated 20 April 1670 (Comm. Epistolicum, XV):

I cannot understand the series sent by you for the zone of a circle, namely:

$$2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} - \frac{5B^9}{576R^7} - , \&c.$$

If this be correctly transcribed I suspect that it is not a legitimate series.

If  $x$  be written for  $\frac{B}{R}$ , the expression quoted above by Gregory takes the form

$$2R^2 \cdot \left( x - \frac{1}{2} \frac{x^3}{3} - \frac{1}{8} \frac{x^5}{5} - \frac{1}{16} \frac{x^7}{7} - \frac{5}{128} \frac{x^9}{9} \right), \&c.,$$

and the history of the discovery of the series within the brackets, which expresses the integral of  $(1-x^2)^{\frac{1}{2}}$ , was given by Newton in his letter to Leibnitz of 24 October 1676, of which I have given a translation in a previous contribution. It will be remembered that this was the very first series obtained by Newton and that it was immediately connected with the discovery of the Binomial Theorem. That it continued

to be for some time a puzzle to Gregory is shown by the following extract from his letter of 23 November 1670 to Collins (Rigaud, v. ii, p. 203) :

I suppose these series I send you here enclosed may have some affinity with those inventions you advertise me that Mr. Newton had discovered. It was upon this account I so often desired you to communicate the same unto me. I shall also give here an approximation for the sines.

It is unnecessary to reproduce the approximation here. He continues :

It were no hard matter to bring from these several approximations for the segments of a circle, but it were to no purpose seeing I cannot take away the alternate powers as Mr. Newton doth in his series (if it be one, for the truth is I cannot reduce it to any of mine), yet I imagine that my series may be as shortly performed as his, seeing that my continued proportion (*caeteris paribus*) is much greater than his, and hence the terms of the series vanish much more quickly. . . .

The letter from which this extract is taken must have been a communication of great length. Unfortunately the original has disappeared, and we have only extracts made by Collins which occupy more than ten pages of Rigaud. From a comparison with portions of the same letter quoted in the *Commercium Epistolicum* it would appear that the original was in Latin ; but Collins's extracts, which have the character of rough notes made for his personal use, are partly in English and partly in Latin, half a sentence sometimes being in one language and half in the other. In my quotations from Rigaud I have turned the Latin wherever it occurs into English.

The following portions of the letter are of special interest because they constitute the evidence which is adduced to prove the priority of Gregory's discovery of the fundamental formula of interpolation (Rigaud, v. ii, p. 208) :

*A method of interpolation.*

In the end of my Geometrical Exercitations I fail exceedingly . . . . . and hence in place of anything I have described there, if we put

$$AP = PO = c$$

$$PB = d$$

and make the leading terms of the 1st, 2nd, 3rd, 4th, 5th, differences =  $f, h, i, k, l$ , then taking all the differences to be of the positive sign, ABP will be

$$= \frac{dc}{2} - \frac{fc}{12} + \frac{hc}{24} - \frac{19ic}{720} + \frac{3kc}{164} - \frac{863lc}{60480}, \text{ \&c., in infinitum.}$$

However the differences be affected I can easily square the figure, and by this means all figures imaginable. I cannot also but advertise you that Mr. Mercator's quadrature of the hyperbola is a consequtary of this.

I remember you did once desire of me my method of finding the proportional part in tables, which is this. In figure 8 of my Exercitations in the straight line AI let there be imagined a part Aa to which  $a\gamma$  is perpendicular,  $\gamma$  being on the curve ABH. Giving the same meanings as before to the letters  $d, f, h, i, k, l$ ,

let there be an infinite series,  $\frac{a}{c}, \frac{a-c}{2c}, \frac{a-2c}{3c}, \frac{a-3c}{4c}, \&c.$ ; then

calling the product of the first 2 terms,  $\frac{b}{c}$ ; of the first

3 terms,  $\frac{k}{c}$ ; of the first 4 terms,  $\frac{1}{c}$ ; of the first 5 terms,  $\frac{m}{c}$ , &c.

in infinitum; the straight line  $a\gamma$  will be  $= \frac{ad}{c} + \frac{bf}{c} + \frac{kh}{c} + \frac{li}{3} +$ ,

&c., ad infinitum. This method, as I apprehend, is both more easy and universal than either that of Briggs, &c., and also performed without tables.

The diagram used by Gregory is not given but can easily be imagined. AB is a curve of which the end A is at the origin of the axes of co-ordinates, PB is the ordinate at the point P, and  $c$  is the common distance between the ordinates. If we write  $v, \Delta v, \Delta^2 v, \Delta^3 v, \Delta^4 v, \Delta^5 v$ , in place of  $d, f, h, i, k, l$ , Gregory's formula for the area ABP in the first part of the above extract becomes

$$c\left(\frac{1}{2}v - \frac{1}{12}\Delta v + \frac{1}{24}\Delta^2 v - \frac{19}{720}\Delta^3 v + \frac{3}{164}\Delta^4 v - \frac{863}{60480}\Delta^5 v + \right), \&c.$$

The expression within brackets, after altering in the denominator of the 5th term 164 to 160, is the integral between the limits 0 and 1 of the series.

$$X_1 v + X_2 \Delta v + X_3 \Delta^2 v + X_4 \Delta^3 v + X_5 \Delta^4 v + X_6 \Delta^5 v +, \&c.,$$

where  $X_n$  stands for

$$\frac{X(X-1)(X-2) - (X-n+1)}{n}$$

The relationship of this series to the fundamental interpolation formula of Newton,

$$u_0 + x\Delta u_0 + x_2\Delta^2 u_0 + x_3\Delta^3 u_0 + x_4\Delta^4 u_0 +, \&c.,$$

is seen at once when we remember that Gregory arranged his diagram so that  $u_0$  should be zero, and that he accordingly



made  $\Delta u_0$  the last term of the series of primary values. Writing  $\Delta u = v$ , Gregory's form is at once obtained.

In the second part of the extract,  $\alpha$  is the length of the abscissa  $A\alpha$ , and when we make the same substitution as above and write  $x$  for  $\frac{\alpha}{c}$ , his formula for the ordinate  $\alpha\gamma$  takes the form

$$(xv + x_2\Delta v + x_3\Delta^2v + x_4\Delta^3v +), \&c.,$$

which, as we have just pointed out, is the equivalent, on Gregory's assumption, of the Newtonian form

$$u_0 + x\Delta u_0 + x_2\Delta^2u_0 +, \&c.$$

In the next extract Gregory gives the Binomial Theorem (Rigaud, v. ii, p. 209):

*To find the number of a logarithm.*

The logarithms of the two numbers  $b$  and  $b+d$  being  $e$  and  $e+c$ , it is required to find the number whose logarithm is  $e+a$ .

Let there be a series of continued proportionals,  $b, d, \frac{d^2}{b}, \frac{d^3}{b^2}, \&c. \dots$  and another series,  $\frac{a}{c}, \frac{a-c}{2c}, \frac{a-2c}{3c}, \frac{a-3c}{4c}, \&c.;$  and let the product of the first two terms of this series be  $\frac{f}{c}$ ; of the first three,  $\frac{g}{c}$ ; of the first four,  $\frac{h}{c}$ ; of the first five,  $\frac{i}{c}$ , &c., the number whose logarithm is  $e+a$  will be

$$b + \frac{ad}{c} + \frac{fd^2}{cb} + \frac{gd^3}{cb^2} + \frac{hd^4}{cb^3} + \frac{id^5}{cb^4} + \frac{kd^6}{cb^5} +, \&c.$$

Hence by the application of a little industry, any pure equation is solved without any trouble.

Here, if we take  $g$  as the base of logarithms,  $g^e = b$ , and  $g^{e+c} = b+d$ , so that  $g^c = 1 + \frac{d}{b}$ . The number sought is  $g^{e+a} = b \times g^a = b \left(1 + \frac{d}{b}\right)^{\frac{a}{c}}$ , and writing  $n$  for  $\frac{d}{b}$  and  $x$  for  $\frac{a}{c}$ , Gregory's formula is

$$b(1+n)^x = b(1 + x_1n + x_2n^2 + x_3n^3 +), \&c.$$

By a pure equation is meant such an equation as  $x^n = a$ , where to find  $x$  all that is required is the extraction of the  $n$ th root of  $a$ . Equations involving two or more different powers of the unknown quantity were called affected equations.

Next follows the solution of a question in Compound Interest which had been put to Gregory by Collins (Rigaud, v. ii, p. 210):

An example was desired about finding the first of 364 means between 100 and 106.

$$\text{Put } b = 100$$

$$d = 6$$

$$e = 0$$

$$a = 1$$

$$c = 365$$

The first series is 100, 6,  $\frac{36}{100}$ ,  $\frac{216}{10,000}$ ,  $\frac{1,296}{1,000,000}$ , and these 4 terms are the 1st, 2nd, 3rd, and 4th differences.

The second series is  $\frac{1}{365}$ ,  $-\frac{364}{730}$ ,  $-\frac{729}{1095}$ ,  $-\frac{1094}{1460}$ , &c.

Hence,  $\frac{f}{c} = -\frac{182}{133225}$ ;  $\frac{g}{c} = \frac{132678}{145881375}$ ; it follows therefore from what precedes

$$b = 100 \qquad = 100 \cdot$$

$$\frac{ad}{c} = \frac{6}{365} \qquad = \cdot 01647123$$

$$\frac{fd^2}{cb} = -\frac{6552}{13322500} \qquad = -\cdot 00049199$$

$$\frac{gd^3}{cb^2} = \frac{28,658,448}{1,458,813,750,000} \qquad = \cdot 00001964$$

all of which added together make the  
mean proportional sought  $= 100 \cdot 0159919$

The first term of the second series is given in Rigaud as  $\frac{1}{364}$ , but it is clear that this is an error.

The value required is that of  $100 \times (1 + \cdot 06)^{\frac{1}{365}}$ , and the actual sum of the numbers given above is 100·0159989. There are however errors in the work and a sufficient number of terms is not taken. The corrected figures and the figures of the additional terms are as follows:

$$\begin{array}{r} 100 \cdot \\ \cdot 01643836 \\ - \cdot 00049180 \\ \cdot 00001965 \\ - \cdot 00000088 \\ \cdot 00000004 \\ \hline 100 \cdot 01596537 \end{array}$$

The student who cares to verify the calculation will find it convenient to make use of Erskine Scott's 10 place logarithms (2nd edition, 1892, pp. 386 and 392).

The final extract from the same letter is the fully worked out example which was mentioned by my Edinburgh critic (Rigaud, v. ii, p. 211):

*An example of interpolations.*

The cube of the number 23 is required.

Number	Cubes	1st Diff.	2nd Diff.	3rd Diff.
10	1000			
15	3375	2375		
20	8000	4625	2250	750
25	15625	7625	3000	750
30	27000	11375	3750	

$$c = 5$$

$$a = 23 - 10 = 13$$

$$d = 2375$$

$$f = 2250$$

$$h = 750$$

The infinite series  $\frac{a}{c}, \frac{a-c}{2c}, \&c.,$  is  $\frac{13}{5}, \frac{4}{5}, \frac{1}{5}, \frac{1}{10}.$

The products are  $\frac{13}{5}, \frac{52}{25}, \frac{52}{125}, \frac{26}{625}.$

And

$$1000 = 1,000$$

$$\frac{13}{5} \times 2375 = 6,175$$

$$\frac{52}{25} \times 2250 = 4,680$$

$$\frac{52}{125} \times 750 = 312$$

$$\text{Cube of } 23 = \underline{\underline{12,167}}$$

Gregory's next letter, dated 19 December 1670, is of extreme importance. After despatching his communication of 23 November from which the above extracts have been

taken, he returned to that series of Newton's which had so greatly baffled and perplexed him, and this time light broke upon his mind. In the version of the letter given by Rigaud, from a paper in the handwriting of Collins, the word "consectary" which occurs here and in other parts of the correspondence is a transference into English of the Latin *consectarium* which has the meaning of a necessary consequence. The reference to the proposition concerning Logarithms is to the extract given above from the November letter in which, as I have pointed out, Gregory had propounded what is in effect the Binomial Theorem. The letter as given by Rigaud (v. ii, p. 212) runs as follows:

Sir, In my last to you I had not taken notice that Mr. Newton's series for the zones of a circle (which you sent me a long time ago), together with an infinite number of series of the like nature may be a consectary to that which I sent you concerning logarithms; namely, given a logarithm to find its number, or to convert the root of any pure power into an infinite series.

I admire much my own dullness, that in such a considerable time I had not taken notice of this; nevertheless that I had taken much pains to find out that series. But the truth is, I thought always (if so be it were a series) that I might fall upon it by some combination of my series for the circle, seeing I had such infinite numbers of them, not so much as once desiring any other method.

Putting R for the radius and B for the breadth of the zone, the area thereof producing the series a little further is, namely,

$$= 2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} - \frac{5B^9}{576R^7} - \frac{7B^{11}}{1408R^9} \\ - \frac{21B^{13}}{6656R^{11}} - \frac{11B^{15}}{5120R^{13}} - , \&c.$$

On the same suppositions the arc whose sine is B

$$= B + \frac{B^3}{6R^2} + \frac{3B^5}{40R^4} + \frac{5B^7}{112R^6} + \frac{35B^9}{1152R^8} + , \&c.$$

I could give you several other series of this nature but perchance you know more of them than myself. . . .

The muffled English of this version by Collins gives no idea of the vigour and vividness of the original letter in Latin which has fortunately been preserved in the *Commercium Epistolicum* (No. XVIII), and is there stated to be taken from Gregory's autograph. On account of the interest and importance of the letter I give the original Latin:

Quum postremas ad te dedi literas, nondum animadvertissem D. Newtoni seriem de circuli zonis, quam jamdudum ad me misisti, una cum infinito istiusmodi serierum numero, consecrarium illius esse posse, quam misi de logarithmis : nempe dato logarithmo invenire ejus numerum ; vel radicem potestatis cujuscunque purae in infinitam seriem permutare.

Me sane tam tardi fuisse ingenii miror ; qui tanto temporis spatio hoc non animadverteram, quum tamen multum olei et operae in ista serie expiscanda impenderam. At ut ingenue fateor, semper in animum induxeram, si modo series esset, me in eam incidere posse, ope aliquarum e seriebus meis pro circulo inter se combinatis quarum quidem plurimas ad manus habeo ; neque ullam aliam desideraram methodum. . . . . Plures hujusmodi series proferre possem, sed tu fortasse plus meipso de his rebus nosti. . . .

“Me sane tam tardi fuisse ingenii” : He reviles himself as heartily as a golfer who suddenly discovers that he has played on to the wrong green.

This letter was a frank acknowledgment of Newton's priority, and Gregory made no subsequent claim to any credit for his independent discovery.

Hutton in his *Mathematical and Philosophical Dictionary* gives the version of the story preserved in Gregory's family as communicated to him by Professor Thomas Reid, who has already been mentioned as one of Gregory's collateral descendants. He says :

That in 1670 having received in a letter from Collins a series for the area of the zone of a circle, and as Newton had invented an universal method by which he could square all curves, Geometrical and Mechanical, by infinite series of that kind, Gregory after much thought discovered the universal method or an equivalent one. Of this he perfectly satisfied Newton and the other mathematicians of that time, by a letter to Collins in February 1671. He was strongly solicited by his brother David to publish his *Universal Method of Series* without delay, but excused himself upon a point of honour, that as Newton was the first inventor and as he had been led to it by an account of Newton's having such a method, he thought himself bound to wait till Newton should publish his method. I have seen the letters that passed between the brothers on this subject.

Reid appears to have made a slip in giving February 1671 instead of 23 November 1670 as the date of Gregory's letter to Collins.

Collins himself was quite clear as to Newton's priority. I have already quoted from his letter of 24 December 1670 to

Gregory. He wrote to Berter, the French mathematician, on 21 February 1670-1 (Comm. Epistolicum, XXI) :

About four years ago a general analytical method was discovered by Isaac Newton for the quadrature of all curves. . . . James Gregory, in Scotland, has quite recently hit upon the same method.

On the 30 July 1672 he wrote to Newton (Comm. Epistolicum, XXV) :

I believe that Gregory has expended no small labour in investigating series for the extraction of roots in (Algebraical) symbols after the manner used by Viète (in numbers). However, he is resolved not to write anything on the subject before you, the inventor of this method, have made public your own conclusions, but is intent upon other subjects in the meantime.

On the 26 July 1672 he had written to Strode (Comm. Epistolicum, XXIV) :

In the month of September 1668 Mercator published his *Logarithmotechnia*, which contains an example of this method, *i.e.*, of infinite series, in the case of one figure only, namely, the quadrature of the hyperbola. Not long after the public appearance of that book I sent a copy to Wallis at Oxford, who at once gave a review of it in the *Philosophical Transactions*; also a copy to Barrow at Cambridge, and in return he immediately sent some papers of Newton, who had just succeeded Barrow in the public mathematical lectures. From these papers and from other papers previously communicated by the author to Barrow, it appears that that method had been thought out by the said Newton some years before, and had been applied by him in a universal manner; so that by its use there can be obtained for any curved figure that may be proposed, which is defined by one or more properties, the quadrature or area of the said figure, accurately if that be possible, or else to an infinitely near approximation. . . .

After Gregory had learned that this method used by Mercator in the *Logarithmotechnia* and employed for finding the area of the hyperbola, which Gregory himself had augmented, had now been generalized and applied to all figures; he arrived at the same result by intense study and laboured hard in expounding it.

Each of them, Newton and Gregory, has the intention of elaborating this method. But Gregory does not consider it right to anticipate Newton, its first discoverer.

Gregory continued to correspond with Collins until the month of his sudden death; through Collins he was in friendly communication with Newton, principally on the subject of the telescope, and there is no trace of any controversy or difference between them on the matter in



question. They do not appear ever to have met, though Newton in a letter dated September 1673, understanding that Gregory was then in London, expresses the hope that "Mr. Gregory will be pleased to favor me with a visit" on his way into Scotland.

There is one point on which it may be useful to add a remark. Gregory in his letter of 23 November 1670 incidentally gave the Binomial Theorem as a particular case of his general interpolation formula. It was a perfectly familiar idea to the mathematicians of the time that the coefficients of the powers of  $x$  in the expansion of  $(a+x)^n$ , were identical with the coefficients in the expression for  $u_n$  in terms of  $u_0$  and its differences, the coefficients being taken from a table of figurate numbers. Mercator in his *Logarithmotechnia*, which we know to have been in the hands both of Gregory and of Newton, after describing how to obtain the expression for  $u_n$ , says that the table of figurate numbers provides the necessary coefficients "in just the same way as the same figurate numbers provide the coefficients of the powers generated from a binomial expression"; and Mercator was simply expounding the common knowledge of his time.

But the table of figurate numbers did not enable mathematicians to deal with other than integral values of  $n$ . The general formula for the coefficients was not known; and it was by enunciating the general formula that Newton, and after him Gregory, made the great step forward which enabled them at one and the same time to express the fractional powers of a Binomial in an infinite series and to obtain the general interpolation formula of Finite Differences.

*A Note on Mr. King's method of Graduation and its relation to Graphic Method.* By G. UDNY YULE, M.A., C.B.E.

THE method of graduation to which I refer is that described by Mr. King in *J.I.A.*, vol. xliii, pp. 109 *et seq.* (*cf.* especially §§ 8-14), and used by him for the graduation of ages at the Census of England and Wales, 1911, and in the construction of English Life Tables No. 7 and No. 8, and others included in Cd. 7512, 1914.

In Mr. King's notation  $u$  is the function to be graduated,

say, numbers at single years of age,  $w$  denotes the sum of, say, five values of  $u$ ,  $w_0$  being the sum of  $u_0, u_1, u_2, u_3, u_4$ ;  $w_5$  the sum of  $u_5, u_6, u_7, u_8, u_9$ ; and so on. Then we have approximately for a graduated central value of  $u$ ,

$$u_7 = \cdot 2w_5 - \cdot 008\Delta^2 w_0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The result can obviously be generalized without any difficulty. Following Mr. King's method, let  $y$  be the finite integral of  $u$  taken negatively, so that  $y_t = u_t + u_{t+1} + \dots$ . Let  $w_t$  denote the sum of  $x$  values of  $u$ , beginning with  $u_t$ . For the purpose of differencing let  $x$  be regarded as a unit, so that  $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$  denote the differences of  $y$  obtained from  $y_0, y_x, y_{2x}, \dots$ . Then

$$\begin{aligned} y_{\frac{3x-1}{2x}} &= y_0 + \frac{3x-1}{2x} \Delta y_0 + \frac{(3x-1)(x-1)}{8x^2} \Delta^2 y_0 \\ &\quad - \frac{(3x-1)(x-1)(x+1)}{48x^3} \Delta^3 y_0 \\ y_{\frac{3x+1}{2x}} &= y_0 + \frac{3x+1}{2x} \Delta y_0 + \frac{(3x+1)(x+1)}{8x^2} \Delta^2 y_0 \\ &\quad - \frac{(3x+1)(x+1)(x-1)}{48x^3} \Delta^3 y_0 \end{aligned}$$

Whence, for the central value of  $u$  determined from three groups of  $x$  years each, we have

$$\begin{aligned} u_{\frac{3x-1}{2}} &= y_{\frac{3x-1}{2x}} - y_{\frac{3x+1}{2x}} \\ &= -\frac{1}{x} \Delta y_0 - \frac{1}{x} \Delta^2 y_0 + \frac{x^2-1}{24x^3} \Delta^3 y_0 \\ &= +\frac{1}{x} w_0 + \frac{1}{x} \Delta w_0 - \frac{x^2-1}{24x^3} \Delta^2 w_0 \\ &= \frac{1}{x} w_x - \frac{x^2-1}{24x^3} \Delta^2 w_0 \end{aligned}$$

or, as I prefer to write it

$$u_{\frac{3x-1}{2}} = \left[ \frac{w_x}{x} \right] - \frac{x^2-1}{24x^2} \Delta^2 \left[ \frac{w_0}{x} \right] \quad . \quad . \quad . \quad . \quad (2)$$

In this expression the first term, the average value of  $u$

over the interval, may be regarded as a first approximation to the central value, the second term as a correction. Obviously the central value of  $u$  will not correspond to an integral year unless  $x$  is odd: if  $x$  is even, the central value falls between two years. The coefficient of the second term for  $x=2$  is  $1/32$  and rapidly approaches its limiting value of  $1/24$  as  $x$  increases. The values are:

$x$	Coefficient of $\Delta^2 \left[ \frac{w_0}{x} \right]$	
	As fraction	As decimal
2	$1/32$	.03125
3	$1/27$	.0370370
4	$15/384$	.0390625
5	$1/25$	.04
6	$35/864$	.040593
7	$2/49$	.0408163
8	$63/1536$	.0410156
9	$10/243$	.0411523
10	$99/2400$	.04125
11	$5/121$	.0413223
Limit	$1/24$	.0416667

In its general form the correction is closely related to a graphic method given many years since by Lord Rayleigh (Phil. Mag., vol. xlii, 1871, pp. 441-4, and Collected Papers, vol. i, p. 135, "on a correction sometimes required in curves "professing to represent the connection between two physical "magnitudes"). The correction is this. Plot the curve given by the quantities like  $w/x$ , i.e., values averaged over a range, say,  $\pm h$ : such values might be given, for example by a line thermopile of width  $2h$  used to measure temperature along a spectrum. Let R (Fig. 1) be any point on the false curve so obtained. Let PM, QN be ordinates at the distances  $\pm h$  from the ordinate Rm. Join PQ cutting Rm in S. Take Rp equal to one-third of RS. Then  $p$  is the corrected point,  $p$  and S always lying on opposite sides of R. It will be noted that in this construction PM and QN are, in the actuarial case, unknown quantities, representing bounding ordinates of a  $w$ -group. In Lord Rayleigh's case, since many values—overlapping values—of  $w/x$  are obtained by observation, the ordinates may be supposed for graphic purposes to be known.

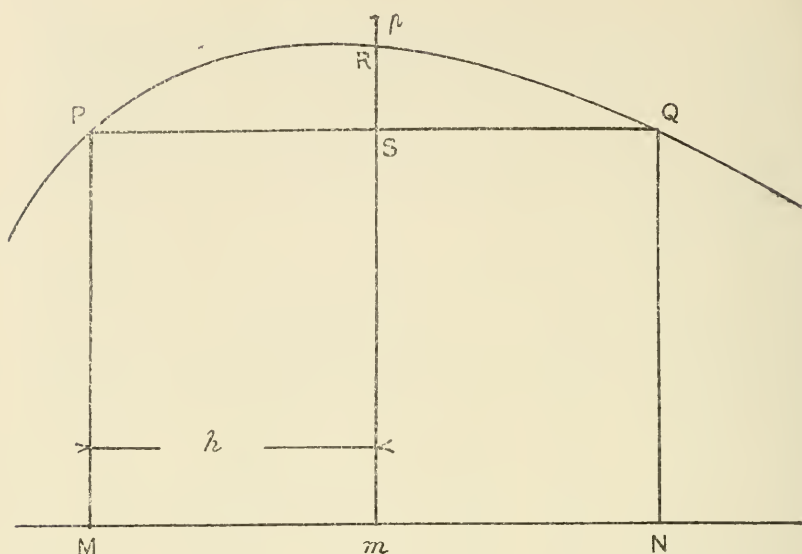


FIG. 1.

The proof given is as follows: Let  $y$  denote an ordinate of the *true* curve; then

$$\int_{-h}^{+h} y \cdot dh = \int_{-h}^{+h} \left( y_0 + \frac{dy}{dx_0} h + \frac{1}{2} \frac{d^2y}{dx_0^2} h^2 \right) dh$$

where  $y_0$  is the ordinate of the true curve at  $m$  and  $x_0$  is the abscissa of  $m$ . That is

$$\text{area} = 2h \left( y_0 + \frac{h^2}{6} \frac{d^2y}{dx_0^2} \right).$$

But if  $y'$  be the ordinate of the observed curve, by definition we must have  $2hy'_0$  equal to this area. That is

$$y'_0 = y_0 + \frac{h^2}{6} \frac{d^2y}{dx_0^2},$$

and, generally,

$$y' = y + \frac{h^2}{6} \frac{d^2y}{dx^2}.$$

Hence

$$\begin{aligned} \frac{d^2y'}{dx^2} &= \frac{d^2y}{dx^2} + \frac{h^2}{6} \frac{d^4y}{dx^4} \\ \frac{h^2}{6} \left( \frac{d^2y}{dx^2} \right) &= \frac{h^2}{6} \left( \frac{d^2y'}{dx^2} - \frac{h^2}{6} \frac{d^4y}{dx^4} \right) \\ &= \frac{h^2}{6} \frac{d^2y'}{dx^2} \end{aligned}$$

if we may neglect the term in  $h^4$ . Whence finally

$$y_0 = y'_0 - \frac{h^2}{6} \frac{d^2 y'}{dx_0^2} \quad (3)$$

But

$$\begin{aligned} Sm &= \frac{1}{2} (PM + QN) \\ &= \frac{1}{2} \left( y'_0 - \frac{dy'}{dx_0} h + \frac{d^2 y'}{dx_0^2} \frac{h^2}{2} \right) \\ &\quad + \frac{1}{2} \left( y'_0 + \frac{dy'}{dx_0} h + \frac{d^2 y'}{dx_0^2} \frac{h^2}{2} \right) \\ &= y'_0 + \frac{d^2 y'}{dx_0^2} \frac{h^2}{2} \quad (4) \end{aligned}$$

Therefore, since  $Sm = y'_0 - RS$ ,

$$y_0 = y'_0 + \frac{1}{3} RS \quad (5)$$

As pointed out above, to get this correction Lord Rayleigh used two points which, in the actuarial case, cannot be supposed to be known. Let the figure be extended, and let AB, CD (Fig. 2) be the ordinates of the "false curve" at distances of  $\pm 2h$  from Rm. By the same line of argument, if we may still neglect powers of  $2h$  beyond the second,

$$Tm = y'_0 + \frac{d^2 y'}{dx_0^2} 2h^2$$

corresponding to equation (4) above. Whence

$$RT = y'_0 - Tm = - \frac{d^2 y'}{dx_0^2} 2h^2$$

and

$$y_0 = y'_0 + \frac{1}{12} RT.$$

This is the limiting value of Mr. King's correction. We evidently have without approximation, if

$$AB = \frac{w_0}{x}, \quad Rm = \frac{w_x}{x}, \quad \text{and} \quad CD = \frac{w_{2x}}{x},$$

$$RT = \frac{1}{2x} \{ \overline{w_x - w_0} - \overline{w_{2x} - w_x} \} = - \frac{1}{2} \Delta^2 \left[ \frac{w_0}{x} \right]$$

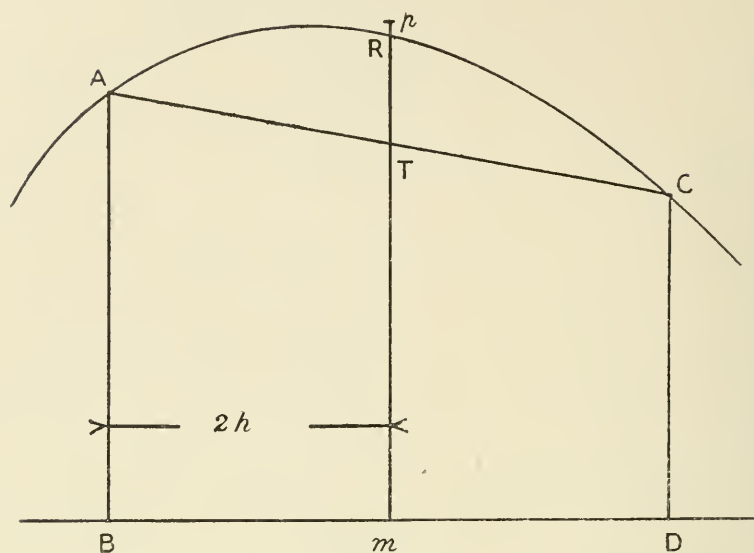


FIG. 2.

Hence Lord Rayleigh's method would give for the actuarial case: Erect ordinates at equal distances giving the average number of persons per year ( $w/x$ ) in the groups. Let AB, Rm, CD, be three successive ordinates. Join AC cutting Rm in T. Produce TR by  $1/12$  of itself to  $p$ , then  $p$  is a point on the graduated curve,  $p$  and T lying on opposite sides of R.

Mr. King's method, applied to graphic construction, would only substitute for the fixed coefficient  $1/12$  a coefficient varying with the number of years in the group, taking the value  $\cdot 08$  for a five-year group, and only dropping to the limit of  $1/12$  or  $\cdot 08333 \dots$  for a group covering an indefinitely large number of years or other unitary groups. The difference between the results given by the two methods on a graphic scale would usually be hardly appreciable.



*Actuarial Aspects of Industrial Assurance, with special reference to the Report of the Departmental Committee on the Business of Industrial Assurance Companies and Collecting Societies.*

[Abstract of a Discussion at the Institute on 10 May 1920.]

SIR JOSEPH BURN, K.B.E., in opening the discussion, said that it was impossible, in his opinion, to deal adequately with the subject of industrial insurance without first considering one or two fundamental questions. The first was whether industrial insurance was desirable. There was undoubtedly a general demand for industrial insurance. It had promoted the saving habit. If insurance was desirable for the upper classes of society, as he was convinced it was, then surely it was also desirable for the great body of wage-earners who were for the most part receiving their wages weekly and who wished to pay small sums weekly or monthly provided the way was made easy for them by means of collectors at their own homes. It might be argued that although industrial life assurance in its simplest form was desirable, endowment insurance raised a somewhat different question. It must be admitted that in the case of endowment insurance the higher expense-ratio of industrial, as compared with ordinary, business made the question more difficult, but the fact remained that people desired endowment insurances: they wished to save by small instalments, and there was a very great number of cases where they would not do so if they had not the opportunity of industrial endowment insurance.

If it were conceded that industrial insurance was desirable, the feature of high expense must be admitted as necessary, and capable only of modification. Industrial insurance would always be expensive. But could it be made cheaper? He thought it could. It must be remembered that prices had risen, and might rise still further. Everything which contributed to the expense of industrial insurance had increased in cost, and consequently the fact had to be faced that in order merely to keep the expense ratio down to what it had been in the past, some powerful means of economy must be found. Fortunately, as it seemed to him, the trouble itself provided to some extent its own cure. If the standard of prices had now doubled, it followed that the minimum amount of life insurance required had also doubled. Before the war, generally speaking, people in this country were holding far less life insurance than they should have done, and consequently at the present time they were holding far less than half as much as they should hold. Sooner or later it would be found that the agent would secure at least twopence in the place of each penny that he formerly collected. His work, however, would not have doubled, and consequently his rate of remuneration for services rendered would automatically increase. It was true that the movement which would place the average amount of insurance held more nearly in agreement with the standard of prices must take time; but so far as the outdoor staff

of industrial insurance companies was concerned they would find a larger amount of new business available, and in some form or other they would receive—and, in his opinion, they must receive—the equivalent of a new business commission, which would, as indeed it already had, help every honest worker to bridge over the interval which otherwise would have entailed very grave hardship. However, that corrective, which appeared to him to be a necessarily automatic one, did not mean that industrial insurance companies could or would be content to refrain from making the most strenuous efforts to economise, re-organise and generally improve the machinery of their business. Effective steps in that direction were being taken before the war, and all well-wishers of industrial insurance were convinced that those efforts would not be decreased but would be increased to the greatest possible extent.

Another method by which it might be hoped there would be some decrease in expense would be, he thought, the extension of the block system. It seemed a little curious that such brief reference had been made to that in the Report of the Committee of Enquiry. On page 3 the Committee stated: "Evidence was given before the Committee that convenience and economy in collection could be improved by the introduction of the block system, under which an agent is assigned an exclusive area for his operation. This system is still in the experimental stage, but the Committee suggest that it should be introduced and extended wherever practicable. They are of opinion that by the general adoption of this system a large amount of unnecessary labour and expenditure might be avoided." Then, on page 11 (where the question of lapses is referred to), it was stated: "They are of opinion, therefore, that the end in view can only be reached by the abolition of procuration fees, and recommend the prohibition of the payment to agents of procuration fees for new business, and the substitution of a minimum weekly wage, based on a fixed collection per week, with a commission on all sums collected above that amount." The latter statement appeared to him somewhat vague. To those who had given a great deal of careful study to the matter, and who had the added advantage of a wide practical experience, it seemed that great economies could be effected by blocking out all insurance areas and employing salaried agents for the collection of industrial insurance premiums. The collectors, it seemed to him, should be paid suitable and generous salaries in accordance with the work they performed. Their duties to a great extent were those of collecting clerks, and the amount of remuneration to which they were entitled obviously could not in all cases be stated as a fixed percentage of the amount collected any more than could the salary of a tram conductor or, possibly, of a railway booking clerk. Their work would depend largely upon the concentration of the business, on the average amount of the premiums to be collected, and on the class and attitude of the assured persons. It did not seem to him possible or desirable, in the interests of all the parties concerned, to attempt to make a rigid rule as to the weekly amount to be collected by any one man. It

was quite possible that a man who might be an excellent collector might not be so suitable for work in connection with new business. He used that general expression because there was other important work in connection with new business quite apart from that of canvassing. The duty of all industrial insurance companies was to find the most effective and the most economical way of doing all the work that was necessary. It did not seem possible that any Committee of Enquiry, however sincere in its efforts, would be able usefully to instruct those who had a life-long experience and who were trying honestly to do their best. Still less did it seem likely that such instructions could be set out in the rigid wording of an Act of Parliament without hindering the very objects which it was intended to promote.

The subject of industrial insurance expenses was such an important factor in an actuarial valuation, that it seemed to him better to deal with that matter first. It was well known that in ordinary business the factor of initial expense was of some importance in connection with the basis of valuation. It was of even greater importance in the case of industrial insurance. It must be remembered that in order successfully to carry on industrial insurance an organisation must be built up, and that could only be done at considerable expense. Then, again, the early lapse of a policy entailed far greater proportional loss in the case of an industrial company than it did in the case of an ordinary company. An ordinary company, having issued a policy, say, for £1,000, and having obtained the first annual premium, could contemplate the possibility of lapse in quite a different way from an industrial insurance company which, after a considerable amount of labour and expense, had in the end succeeded in obtaining, perhaps, one penny.

The question then arose whether a standard valuation was desirable. That was a very important matter, requiring the most careful consideration by those most capable of appreciating the very many difficulties involved. In the past the Institute of Actuaries had maintained that the best safeguard for the public was publicity. Was that well-considered decision to be reversed, and, if so, where was its reversal likely to lead? Was publicity to be held to be a sufficient safeguard for ordinary policyholders and not a sufficient safeguard for industrial policyholders? When was a company insolvent? He did not think the last question could be answered in the same terms in the case of an industrial company as in the case of an ordinary company. Generally speaking, the expense loading of an industrial insurance premium was less than the average annual rate of expense, including all costs of new business. The flow and proportion of new business was a very important factor and a very difficult one to deal with. Surely merely to set up one fixed standard of solvency would be most mischievous. On page 6 of the Report, the Committee said: "The differences between the respective bases of valuation are the more unfortunate, in that they appear to have misled the agents into thinking that certain companies place to reserve a greater part of the premiums than the liabilities require."

In the evidence tendered to the Committee, it was submitted that the reserves should be lessened in order that the remuneration of the agents might be increased." It seemed likely that a fixed standard valuation basis would only tend to increase those mischievous demands. The actuary should be, and generally is, always anxious to strengthen his reserve bases and to advise increased benefits only when he felt confident that such might safely be done. Would it not, in any case, be necessary to have varying standards of solvency settled, not by Acts of Parliament, but by competent judges who would be able to give full weight to the particular circumstances? Or would it possibly be better if, in certain cases, a company or society should be required to prove its solvency? Theoretically, he supposed the most nearly perfect basis of valuation would be one where the actuary made the closest possible estimate of every factor; that was to say where he made the best possible estimate as to probable mortality rates, interest rates, expense rates, lapse rates, &c. He did not think there were many actuaries, however, who would advise such a basis. As to interest, who could say what would be the net interest yield in the next fifty years, or for that matter the next five years? As to lapses, it was well known to actuaries that there would always be lapses, but neither their extent nor their effect could be closely measured. As to expenses, he had already indicated some of the difficulties. To some, a net premium valuation was all that was required. Personally, he was glad that he had never been called upon to make any other than net premium valuations. But would it be right to insist on a net premium valuation in every case? Would such a course result in the equitable treatment of the greatest number of industrial policyholders? Then, should the inclusion of negative values be permitted? He considered certainly not, but some eminent members of the Institute had not held the same opinion. Should extension expenses or other similar items be included as assets in the insurance funds? Again, he thought the answer was necessarily in the negative, but in the case of a young company was it not permissible to use some portion of the shareholders' capital in building up a sound and suitable organization? On page 13 of the Report, reference was made to cases where debentures and loans had been issued and charged on the life assurance fund. That surely was a proceeding which must be condemned by any competent actuary. The situation must have become desperate if no other alternative was possible.

The position of the actuary to an industrial insurance company was one of peculiar responsibility. He should be prepared and should be given the fullest opportunity to speak out boldly what he believed to be the truth on every occasion. With every copy of the Institute of Actuaries' *Journal* there was a reminder of what the members of the Institute were expected to do, and those noble words of Bacon were: "I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto." In those words



if carefully studied, would be found the whole truth. It should be noticed that in a profession one expected, and was entitled to receive, countenance as much as profit. Actuaries were not to be merely clerks or automatic calculating machines but expert advisers who had been admitted as Fellows to an honourable Institute, to which at all times they should be prepared to render an account. The President would, he felt sure, agree with him that the Institute also had its responsibilities. Having admitted its Fellows as qualified actuaries it must be prepared at all times to uphold the honour of the profession, to see that professional services were well and honourably rendered, and to give every possible assistance and support to its members in order to enable them to act up to Bacon's inspiring words, both in spirit and in truth.

Upon the basis of valuation the treatment of policyholders with regard to the granting of free policies and surrender values was largely dependent. It would be evidently unsound to grant any form of surrender values in excess of the reserve held. Having regard to the effect of selection—that was to say, to the certainty that persons in a very bad state of health were less liable to surrender—he thought that it was equally evident that any form of surrender value allowed should have a cash value less than the reserve held against any particular policy. Experience indicated that in most cases surrender value, in the form of a free, paid-up industrial policy, was preferable to a cash allowance, although the growing popularity of endowment insurance had necessitated some modification of that rule. Perhaps one of the most remarkable passages in the Report of the Committee of Enquiry was on page 5 where it said: "Industrial policies generally carry no right to surrender value or a free, paid-up policy upon lapse. In the case of the Prudential, it is stated to be the general practice of the company to grant free, paid-up policies after premiums have been paid for five years, provided that the life assured has attained the age of 15, and under these arrangements free policies have, it would seem, been regularly granted, where applied for." Why so grudging an admission of a fact known to everybody? A phrase like that did not tend to increase confidence in the belief that the Committee properly appreciated the rights of industrial policyholders, or that the Committee were sufficiently anxious to encourage the best form of progressive development. Perhaps that was the inevitable result of having only one professional expert on the Committee, who no doubt, at times, found it necessary to give way to the majority.

The Committee also stated on page 8 of the Report: "They think that, in general, the policyholders should receive a substantially greater part of the profits than is accorded to them at the present time." It might be argued that there was no right to any bonus, and that a bonus, if granted, must be regarded as a gift. But bonuses had, in fact, been granted to industrial policyholders in various forms. Rates had been revised and sums assured made subject to increase, and those alterations had been made retrospective, the result being a form of bonus policy free of all premiums

after a certain duration and a certain age attained, and there had been other forms of bonus instituted at different times. But the most conspicuous case of an industrial bonus declared, as a direct result of the profit disclosed in the annual valuation, was that of the Prudential, the bonus in this case being given in the form often referred to as a mortuary benefit. It was a bonus paid as an addition to the sums assured in all cases of claims where such claims resulted from death or the maturity of an endowment. That method was not free from possible criticism, as might be seen from the Report, but in practice he thought it was the best possible under the circumstances. It had to be remembered that one of the main considerations in any method adopted must be simplicity, and the saving of any considerable labour and expense. Before hastily concluding that a reversionary bonus with full cash reserve would be preferable, there were many points to be considered, which it would be impossible for him to deal with that evening.

The Committee of Enquiry was appointed, as stated on page 1 of the Report, to "enquire into the business carried on by industrial assurance companies and collecting societies." In order to form any correct opinion on the possibilities of industrial insurance business it seemed evident that a matter absolutely of first-class importance was the effect of the war, and one effect of the war was that the burden of the Courts (Emergency Powers) Act was thrown upon the societies. At the outset they were informed that there would be no loss to them, as they could always deduct the premiums from the claims. The Act was a great strain on industrial companies and it still remained. Was that Act just, so far as the insurance companies were concerned? The Report referred to this matter on page 14 in the following terms: "The question which arises under this Act is a complicated one. Suggestions for dealing with the problem were supplied to the Committee by the Prudential Company and by the Association of Industrial Assurance Companies and Collecting Friendly Societies. The Committee desire to express their obligation to the company and the association for the assistance which they have given. . . . When the Act comes to an end the Committee recommend, as a fair way of meeting the difficulties which must arise, that policies subject to the Act shall be kept in force for six months after the Act has ceased to operate and that the companies shall be under obligation to notify every policyholder to whom the Act applies his right to secure the maintenance of the policy by payment of the arrears, the amount of which should be stated on the notice. Alternatively, the companies should give the option to the policyholder of maintaining the policy in force for a reduced amount, or in the case of an endowment assurance for an extended period, subject to the cancellation of the arrears, on the terms of a scheme to be approved by the controlling authority." There seemed no recognition of how the Act weakened the resources of the company at the most critical time, and consequently how the capabilities of granting future concessions were also weakened. Industrial insurance companies and societies paid many millions



of pounds in the shape of war claims for which they were not liable and for which they received no premiums, that was to say the premiums they received were inadequate for the war risk in the same way that pre-war marine insurance rates were inadequate for the risks run during the submarine campaign. Did the members of the Committee commend that action or not? It was difficult to judge from the Report. Had war claims not been paid, or had adequate extra premiums been charged, the industrial bonus could not only have been maintained, but increased, with less strain on the companies' funds than the course they had taken had involved. Not only so, but had there been no war it was impossible to say how far the expenses of the business would have been reduced.

MR. A. D. BESANT said that in the interesting and weighty speech to which they had just listened Sir Joseph had dealt in some detail with the various technical questions affecting industrial insurance. Some of those questions were of great interest to actuaries engaged in that particular field but did not largely concern the Institute as an Institute, and he did not propose to discuss them in any detail. A question in his opinion of far wider importance was how did the Report of the Departmental Committee upon the industrial life assurance system affect the Institute, both in its corporate capacity and as regarded its individual members. The most important aspects of that question were the valuation question and the amalgamation question. He might begin by summing up as shortly as possible the main impressions gathered from a careful study both of the Departmental Committee's Report and of the minutes of evidence that had been published. It could not be denied, he feared, that that evidence showed the existence of abuses in certain directions, but of course it was necessary to keep a sense of proportion and to remember that those abuses were not universal—they were far from universal—and he hastened to say that many of the actuaries concerned had steadily and resolutely laboured, as Sir Joseph Burn had said, to improve the financial position of the companies with which they were connected. But having admitted that, the Committee undoubtedly did comment in the most adverse terms on unduly large payments to shareholders and to officials and directors and agents. They called attention to cases in which the expense of management had been unduly weighted as against industrial policyholders in favour of the ordinary policyholders, and they summed up their views by stating in Section 25 of the Report: "The industrial assurance system lends itself to abuse in many directions", and they went on to say that "the public is defenceless against the machinations of any group of adventurers who have neither money nor reputation to lose and whose single purpose is to exploit the system for their own benefit." In that connection he would ask the members to read over the detailed evidence given in cross-examination by Mr. Mashford, who dealt with typical cases of complaints made by policyholders. That evidence was set forth on pages 91 to 104 of the Minutes of Evidence. Section 26 of the Report dealt with abuses of amalgamations and transfers of business

and with the evasions of the safeguards which were supposed to be provided by the Act of 1909. In the next section cases were dealt with of abuses of valuation, such as the introduction of negative values and organization expenditure as assets. Finally, in Section 28, the Committee bluntly said: "In the public interest there is need for many reforms and for increased control." He had to confess that the effect left on his own mind after studying the Report was one of bewilderment and to some extent of humiliation. The Report undoubtedly reflected both on the Institute and on the profession of the actuary, and the question at once arose as to what the Institute could do to ensure the observance of a high standard of professional conduct amongst its members—what it could do to support its members in obtaining a high standard of valuation, and eliminating the abuses of valuation—and, in the last resort, what disciplinary powers it could exercise in vindication of its good name. Those were hard questions, and questions which no individual member of the Institute could fully answer, but they were questions which he believed it was the duty of the Council to consider, and to consider with a full sense of its responsibility both to the public and to its members. He ventured to think that in the discharge of that duty fearlessly the Institute would best justify its existence in the eyes of the public.

With regard to the position of the actuary in the field of industrial assurance, it would seem that he did not command generally the same authority with his directors or the management as did the actuary in the case of ordinary companies. The Institute must do what it could to strengthen the official position of the industrial actuary, and he hoped that the Council would at once make it perfectly clear that any industrial actuary would at once receive the full support—the active and united support—of the Institute in any firm effort that he made to resist any improper practices or any pressure that might be put upon him. The Institute was established, as its Charter stated, for the purpose of elevating the attainments and status of all engaged in occupations connected with the pursuits of an actuary, and, moreover, the Institute possessed full disciplinary powers under its by-laws for dealing with the case of any member who might be charged with conduct discreditable to the Institute. If, as the Departmental Committee recommended, legislation was now urgently needed to abolish the abuses which had been exposed and condemned, then let the Council of the Institute forthwith appoint a special committee of actuaries to advise upon the whole question, and to guide public opinion as to the lines upon which that legislation should go. It would surely be better for the profession as a whole that the Institute should itself impress upon its members their duty towards the public, rather than that it should suffer dictation from any Government department. The recommendations made by the Committee were all of very far-reaching importance and deserved most careful attention, particularly those recommendations which dealt with industrial insurance valuations. They were set forth in full detail in Section 47. They

were of the greatest importance, being in fact, revolutionary in their character, and he thought it was incumbent on the Institute to discuss them in detail later on.

He ventured to hope that under the lead of the President and of his successor, Sir Alfred Watson, the Institute of Actuaries would take action in its corporate capacity, so as to protect the interests of the public and to ensure the very highest standard of professional conduct amongst its own members.

MR. C. W. KENCHINGTON thought that if it had not been for the report of the Departmental Committee, the present discussion on the subject of actuarial aspects of industrial assurances might not have taken place on that particular occasion. It would, perhaps, have been well if in times past the Institute had discussed the actuarial aspects of industrial insurance a little more than had been done, because on looking through the pages of the *Journal* he could not help being struck by the comparative paucity of papers on that subject, although, of course, it must be borne in mind that the subjects in which the actuaries of industrial insurance offices were concerned did not appeal to the majority of the members of the Institute. It might be recalled, however, that if it had not been for the cessation of the meetings, there would have been a paper on industrial assurance early in 1916.

The Report of the Committee dealt principally with matters of administration, and it was a little difficult to steer entirely clear of such matters when talking about the actuarial aspects of the business; so much was that in evidence, that Sir Joseph Burn that evening had gone at rather greater length than some members perhaps anticipated into the administrative aspect of the subject. He, himself, particularly desired to confine his observations to the question of valuation. He did not intend to say anything about surrender values and free policies, which were matters essentially of actuarial interest. In the first place he wished to draw attention to the very meagre extent of the evidence put before the Committee of Enquiry on the subject of valuations. There were no questions on the subject addressed to the offices in the long questionnaire that was sent out. With regard to the oral examinations of witnesses, there were altogether nine actuaries called as witnesses, one of whom was not an industrial office actuary at all, one appeared purely as the secretary of his own company, and of the remainder only three were asked any questions whatever on the subject of valuation. One of them satisfied Sir Alfred Watson, apparently, as to the stringency of his valuation, and another was merely asked whether he was satisfied that his valuation was all that it ought to be, as a method of drawing attention to another point upon which Sir Alfred was at that particular time examining him. It was only in the case of one office that the valuation basis was really seriously questioned, and although it was not for him to defend that particular office he would point out that it was one which had comparatively recently acquired other businesses. In view of the fact that so little direct evidence was before the Committee it seemed rather curious that

so revolutionary a set of proposals should have been put forward. In saying that those proposals must be fought he did not wish it to be considered for one moment that he desired to bolster up unsound or weak valuations; in fact, his record would be seen to be against that when he pointed out that for two years successively he himself brought out deficiencies in industrial assurance valuations and publicly showed them in the published accounts of the society he happened to represent. The chief objection to the recommendations was that they went entirely against the acknowledged principle of freedom and publicity. If it was agreed that that policy of freedom and publicity had worked to the advantage of ordinary offices, so, he ventured to assert, it had worked to the advantage of the industrial offices, and by that he meant to those assured in the industrial offices throughout the country. The position was perhaps not quite so plain with regard to collecting friendly societies, because their reports were not so readily available as those of the companies. How was that principle of freedom and publicity operating to the advantage of industrial offices? Even the largest office of all was far stronger to-day than it was in the time of the Assurance Companies Act of 1870, and he ventured to assert that hardly a year had passed but what some strengthening had taken place in connection with that particular office. In the smaller offices progress had been even more marked. Some had indeed been rescued from destruction. The war period had brought claims for losses which had weighed a very great deal more heavily upon the industrial offices than upon ordinary offices, and inevitably the effect had been to retard progress. It was not possible to say that all was well. There were still some few offices whose reserves were not as strong as their directors, let alone their actuaries, would desire, but progress had been made and with freedom progress would continue to be made.

With regard to the recommendations as to standardization of valuations, the Report did not in so many words recommend a standard basis of valuation applicable to all companies and societies alike, but it left the matter in doubt by describing the method to be adopted as "the basis to be prescribed by the controlling authority." He wished that Sir Alfred Watson had been present that evening to offer some enlightenment on the subject, because it was a matter about which actuaries of industrial offices felt considerable concern. Some of the dangers of a standard valuation had been already referred to, but it might perhaps be added that there was the danger of the possible elimination of the actuary, as it might be thought that the work of valuation, when once standardized, was one purely of machinery. That would be an entire misapprehension of the actuary's duties and responsibilities. The matter was one of importance to the Institute as a body, and not only to the actuaries of industrial offices. It was as it were the thin end of the wedge, and they could not say what might happen in the future if Government control were brought about in that way.

There might be some doubts as to whether the publicity given to the returns of the industrial offices was as sufficient as it was



thought to be in the case of ordinary offices. But surely it would be possible to obtain a remedy, if it was thought that the particulars were insufficient, by modifying the questions that were incorporated in the Schedules to the Assurance Companies Act, 1909. His last point was as to what could be said to be a desirable basis for the valuation of the liabilities of industrial offices. Sir Joseph had referred to his great good fortune in not having had to make a valuation by any except the orthodox net premium method. Some actuaries held that that method, particularly when based upon a 3 per-cent rate of interest, resulted in unduly high reserves. That, perhaps, was a matter of professional opinion, but certainly one of the effects of the great strength of Sir Joseph Burn's office had been that some other offices had felt bound by competition to raise their benefits, when it would have been wise had they not done so for a period of years. It appeared to him that the method required was one which would make some reasonable allowance for the expenses of new business. He had been connected with an industrial company as successor to the late Mr. H. W. Manly, and in one of his last business interviews, before his death, Mr. Manly pointed out that, although the method of valuation he had been adopting was one which excluded the first five years of duration, so that no reserves began to be built up until five years had expired, he thought that too long a period, and was working for a three years' exclusion. Three years' exclusion on the basis of a 3 per-cent rate of interest was probably very close to one-and-a-half or two years' exclusion on the basis of a rate of interest more nearly approaching the experienced rate, and there were certainly difficulties to contend with if the period of exclusion were made too long, because the constancy of the surplus or the capability of being able to make up the reserves depended to no little extent upon the flow of new business which might, in certain circumstances, as during the war, be upset by conditions entirely outside the control of the office or of the actuary. He had intended, in closing his remarks, to appeal to the Institute to stand by the actuaries of the industrial offices against what many of them felt to be the unmerited slur cast upon the business in which they were engaged by the Report, but he felt that that appeal had been made much better by Mr. Besant, and he wanted to thank him for having spoken in the way he had done.

MR. S. G. WARNER did not think it was possible to avoid the conclusion that the Report of the Committee was in the nature of an indictment and that industrial insurance was more or less on its defence. Further, that indictment—whether just or unjust—represented the crystallization of a great amount of popular opinion on the subject. In the course of a good deal of communication with social workers among the poorer classes of the community, he had not heard many complimentary references to industrial agents or collectors or to the beneficial effect of the system on the population. Those people might be entirely misinformed, but there was a popular impression to the prejudice of the whole system. In those circumstances, he thought those who were connected with industrial

insurance would be well advised not to adopt altogether too critical or too *non possumus* an attitude to the recommendations of the Committee. In the first place, there was no reason to suppose that the members of the Committee entered upon their task with a prejudice against the system of industrial insurance. They heard a great mass of evidence and gave very careful consideration to it and produced a Report which, while it certainly pointed out some things which they believed to be defects, did not make any attack on the general principle. Therefore it would be bad policy, if nothing else, to take the attitude of too violent a depreciatory criticism of what the Committee had to say. An attacked institution which felt it had a perfect reply would be comparatively calm in the circumstances and prepared to admit much while at the same time it had a good defence against much else. On the other hand, he felt strongly that those who had no actual experience of industrial insurance should also take a fair and sympathetic attitude towards what was undoubtedly a great institution, involving large sums of money, and which certainly had had to a large extent a beneficent influence on the community; and they should adopt a friendly attitude and one of comradeship and sympathy towards those of their colleagues who were engaged as actuaries in the difficult work of endeavouring to guide the administration of that great branch of insurance business.

Something had been said that evening in praise of the well-known formula "Freedom and publicity." That seemed to be recognized as the principal article of the Institute's faith. Freedom and publicity since the Act of 1870 were assumed to have produced such good results that they must continue to stand by them. There was a good deal of truth in this, but not all the truth. They must not make a fetish of the phrase. The beneficent effects on ordinary life assurance were pointed out, and it was certainly true that there were such, but there was a great difference between the *clientele* of ordinary life offices and that of industrial offices. The industrial offices had an immensely larger constituency and a very imperfectly educated one. People who knew very little about the matter were not always the best judges of their own interests or as to how those interests were being conserved. Freedom and publicity had on the whole acted well as regards ordinary life assurance, because the average ordinary policyholder was a shrewd business man appreciating caution and the necessity for making good reserves. That was not necessarily the case in a much larger and less educated community, and therefore, it did not by any means follow that certain of the recommendations of the Committee which seemed to some extent to interfere with the formula of "freedom and publicity," should not be adopted because they did not apply equally to the field of ordinary life assurance. Anyone who had knowledge of the history of how the system of industrial insurance had grown up, or had made any attempt to introduce thrift of that particular kind amongst the working classes, knew how very difficult it was to instil sound financial ideas into their minds. As Mr. Moffrey's



book on the Oddfellows' Society showed, there was a mass of misunderstanding that reformers and pioneers in this country had had to remove. The great men who fought for correct principles of valuation and solvency, in the midst of the most extraordinary ignorance and opposition from their fellows, deserved great credit. In an ideal world he thought that industrial insurance would be better managed mutually than by a joint stock company seeking profit. There were fewer dangers in the path.

Industrial insurance was peculiar in the fact that it depended on a great army of agents, not agents as the term was ordinarily understood by the actuary in the insurance world, but whole-time agents. Some of the evidence given before Lord Phillimore's Committee on the question whether the industrial insurance companies could afford to give their outside representatives a war bonus was suggestive as regards the difficulties besetting the administration of the business. A company had introduced a new table which gave specially good benefits to the public, but which for that reason did not admit of quite so high a commission to its agents, and it tried to get them to take up the table by pointing out to them that, although the commission on each individual case was less, the scheme was so attractive and so sound that probably they would be able, by doing a larger volume of business, to earn a larger total commission. It was curious to find the attitude taken up by the agents with perfect honesty and straightforwardness, that they should not be expected to take up a scheme which paid a reduced commission, no matter how attractive were its terms to the insurer. That was an illustration of some of the difficulties with which those who were administrators of industrial insurance had to contend, and which made their task more complicated.

He most thoroughly and cordially agreed with Sir Joseph Burn with regard to the Courts Emergency Powers Act. The references to that Act in the Report were distinctly inadequate. It seemed to him to have been a piece of panic legislation, passed in the most hasty and ill-considered way, and based on two main assumptions — one that the war would be short and the other that there would be a large amount of unemployment and consequent distress. Both were falsified by events. The Act placed the companies in a most unfair and disadvantageous position. As a member of Lord Phillimore's Committee, he did his best to make that clear, but was bound to say it was most difficult to convince people who had not had the elements of actuarial training that there could be any loss with regard to unpaid insurance premiums unless there was an actual claim by death. The idea that even if the assured did not die the company had been running a risk for them which had a money value seemed to quite intelligent people a thing impossible of comprehension. He cordially agreed that the provisions of the Act with regard to industrial assurances were unjust. He did not know whether the Institute had had the subject before it in any public way before, but it should be put on record that that was the members' opinion, because it was most injuriously affecting one

great branch of life assurance business, and was in defiance of the most elementary principles of actuarial science.

MR. J. MURRAY LAING said that, although he was the actuary of a large industrial assurance company, he was in the somewhat unique position of having been longer in the service of ordinary life offices than he had yet been in industrial work. He had not been long enough away from the ordinary offices to forget the views he used to hold on industrial assurance, but quite long enough to learn that many of those views were wrong. Mr. Besant had spoken strongly on the responsibilities of the actuaries employed in industrial offices. So far as those known to him were concerned, he had no hesitation in saying that they realized the great responsibilities which rested upon them, and had done all they possibly could to improve the position of their offices when such improvement was necessary. He was not aware of any actuary who required the Institute, or any other body, to say to his directors what he ought to say himself. He, personally, had never had any hesitation in saying what he thought, and he had never had any opposition from those in charge of his company to any suggestion he had made. But it was suggested that was not the common experience. In the Report the remarks of a derogatory nature frequently referred only to the smaller companies. There were large numbers of very small societies, and he did not think they always employed actuaries. If evils did exist to any marked extent it was the duty of every Fellow of the Institute to do what he could to improve matters.

With regard to the subjects on which he used to hold wrong views, one was the bogey of the expense rate. Everyone who was not in an industrial office considered that the expense incurred in the business was far too high, but with his knowledge of both sides he certainly did not agree with that. That, however, did not mean that he did not think the expense should be reduced, as wherever it was possible that should be done. But when regard was had to the services rendered respectively by the industrial agents and the ordinary agents, he did not think the comparison was against the industrial companies. The Report said that the premiums collected amounted to £25,000,000 per annum. The average premium being under  $2\frac{1}{2}d.$ , that would represent 2,500 million calls per annum. And, expenses being 40 per-cent, the cost of running the business would work out at  $1d.$  per transaction. Those figures, however, required some modification. Agents very often collected on more than one policy in a house; they did not always collect singly each premium, nor did they always collect every week, although the bulk of them did so. If the  $1d.$  were doubled and the average cost per call put at  $2d.$  it would not affect his argument, as he was going to make a comparison in which he had plenty of margin. On reference to the Report on the working of the Post Office Savings Bank, he found that the cost of each transaction was  $4\frac{1}{2}d.$  in and  $4\frac{1}{2}d.$  out, so that to pay in any sum of money and draw it out cost  $9d.$  The Post Office Savings Bank was not doing for the public all that the agent of an industrial company did for a

policyholder. The agent called at the home and collected the premium, and when a death occurred he called and paid the claim. In the Post Office Savings Bank the depositor had to go to the Post Office to pay in his money or to take it out. Therefore, even a comparison with the Savings Bank was not quite fair to the industrial offices. In any comparison of expenses with premiums there were included the expenses incurred in the payment of claims, so that he had to put on to the Post Office item of  $4\frac{1}{2}d.$  per deposit something for the withdrawals. He found that the withdrawals were one-half of the deposits in number, so that adding one-half to the  $4\frac{1}{2}d.$  there was something like  $7d.$  to be taken into account in comparison with the industrial offices'  $2d.$ , which was surely a tribute to the industrial offices. Some years ago the Fabian Society started a very thorough investigation into the business of industrial insurance, and they were under the impression that they would be able to prove it was very extravagantly managed. Their report contained a statement that it cost  $1d.$  per transaction. They compared that with the cost in an ordinary office, which they found came to  $1s.$ , and they suggested it would not be a bad idea if the ordinary offices found out how it was that the industrial could work so cheaply. A point often overlooked was that industrial assurance was necessarily younger than ordinary life assurance. The big business of to-day had been of very rapid growth, so that a considerable proportion of the expenditure had arisen under the head of new business. It was common to both kinds of life assurance companies that business was more costly in its first year than subsequently. He was well aware that there was a limit beyond which a life office, either industrial or ordinary, was not entitled to go in its transaction of new business, but within limits it was the case that the cost was greater, and that should be borne in mind in comparing costs.

Another matter on which he had held wrong views was with regard to solvency. When he first went into an industrial insurance company he had the old idea that, unless an office was able to make a net premium valuation on the basis of rates of interest and mortality well within those being experienced, it ought to be closed down. He knew of a very large office which was in a strong position to-day, but which 25 years ago was in an almost helplessly weak condition. Had he then been of the age he was now, and without the experience he had had since entering the industrial business, he would certainly have advised closing the office down, thus doing incalculable harm to hundreds of thousands of poor people. Fortunately an eminent actuary on that occasion was called in, with the result that by careful management and the process of gradually building up strength, the company had been saved. Now obviously it was only possible, at the outset in that case, to produce an appearance of solvency by employing methods in the valuation which one would not employ if valuing for the purpose of distributing a surplus. There were two distinct points always to consider. Were they valuing as a test of solvency, or with a view to the distribution of

surplus? He thought that to suggest taking one standard valuation would be altogether wrong. The chief point was whither the office was tending—was it going towards higher strength, or going downhill, or was it remaining stationary? The vitality of industrial insurance was remarkable, and if the position showed that an office was consistently improving from year to year there should be no thought of closing it down even if, at the moment, its position was rather weak. He said that now, because he would have been prepared to take the other view had he been asked for an opinion some years ago.

MR. C. S. KELHAM wished to draw attention to the schedules contained in the Appendices to the Report. The Committee stated that they were unable to publish the whole of the written evidence, although they had published very complete minutes of the oral examination of witnesses. The information supplied had been condensed in the schedules. The period taken for the statistical information covered ten years. It was unfortunate that a large part of that period was quite abnormal. The schedules would require therefore, to be read with great care, although perhaps it was hardly necessary to point that out in the Institute. As an instance, in Appendix B, where the expenses of the industrial offices and collecting friendly societies were shown, the expenses included dividends and in the case of one company that had largely reduced its dividends for the period in question this had the effect of allocating the dividends earned in one year to the premium income of the following year, and consequently the ratio had been overstated.

MR. H. H. AUSTIN said that the Report should be read and judged in connection with the evidence given. The tone of the Report was certainly very much in the nature of an indictment, but he was inclined to think that a wrong impression might easily be obtained by reading the Report alone. He might mention, as an example, the question of lapses. The information given to the Committee was not published in the form in which it was given; detailed information was published as to the lapses for one year, 1913, and the rest of the information was lumped together in an aggregate table covering ten years. That gave no weight to and completely masked the very great change that had taken place of recent years in the matter of lapses. Then, again, in Section 24 the subject of surrender values and free paid-up policies was introduced with the statement: "Industrial policies generally carry no right to surrender value or a free paid-up policy upon lapse", and the Report then went on to mention that the cases of the Prudential and the Aberdeen and Northern Friendly Society were exceptions from which it might be inferred that these were the only exceptions. In the case of his own company, which was by far the largest transacting industrial insurance business after the Prudential, for many years free policies had been granted after five years' duration. He considered the statement, therefore, misleading and inaccurate. On the subject of bonuses and distributions to the public, the Report stated on page 5 that "in the case of the other companies



no attempt had been made to give policyholders a share in the profits, although in one case . . . it was claimed that certain additions made to the policies . . . under a general revision of the tables were in the nature of a bonus distribution." In that case an increase of benefits was given to the whole of the policies in force at that date, it was paid out of accumulated profits, and he thought it must be admitted it was a bonus in every sense of the word. If an indictment was made by such a Committee, at any rate it should be studiously fair and just.

The complaints dealt with in the Report were of two kinds: complaints on the part of individual policyholders and complaints on the part of the public generally in regard to matters of valuation, reserves, and probable returns. When it was considered that the bulk of the population was insured in these offices and a great many of them were ill-educated people who were quite unfamiliar with the principles of insurance, a certain number of complaints from individuals were obviously to be expected, and he thought the evidence was rather astonishing in that it showed so few. The Institute was more concerned with complaints of the second kind. The companies—not only the actuaries but the companies as a whole—were far from taking up an attitude simply of criticism of, or resistance to, the suggestions that were made. One of the most important recommendations of the Committee was on the subject of the standard valuation. He thought that the arguments against a standard valuation for ordinary companies applied to industrial assurance with even greater force. There would be a strong tendency for the minimum to become the maximum. Industrial offices were already much criticized by members of the field staff for accumulating too much in the way of reserves and frequently an actuarial defence of such reserves was treated by them with scant respect. It was beyond their comprehension, and they were convinced that the reserves were not necessary. The institution of what was in effect a Government standard would undoubtedly very strongly reinforce that criticism, and he thought in many cases it would be almost impossible for companies to keep a reserve stronger than the standard laid down. A standard valuation would also appear to lack sufficient elasticity to meet the varying circumstances of different offices, although he was not clear as to what was meant by the suggested minimum basis. It might be a basis having some relation to the age and constitution of the business in question—rather an important point in industrial business—or to the earned interest rate or to the experienced mortality rates and so on, but he thought that such a basis would simply raise fresh difficulties without removing the main objections of principle. For example, a company recently established and whose solvency was in question would have a small proportion of invested assets, and that would probably, by any formula prescribed for the determination of the rate of interest, bring out a low rate of interest and entail a high reserve, while statistics of its mortality experience would be probably non-existent.

It had been mentioned that freedom from official trammels and

publicity had been guiding principles in the past which had worked very satisfactorily, but he did not think these principles could be assumed without question to be sufficient for all cases. Small collecting friendly societies were in a class by themselves. There was undoubtedly great danger, if the Report were not read with care, that criticisms which applied to such societies would be supposed to apply to industrial insurance institutions which did not come in the same category. In the one case a deposit of £20,000 was necessary, and in the other it was not, and where it was not necessary many abuses arose, and he thought that to some extent the whole business was prejudiced by that. It would not be necessary to take into account the possible formation of new concerns, provided steps were taken, as suggested in the Report, to bring Friendly Societies into line by a deposit. The business of industrial insurance was so difficult and so likely to be unremunerative, that the formation of new companies was improbable. Any further development would be probably on the part of existing insurance institutions, and would therefore have a good backing, but even if limited to existing institutions a standard valuation as a test for solvency seemed a very clumsy and inefficient instrument. The recommendation did not suggest that steps should be taken with a view to winding-up simply by reason of a deficiency being shown on the prescribed valuation basis, but that the controlling authority in such a case should be empowered to apply to the Court for a winding-up order if, after investigation, it was satisfied that the existence of the company should be terminated. That power, he thought, could be just as well exercised without any reference to a prescribed standard valuation. It was, indeed, admitted in the Report, by implication, that the effect of a valuation falling short of a standard was not necessarily sufficient ground for taking action. A very brief glance at the history of industrial insurance appeared to show that the well-established concerns gained firm foothold and financial stability in times when the benefits were less in relation to the premiums than they were at the present day, although they had other difficulties to contend with, and a good deal of money was put into them which was lost or was unremunerative. As those concerns progressed they gave, under stress of competition and to attract patronage, extended benefits and privileges, and at the present day premiums were very much more closely adjusted to benefits than they had been, and in fact were very closely adjusted.

With regard to shareholders' dividends, payments to directors and grants of free policies and surrender values, he thought that in the case of many companies a very small deduction from the premiums, probably 2 per-cent or 3 per-cent, would more than extinguish any dividends to shareholders or payments to directors, and probably over the whole business 5 per-cent on the premiums would more than cover the whole of the dividends paid. Smaller companies now, for the purposes of competition, found it necessary in most cases to give benefits similar, in the matter of sums assured, to those of the better-established institutions, although there might



be a set-off in some cases, in that they did not grant the same benefits in regard to cash surrender and free policies.

It was important to bear in mind that the present was a period of great change; he did not suppose, in the whole history of industrial assurance, there had been a time when so many forces making for rapid change had been operating. There was very diverse pressure put upon the companies both to reduce and to increase expenditure. From the Press, the public, Parliament and various quarters, there was a general statement that the expenses were too high and that they must be brought down, and he was not sure that that criticism had not been invited by admissions on their part that expenses were too high and ought to be reduced. Naturally it was to the interest of all companies to reduce expenses, and nothing should be left undone to that end, but for the service that was rendered he did not think it could be said the expenditure at the present time was high. If the views of agents and people who served the public were taken, they would say that industrial insurance was the cheapest thing the working man had. In the case of particular companies there was a rapid concentration of debits into fewer hands and great extension of monthly business. Generally there was a great rise in the rate of interest obtainable on investments, and from his own observation there was at the present time a considerable variation of a favourable character from the normal standards of mortality. Further, as Sir Joseph Burn had brought out, the average premium per policy had increased very rapidly. All the considerations mentioned needed to be brought into calculation when there was any question as to the necessity of winding up a company. A long time ago he had occasion to look into the affairs of a company and he found that over a period of two or three years, having started with very weak reserves, their real assets had not increased at all. In the absence of some very favourable circumstances that would seem to be a case calling for serious consideration as to whether in the public interest the company should be allowed to continue. Incidentally he might say that that company has since been taken vigorously in hand and was in a very different position to-day. To wind up a company which, if left alone, would develop strength and stability, would cause a shock to the whole business and would be of no advantage to the public. The point that was made in the Report, that surrender and free policy values should be made obligatory after a period of seven years, needed careful consideration. He did not think there would be any real opposition to such a suggestion, but if reference were made to a paper read at the Institute by Messrs. Ackland and Bacon some time ago it would seem that the inclusion of those items in the valuation made a very appreciable difference to the reserve which it was necessary to keep on the bases they were considering. He would not rule out the possibility that a company might need to take into account the lapse element and to keep negative values in its reserves. As a counsel of perfection he supposed no actuary would advocate that course, but it might be the lesser of two evils as compared with winding up. The position of an actuary in these matters was very often one of great com-

plexity, he might lay down what was a strong standard of valuation and bring out a heavy deficiency, or he might recommend something which he thought would be, and which probably was, sufficient in the long run to safeguard all interests. The first course, a deficiency in the valuation, would probably mean an increase in such an item as establishment expenses in the balance sheet. It might be that it was better to bring such an item plainly into view in the balance sheet rather than have it hidden in the valuation. As against that the actuary who had an admittedly weak valuation would get some chance of bringing pressure to bear to strengthen the valuations, whereas it was quite conceivable that directors might become case-hardened to such an item as extension expenses in the general balance sheet.

The PRESIDENT said that, as he had taken occasion on a comparatively recent date to warn the representatives of industrial offices that they were liable to attack and that the central point of attack would be their expense ratio, he wished now to say, on further consideration of the subject, and in the light of information since given to him to the effect that the industrial offices were themselves now, and at the time when he spoke, taking steps to reduce their expenditure, that he felt that industrial assurance, in the form in which it existed, could not be conducted except at a very high ratio of expense, and the question for consideration was whether the benefits which it offered to the public were commensurate with the unavoidable expense. He believed that they were. When he heard from Mr. Laing the figures which he had given, comparing expenses with the volume of transactions, he was surprised that it was possible to conduct so vast a business at so low a charge per transaction. If the facts were better appreciated amongst the public and amongst those who criticized industrial assurance companies, it would be very much to the advantage of the companies; and the discussion, in bringing out that fact, had done very much to assist the companies in their difficult task. Sir Joseph Burn had asked whether it was not fair to use some portion of the shareholders' money in building up an organization. To the question put in that form he thought the general answer must certainly be yes, but that did not exhaust it, and the reply must be qualified by saying that the question wore a different aspect and was capable of a different answer if that part of the shareholders' money which had been so used, and in addition other intangible assets such as negative values or an allowance for the lapse element, were brought into account as divisible surplus, when that surplus was to be distributed amongst policyholders, and still more when it was to be given to the shareholders as dividends. When it came to that point the matter required very careful consideration, and must be faced with all the courage of which the responsible actuary was capable. He was not well enough informed as to the position of actuaries in industrial offices generally to say whether the reflections which were implied in the Report were justified or not, but he should like to endorse what Mr. Besant had said, that in any difficulties which might face actuaries in the conduct of their

business, if they were satisfied that within the limits of their responsibilities they were doing all that they could, and all that they properly ought to do, in resisting any undue pressure which might be brought upon them, they would be assured of the support of the Institute in any action they might take. He had been very much surprised to read in the Report the paragraph relating to the Courts (Emergency Powers) Act. All the members knew what an infinity of mischief that Act had done to industrial offices, and that there should be any attempt to carry on that mischief, even temporarily, and to remedy it in the manner which was suggested in the Report was quite unreasonable. He wished to thank Sir Joseph Burn for opening the discussion, which had proved an interesting one, and one which should be valuable, not only to the members, but to the public generally.

### *Stirling's approximate formula for $n!$*

[In amplification of the Editorial Note on p. 106 of this volume we have pleasure in publishing the following further contribution from Mr. Lidstone.—Ebs. J. I. A.]

THE point of Cesàro's proof and of the extension given by the writer lies in expressing the upper and lower limits of  $\Delta \log \psi(n)$  in the form  $U_n - U_{n+1}$ . Since the note was closed for press, it has been found that by the same methods we may replace the inferior limit found for  $n!$  by another which is both nearer the true value and in closer agreement with that found by higher methods. We have

$$\begin{aligned} \frac{1}{n^2 + n + 1} &= \frac{1}{n(n+1)} \left[ 1 - \frac{1}{n^2 + n + 1} \right] \\ &> \frac{1}{n(n+1)} \left[ 1 - \frac{1}{n^2 + n - 2} = 1 - \frac{1}{(n-1)(n+2)} \right] * \\ &> \frac{1}{n(n+1)} - \frac{1}{(n-1)n(n+1)(n+2)} \\ &> \frac{1}{n} - \frac{1}{n+1} - \left[ \frac{1/30}{(n-1)n(n+1)} - \frac{1/30}{n(n+1)(n+2)} \right] \\ \frac{1/12}{n^2 + n + 1} &> \frac{1}{12n} - t(n) - \left[ \frac{1/12}{n+1} - t(n+1) \right] \end{aligned}$$

$$\text{where } t(n) \equiv \frac{1}{360} \cdot \frac{1}{n^3 - n} \text{ and } t(n+1) \equiv \frac{1}{360} \cdot \frac{1}{(n+1)^3 - (n+1)}$$

\* Provided  $n > 1$ , a condition which does not effectively limit the generality of the results.

Hence, using this value of  $t(n)$  and proceeding otherwise as before, we shall find ( $n > 1$ )

$$\begin{aligned} n! &< C n^{n+\frac{1}{2}} e^{-n} \cdot e^{1.72n} \\ &> C n^{n+\frac{1}{2}} e^{-n} \cdot e^{1.72n-1.360(n^2-n)} \end{aligned}$$

the second limit being much closer than that given on pp. 104 and 105.

## LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law.*

Deferred  
assurance policy.  
*Jus quaesitum  
tertii.* THE case of *Carmichael's Executrix v. Carmichael*, 57 S.L.R. 547, which was an appeal from the Court of Session (Scotland) to the House of Lords, is of interest. H. F. Carmichael took out on the life of his son a policy of deferred assurance which provided that up to the son's majority he, the father, might surrender, or in the case of the son's death before that event receive repayment of the premiums paid. The son, if he attained his majority, could maintain the policy by continuing payment of the premium, in which case the sum assured was payable to his executors or assignees on his death, or he might exercise certain options. The father retained the policy in his own custody. The son attained his majority and died before a further premium was due, and a competition resulted between the son's executrix and the father as to the assured fund. The House of Lords, reversing the judgment of the Court of Session, held that the executrix was entitled to the fund inasmuch as the son at his death had a *jus quaesitum* in the policy. The material portions of the policy are as follows: "Whereas Hugh Fletcher Carmichael, consulting engineer, Hong Kong, herein called the grantee, has proposed to effect an assurance with the English and Scottish Law Life Assurance Association for the sum of One Thousand pounds upon the life of his son Ian Neil Carmichael (stated to have been born on the twenty-ninth day of October 1894), hereinafter called the life assured, on the terms hereinafter stated, and has delivered at the offices of the said association a proposal and declaration . . . . And whereas the grantee

“ has paid to the association the sum of Nine pounds ten  
“ shillings premium for the year terminating on the twenty-  
“ first day of October 1904 inclusive, and has agreed to pay  
“ to the association the like premium on the twenty-second  
“ day of October 1904 and on every subsequent twenty-  
“ second day of October up to and inclusive of the twenty-  
“ second day of October 1914. Now this policy witnesseth  
“ that if the life assured shall die before the twenty-ninth day  
“ of October in the year One thousand nine hundred and fifteen,  
“ then the funds and other property of the association shall  
“ be subject and liable, according to the provisions of the deed  
“ of settlement of the association and of the resolutions  
“ endorsed thereon, to repay at one of the principal offices of  
“ the association in Edinburgh, or London, or at the office in  
“ Dublin, to the grantee, his executors, administrators, or  
“ assignees, immediately after the death of the life assured  
“ (prior to the said twenty-ninth day of October 1915) and his  
“ age and the title to this policy shall have been proved to the  
“ satisfaction of the directors of the said association, all the  
“ said premiums so paid by the grantee or his foresaids, but  
“ without any interest thereon, and the receipt of the grantee  
“ or his foresaids shall also be a good discharge to the  
“ association for any surrender value allowed under this  
“ policy prior to the said twenty-ninth day of October 1915.  
“ But should the life assured live until the said twenty-ninth  
“ day of October 1915 and so attain twenty-one years of age,  
“ and he or his assignees shall continue thereafter to pay to  
“ the said association the premium of Nine pounds and ten  
“ shillings on the said twenty-second day of October in each  
“ year during the continuance of this assurance, then the  
“ funds and other property of the association shall be subject  
“ and liable as aforesaid to pay at one of its said offices to the  
“ executors, administrators, or assignees of the life assured  
“ immediately after the death of the life assured (on or  
“ subsequent to the said twenty-ninth day of October 1915),  
“ and his age and the title of this policy shall have been  
“ proved as aforesaid, the said sum of One thousand pounds  
“ of lawful money of Great Britain.”

Lord Dunedin, in the course of a very learned judgment, said: “The question therefore comes to be—Had Ian  
“ Carmichael a *jus quaesitum tertio* under the policy which  
“ passed the proceeds thereof to his executrix? I think it



“ very necessary to begin by pointing out that the expression  
“ *jus quaesitum tertio* is in different cases and different  
“ circumstances used in a varying sense, or perhaps I might  
“ better say, is looked at from a different point of view. The  
“ one sense is meant when the question being considered is  
“ simply whether the *tertius* C has the right to sue A in  
“ respect of a contract made between A and B to which  
“ contract C is no party. The controversy then arises between  
“ C, who wishes to sue, and A who denies his title to do so.  
“ It is here that there is a sharp technical diversity between  
“ the laws of England and Scotland. In England no matter  
“ how much the contract contained provisions for behoof of  
“ C, C could never sue at law. In equity he could sue, but  
“ he could only sue if by the terms of the contract he could  
“ successfully maintain that A was constituted a trustee in his  
“ favour. In Scotland if the provision is expressed in favour  
“ of C he can sue, and this is often designated by saying ‘ He  
“ has a *jus quaesitum tertio*.’ Probably the reason of the  
“ difference indicated lies in the simple fact that in Scotland,  
“ law and equity were never separate. Another familiar  
“ illustration of the same class of difference will be found in  
“ the right of an assignee in Scotland to sue in his own name  
“ —a right which at common law in England, apart from  
“ statute, he did not possess. But, as already stated, in all  
“ this class of cases the controversy is between A and C. B  
“ is either no longer existent, or is so far as he is concerned  
“ quite willing that C should exact his rights . . . . The other  
“ sense of the expression is when the emphasis is, so to speak,  
“ on the *quaesitum*, and when the controversy arises not  
“ between C and A, but between C and B. In such a case A  
“ is willing to perform his contract, and the contract in form  
“ provides that A shall do something for C, but B or those  
“ who represent B’s estate interfere and say that B and not C  
“ is the true creditor in the stipulation. . . . . It is needless  
“ to say that the present case is one of the latter category. . .  
“ Using the letters as above A is here willing to pay. The  
“ controversy is as to whether B or C is entitled to receive. . .  
“ . . . . We are entitled, I think, to look to the nature of the  
“ insurance effected, the incidents which are connected with  
“ it, and the objects which, according to the prospectus  
“ attached to the form which Mr. Carmichael signed, are  
“ sought to be obtained. So doing I find a contract which



“ makes a marked distinction between the period up to the  
 “ majority of the life assured and the period thereafter. Up  
 “ to the majority it is the taker of the policy—the grantee as  
 “ he is called—who engages to pay the premiums, who can  
 “ stop paying if he likes and transact as to a surrender value,  
 “ and who, if the life fails to attain majority is entitled to a  
 “ return of the money paid but without interest. After  
 “ majority the whole scheme alters, the grantee no longer  
 “ engages to pay the premiums, but the life assured is given  
 “ several options. He may elect to continue to pay the  
 “ premiums and keep the policy as an ordinary policy, or he  
 “ may convert it into a paid up policy, or into an endowment  
 “ policy, or he may receive a cash payment. These options  
 “ given to the life assured but not to the grantee are strangely  
 “ inconsistent with the idea of there being no vested right in  
 “ the life assured, and consistent with the idea of promoting  
 “ thrift in the person of the life assured. I next turn to the  
 “ fact that during the period in which action was open to the  
 “ grantee no action was taken. Then comes the fact that  
 “ the son undoubtedly knew of the assurance—a knowledge  
 “ which it is legitimate to conclude came through his father,  
 “ though the proof falls short of direct communication. . . .  
 “ Taking all the circumstances together I come to the  
 “ conclusion that we have here the evidence necessary, when  
 “ taken along with the terms of the document, to show that  
 “ an irrevocable *jus quaesitum* was constituted in favour of  
 “ Ian Carmichael, that the proceeds of the policy which, by  
 “ the conception of the contract, fall to be paid to his  
 “ executors, truly belong to them, and that therefore the  
 “ present appeal should be allowed.”

The liability of an assurance society for the  
Assurance policy.  
 Misrepresentation  
 by agent. misrepresentations of its agent was considered by  
 the Irish Courts in the case of *Bryne v. Rudd and  
 others*, Trustees of the City of Dublin Assurance Society,  
 reported in the King's Bench Division (1920), 2 I.R. 12.

In this case the plaintiff was a railway servant, and  
 the defendants were the Trustees of the City of Dublin  
 Assurance Collecting Society, a friendly society registered  
 under the Friendly Societies Act, 1896. The society employed  
 agents who canvassed for persons to join the society as

members for certain premiums, and among other objects granted endowment policies.

Evidence was offered that one of the society's agents visited plaintiff at his residence in December 1917, and represented to him that if he insured and took out an endowment policy in the said society, and paid a first premium of £7. 5s. 10d., he could obtain an immediate loan of £100 without security and without interest. The plaintiff was desirous of getting the loan in order to purchase certain land which was being distributed amongst the poor people of the district at a reduced price by trustees, who had purchased a large estate. The agent was aware of this, and discussed it with the plaintiff. The plaintiff signed a proposal and application for a policy, which declared that the proposal should be the basis of the contract entered into between him and the society, and that the proposer had signed the proposal, having first read the prospectus. The plaintiff had not in fact read the prospectus, nor were its contents communicated to him. The prospectus contained a statement that loans to members would be granted on approved security after one year's membership. The plaintiff borrowed the said sum of £7. 5s. 10d., and paid it to the society, whose receipt was proved.

An endowment policy, dated 1 December 1917, for the sum of £200, was produced in Court. The plaintiff admitted that a green book was shown to him by the agent, but stated that he did not read it, and that its contents were not communicated to him. No evidence was offered that plaintiff knew that the society was a friendly society within the meaning of the Friendly Societies Acts, and as such could only make loans to members of at least one full year's standing. In February 1918, the plaintiff applied to the society for a loan of £100 on his policy, but was informed that no loans were granted until a policy had been a year in force, and that in any event he had not stated what security he intended to offer.

The King's Bench Division (Gibson and Gordon, J.J., sitting as a Divisional Court), held that the complete absence of authority in the agent to make a representation, or to give the undertaking, made it impossible for the plaintiff either to fix the society with fraud or recover back the cash; and that, viewed as an *ultra vires* transaction, the payment could not found any legal remedy.

The Court of Appeal (reversing the King's Bench Division) held that the society could not be allowed to retain money which had been procured for and received by it under a contract which the assured had been induced to enter into by the fraudulent misrepresentation of its agent; that even though the agent's representation involved the doing of an act which was illegal or ultra vires on the part of the principal, he was nevertheless acting within the scope of his authority in making it, in so far at least as to deprive the society of the right to retain any benefit under it.

Sir James Campbell, Chancellor, in his judgment, referred to the decisions of the Court of Appeal in the cases, *British Workman's & General Assurance Co. v. Cunliffe*, 18 T.L.R. 425, and *Harse v. Pearl Life Assurance Co.* (1903), 2 K.B. 92, and said:

"These two cases are clear and distinct authorities for the proposition that a plaintiff is entitled to recover back from the insurers the premiums he has paid under a contract of insurance into which he has been induced to enter by the fraudulent misrepresentation of the agent of the insurers in reference to a subject which, to some extent, at least, as in the case before us, involved a matter of law, namely, the validity of the contract, or the particular obligations which it imposed upon the company."

The Chancellor also referred to the judgments in *Kettlewell v. Refuge Assurance Co.* (1907), 2 K.B. 242, and *J.I.A.*, vol. xli, p. 574, vol. xlii, p. 401, and vol. xliii, p. 351; *Tofts v. Pearl Life Assurance Co.* (1915), 1 K.B. 189, and *J.I.A.*, vol. xlix, p. 93, and *Hughes v. Liverpool Victoria Legal Friendly Society* (1916), 2 K.B. 482, and *J.I.A.*, vol. l, pp. 39 and 120, and continued: "These important judgments established beyond question that an insurance company is not entitled to retain as against the insured the premiums that he has paid under a policy which he has been induced to effect by the fraudulent representation of the agent of the company, even though such representation amounts to a mis-statement of law, or involved his principal in an obligation which he was by law prohibited from performing. Every line in these judgments is at variance with the reasoning and conclusion of the judgments of Gibson and Gordon, J.J., and completely disposes of the suggestion that a principal is entitled to retain the profits of a contract induced by the fraudulent

“representation of his agent, in any case in which the  
 “representation involves the doing of an act which was illegal  
 “or ultra vires on the part of the principal.”

Memorandum of  
 Association.  
 Alteration of  
 objects clause.

Section 9, sub-section 3 (a) of the Companies (Consolidation) Act, 1908, requires that notice must be given to “any persons or class of persons whose interest will, in the opinion of the Court, be affected by the alteration”, before the Court will confirm alterations of the objects clause in the memorandum of association. In the case of *Hearts of Oak & General Assurance Co., Limited, In re*, reported 89 L.J. Ch. 241, it was held, on a petition by a company whose principal object was general and industrial life assurance to confirm certain alterations, that the sub-section only refers to persons or classes of persons who have interests in the petitioning company and not to those who have interests outside the company, which may be affected; and consequently that a rival society carrying on a similar business to that which the company is seeking to carry on cannot claim to be heard on the ground that its interests will be affected if the petition is granted.

Mortgage.  
 Knowledge of new  
 trustee. Notice.

Where a trustee before his appointment acquires knowledge of an incumbrance on the trust estate in such a way that as a reasonable man he would on his appointment act upon and regulate his conduct in the execution of the trust by the information so acquired, and that knowledge continues to operate on his mind in the same way after his appointment, notice given to him after his appointment by a subsequent incumbrancer will not displace the priority of the earlier incumbrance. This was the decision of Lawrence J., in the case of *The Ipswich Permanent Money Club, Ltd., v. Arthy*, reported (1920), 2 Ch. 257.

The facts are briefly as follows: In 1903 William Spendelow, the sole trustee of a trust estate, mortgaged his reversionary interest in the trust estate to the defendant, Arthy, and in 1907 he again mortgaged his reversionary interest to the plaintiffs without disclosing Arthy's mortgage. William Spendelow's reversionary interest was insufficient to pay both mortgages. In July 1908 the plaintiffs became aware of William Spendelow's fraud, and negotiations then ensued and were still pending between them and Harry Spendelow (the brother of William Spendelow) when Harry

Spendelow on 23 September 1908, procured himself to be appointed a new trustee of the trust estate in order to deal with the situation occasioned by the brother's fraud. On 29 September 1908, the plaintiffs again wrote to Harry Spendelow referring in effect to the fraud and pressing for an answer. Harry Spendelow replied that he could do nothing, and, on 28 October 1908, gave to the plaintiffs and to Arthy notice that he had been appointed a new trustee. Thereupon the plaintiffs through their solicitors on 30 October 1908, gave Harry Spendelow formal notice in writing of their mortgage, and subsequently the transferee of Arthy's mortgage gave to him formal notice of his security. After the reversionary interest had fallen into possession the plaintiffs claimed priority on the ground that their notice to Harry Spendelow after his appointment as trustee was first in point of date, and relied on *In re Dallas* (1904), 2 Ch. 385.

In the course of his judgment Lawrence, J., said :

“ The plaintiffs rest their case entirely upon their  
“ solicitors' letter of 30 October, 1908, which they contend  
“ operated to displace the priority of Arthy's mortgage and  
“ to give priority to the plaintiffs' mortgage. The  
“ submissions made to me in support of this contention were,  
“ first, that the rule in *Dearle v. Hall*, 3 Russ. 1, applies to the  
“ plaintiffs' and Arthy's mortgages; secondly, that the  
“ knowledge which William Spendelow had of the mortgages  
“ created by himself was not an effective notice to operate on  
“ priorities within the rule, and consequently that the rule first  
“ became applicable when Harry Spendelow was appointed a  
“ trustee of the testator's will; thirdly, that knowledge  
“ acquired by or notice given to Harry Spendelow before his  
“ appointment as trustee of the testator's will had no effect,  
“ and therefore the dates of the notices given after his  
“ appointment must determine the order of priority; and  
“ lastly, that the first notice of the plaintiffs' mortgage given  
“ to Harry Spendelow after his appointment was the notice  
“ given by the letter of 30 October, 1908, and consequently  
“ that the plaintiffs' mortgage ranks in order of priority before  
“ Arthy's mortgage. In the next place, the Court of Appeal  
“ in *In re Dallas* has decided that notice to, or knowledge of,  
“ a sole assignor trustee is not an effective notice to  
“ operate on priorities. I am bound, therefore to dismiss  
“ from this case all considerations of the knowledge of  
“ William Spendelow and of the express notice given to him



“ by the plaintiffs’ solicitors on 30 May 1907. It follows from  
“ what I have stated that on the appointment of Harry  
“ Spendelow as trustee of the testator’s will there came into  
“ existence for the first time a legal holder of the trust  
“ property to whom notice could be given so as to operate on  
“ the priorities of Arthy’s and plaintiffs’ mortgages. The  
“ question then arises whether the knowledge which Harry  
“ Spendelow had acquired of Arthy’s mortgage prior to his  
“ appointment as trustee to the testator’s will operated to  
“ protect the priority of that mortgage and to prevent the  
“ notice which the plaintiffs gave of their mortgage on  
“ 30 October 1908, from displacing that priority. In my  
“ opinion this question is not free from doubt, and its solution  
“ is not easy, owing to the admitted difficulty of accurately  
“ defining the principle on which the doctrine of notice as  
“ established by the rule in *Dearle v. Hall* depends: See per  
“ Lord Macnaghten in *Ward v. Duncombe* (1893), A.C. 369.  
“ It is well established that the rule does not render it  
“ necessary for an incumbrancer, in order to preserve the  
“ priority of his incumbrance, to prove that he gave notice  
“ of his incumbrance to the trustee of the fund. It is  
“ sufficient if he can prove to quote the words of Lord Cairns  
“ in *Lloyd v. Banks*, L.R. 3 Ch. 488, 490—‘ that the mind of  
“ ‘ the trustee has in some way been brought to an intelligent  
“ ‘ apprehension of the nature of the incumbrance which has  
“ ‘ come upon the property, so that a reasonable man, or an  
“ ‘ ordinary man of business, would act upon the information  
“ ‘ and would regulate his conduct by it in the execution of  
“ ‘ the trust. If it can be shown that in any way the trustee  
“ ‘ has got knowledge of that kind . . . then I think the end  
“ ‘ is attained, and that there has been fixed upon the  
“ ‘ conscience of the trustee, and through that upon the trust  
“ ‘ fund, a security against its being parted with in any way  
“ ‘ that would be inconsistent with the incumbrance which  
“ ‘ has been created.’ In the present case it has been proved  
“ conclusively that the mind of Harry Spendelow had before  
“ his appointment as a trustee been brought to an intelligent  
“ apprehension of Arthy’s mortgage in such a way that as a  
“ reasonable man he would on his appointment as a trustee  
“ act upon the information he had acquired and would  
“ regulate his conduct in the execution of the trust by such  
“ information.”

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## ACTUARIAL NOTE.

*On obtaining values of Life Annuities at isolated rates of interest. By E. H. LEVER, F.I.A., of the Prudential Assurance Company.*

DR. J. F. STEFFENSEN has shown (*J.I.A.*, vol. li, p. 278) that very fair approximate values of  $a_x$  may be obtained by expressing the life annuity in the form of an annuity-certain, the term of which is  $e_x - i\epsilon_x$ ,  $e_x$  being the curtate expectation of life,  $i$  the rate of interest involved, and

$$\epsilon_x = \frac{1}{l_x} \sum_{t=0}^{e_x-1} l_{x+t} - \frac{1}{2} e_x (e_x + 1),$$

or 
$$\frac{Y_x}{l_x} - \frac{1}{8} - \frac{1}{2} e_x^2$$

where  $Y_x$  has the meaning assigned to it in T.B. notation.

The above result is obtained by solving for  $n$  in the equation

$$a_x = a_{\overline{n}|i} = \frac{1 - v^n}{i}$$

or 
$$n = - \frac{\log(1 - ia_x)}{\log(1 + i)}$$

whence expanding  $-\frac{\log(1 - ia_x)}{\log(1 + i)}$  in powers of  $i$  and neglecting in the result powers above the first the approximation  $n = e_x - i\epsilon_x$  follows directly.

An alternative method is to take known values of  $a_x$  as a basis instead of  $e_x$ . The term of the annuity-certain then becomes  $a_x - ha_x$  where

$$\alpha_x = \frac{vS_x}{D_x} - \frac{1}{2} a_x (a_x + 1)$$

and  $h$  is the difference between the rate at which the annuity value is required and the rate used as a basis. (See *J.I.A.*, vol. li, p. 283).

Leaving for the moment any discussion as to whether the results obtained are sufficiently accurate for practical purposes or whether in any event it would be worth while to tabulate the values of  $\epsilon_x$  or  $\alpha_x$  in the case of a particular mortality table, it is useful to notice that the approximation does in any

case contain the germ of a simple and accurate means of interpolating between known values of  $a_x$  in order to obtain a value at a rate of interest not tabulated.

The practical methods of doing this at present are to interpolate by means of a finite difference formula between such values of  $a_x$  as are known or to make use of the ratio between annuity values of another mortality table.

It is not suggested that the method about to be proposed will entirely supplant those at present in use, as the results by the latter are often sufficiently accurate, but the examples which follow will show the gain in accuracy which the new method gives in any event, and will demonstrate it as providing good results when all other methods fail. The method applies equally to select tables, but perhaps its chief value lies in the fact that it may also be applied with facility and accuracy in the case of joint life and temporary annuities.

Confining our attention to the formula involving the  $\epsilon_x$  function it will be seen that the whole essence of the approximation is that the term ( $n$ , say) of the equivalent annuity-certain can be expressed in the form  $A - Bi$  where  $A$  and  $B$  are functions of  $x$  and as such constant for a particular value of  $x$ .

Ignoring the mathematical form of  $A$  and  $B$ , which is for present purposes immaterial, it follows that with  $i$  as the only variable,  $n$  is a linear function and thus if any two values are known any other value can be accurately obtained by first difference interpolation. Thus suppose the values of  $a_x$  and  $a'_x$  at rates  $i$  and  $i'$  be known and the value of  $a''_x$  at rate  $i''$  be required, and that  $i'' = i + h$  and  $i'' = i + h'$ . The process would be to find first the values of  $n$  at rates  $i$  and  $i'$  from the equations

$$a_x = a_{n|}$$

or

$$n = - \frac{\log(1 - ia_x)}{\log(1 + i)}$$

and

$$a'_x = a_{n'|}$$

or

$$n' = - \frac{\log(1 - i'a'_x)}{\log(1 + i')}$$

Then  $n''$  the term of the annuity-certain equivalent to  $a''_x$  is equal to  $n - \frac{h'}{h}(n - n')$  and  $a''_x = a_{n''|}$  which will give the approximate value required.

As the expression  $A - Bi$  is an approximation only and not exact it is reasonable to suppose that in arriving at a value of  $a_x$  for any particular rate of interest  $i$  the best result will be obtained by using as a starting point known values of  $a_x$  at rates of interest differing as little as possible from  $i$ .

The following examples will show the remarkable accuracy obtained by the method just outlined even when subjected to a severe test. The least satisfactory results naturally occur at the youngest ages and the higher rates of interest.

The numerical work is given in Example I, to illustrate the method clearly.

It may be pointed out that there are two ways in which the values of  $n$  and  $n'$  can be found which satisfy the equations  $a_x = a_{\overline{n}|}$  and  $a'_x = a_{\overline{n'}|}$ . The first and simplest is to enter a table of annuities-certain inversely with the value of  $a_x$  and use first difference interpolation to find the fractional period. The second is to solve for  $n$  in the equation

$$n = - \frac{\log(1 - ia_x)}{\log(1 + i)}$$

and thus obtain a more accurate value.

Logically it would seem that the plan actually used should depend on the method adopted for obtaining the value of  $a_{\overline{n''}|}$  from the value of  $n''$  when this has been ascertained. If, as is proposed, it is decided to use first difference interpolation for the fractional part of  $n''$  to arrive at the value of  $a_{\overline{n''}|}$ , then it would seem better to employ first difference interpolation throughout since by so doing errors would to some extent, if not entirely, counterbalance. This plan has actually been adopted in all the examples which follow.

If the values of  $n$  and  $n'$  were obtained by the more correct formula then logically the value of  $n''$  should be found from the equation

$$a_{\overline{n''}|} = \frac{1 - v^{n''}}{i}$$

The difference in the numerical result, even when the methods are combined, is negligible, only occasionally affecting the third decimal place of the annuity value. The use of first difference interpolation throughout has the advantages of lessening the arithmetical work and obviating the necessity of using tables of logarithms.

Example 1.

Values of  $a_x$  available ... 4 per-cent and 5 per-cent only.

„ „  $a_x$  required ...  $a_{20}$  and  $a_{50}$  at  $4\frac{1}{2}$  per-cent.

Using 4 per-cent and 5 per-cent as the basis and adopting  $H^M$  (T.B.) Mortality, we have

$$a_{20}^{4\%} = 18.662 = a_{\overline{n}|4\%}$$

$$a_{20}^{5\%} = 16.062 = a_{\overline{n'}|5\%}$$

Whence  $n = 34.9898$

and  $n' = 33.3125$

by entering a table of annuities-certain inversely

and  $n'' = \frac{1}{2}(n + n') = \frac{1}{2}(68.3023) = 34.1512$

and  $a_{20}^{4\frac{1}{2}\%} = a_{\overline{34.1512}|4\frac{1}{2}\%} = 17.279$

again by first difference interpolation.

Similarly for  $a_{50}$ .

In this instance the use of the formula

$$n = -\frac{\log(1 - i\sigma_x)}{\log(1 + i)}$$

produces an identical numerical result.

The results are compared below.

Column headed A gives result by new method.

„ „ B „ „ of first difference interpolation between 4 per-cent and 5 per-cent.

*Values of  $a_x$   $4\frac{1}{2}$  per-cent.*

Age	True Value	A		B	
		Value	Error	Value	Error
20	17.279	17.279	.00	17.362	-.083
50	11.923	11.923	.000	11.947	-.024

The results by the new method in this instance are exact and therefore no better results would be obtained even if a more extended range of annuity-values were available and more powerful means of finite difference interpolation applied.

Example 2.H<sup>M</sup> MORTALITY.

Values available ... 3 per-cent and 5 per-cent.

Values required ... 4 per-cent and 6 per-cent.

Column headed A gives values by new method.

“ “ B “ “ of first difference interpolation between 3 per-cent and 5 per-cent.

*Values of  $a_x$  4 per-cent.*

Age	True Value	A		B	
		Value	Error	Value	Error
20	18·662	18·665	—·003	19·063	—·401
50	12·522	12·524	·002	12·624	—·102

The results here are again exact and the same remark therefore applies as in the preceding example.

*Values of  $a_x$  6 per-cent.*

Age	True Value	A		B	
		Value	Error	Value	Error
20	14·035	14·025	·010	13·061	·974
50	10·388	10·381	·007	10·118	·270

The accuracy of the new method in this case is far greater than could be obtained by any other method with the same material and is again exact for all practical purposes.\*

Example 3.

Values available ... 3 per-cent, 4 per-cent,  
5 per-cent, 6 per-cent.

Values required ... 7 per-cent, 8 per-cent,  
which are both well  
outside the range.

\* When, as in this example, other life annuity tables are available at the requisite rates more accurate values would probably be given by reference to such tables. Thus the Carlisle 6 per-cent Table would give for  $a_{20}$  by first difference interpolation of a substituted age, 14·034.—EDS. J.I.A.

Columns headed A give values by the new method based on given values at 5 per-cent and 6 per-cent Carlisle Mortality.

Columns headed B give values obtained by extrapolation from values at the *four* rates given. These approximate values have been obtained by the use of the ordinary formula. It may be pointed out that in this instance much better values could be obtained by assuming, when extrapolating, that second differences are in G.P. Values obtained in this way would show no greater error than those under A.

This example presents a severe test.

*Values of  $a_x$  7 per-cent.*

Age	True Value	A		B	
		Value	Error	Value	Error
20	12.259	12.247	.012	12.192	.067
50	9.749	9.746	.003	9.737	.012

*Values of  $a_x$  8 per-cent.*

Age	True Value	A		B	
		Value	Error	Value	Error
20	10.985	10.951	.034	10.662	.323
50	8.987	8.974	.013	8.933	.054

#### Example 4.

Only values available  $a_x \dots 3\frac{1}{2}$  per-cent.

Values required  $\dots \dots 3$  per-cent and 4 per-cent.

In applying the method in this case one of the values of  $n$  used was necessarily  $e_x$ , the expectation of life.

Column A gives values by new method.

„ B „ „ obtained by Dr. Steffensen by applying the formula  $n = a_x^{3\frac{1}{2}\%} - h\alpha_x$   
(See *J.I.A.*, vol. li, p. 283).



*Values of  $a_x$  3 per-cent.*

Age	True Value	A		B	
		Value	Error	Value	Error
20	22·064	22·054	·010	22·043	·021
50	13·878	13·881	—·003	13·875	·003

*Values of  $a_x$  4 per-cent.*

Age	True Value	A		B	
		Value	Error	Value	Error
20	18·662	18·669	—·007	18·649	·013
50	12·522	12·516	·006	12·519	·003

It is unnecessary to multiply examples. Enough have been given to demonstrate the utility of the method, especially when it is remembered that in practice it is rare to require a value of  $a_x$  at a rate of interest differing by more than 1 per-cent from tabulated values.

It is not apparent that any gain in accuracy would be obtained by using the alternative expression  $n = A' - B'h$  as the basis of the interpolation where  $A'$  and  $B'$  are functions of  $x$  and  $i$  at a known rate. In fact in a few cases that have been tried on this basis the results do not seem to be so satisfactory.

A few examples of the application to temporary and joint life annuities may be instructive.

Example 5.H<sup>M</sup> MORTALITY.

Given  $a_{30:\overline{20}|}$  at 3 per-cent and 5 per-cent.

Required  $a_{30:\overline{20}|}$  at 4 per-cent.

Approximate value by method ...	...	12·499
True value	... ..	12·501
		Error
		·002

Example 6.

Given  $a_{30:\overline{10}}$  at 3 per-cent and 5 per-cent.

Required  $a_{30:\overline{10}}$  at 4 per-cent.

Approximate value	...	...	...	7.769
True value	...	...	...	7.769
				Error
				Nil.

Example 7.

Given  $a_{[40]:[28]}^{\text{F M}}$  and  $a_{[50]:[23]}^{\text{F F}}$  at 3 per-cent and  $3\frac{1}{2}$  per-cent B.O.A.

Required  $a_{[40]:[28]}^{\text{F M}}$  and  $a_{[50]:[23]}^{\text{F F}}$  at 5 per-cent.

This example is chosen so that the method may be compared with that given in the memorandum at the end of the British Offices Life Tables (p. 229 *et seq.*).

Taking the values of  $a_{[40]:[28]}^{\text{F M}}$  and  $a_{[50]:[23]}^{\text{F F}}$  at 3 per-cent and  $3\frac{1}{2}$  per-cent given in the examples in the memorandum (pp. 231 and 232) as correct, the following values result :

	$a_{[40]:[28]}^{\text{F M}}$	$a_{[50]:[23]}^{\text{F F}}$
Value by new method	12.590	11.554
„ method of interpolated equal ages	12.603	11.565
„ Hardy's summation formula	12.592	11.535

As the values by Hardy's summation formula are unlikely to differ from the true values by more than a small third difference error, the above table shows that the new method gives results more accurate on the whole than the method of interpolated equal ages; an even greater advantage is that the new method does not depend for its application upon the existence of a complete table of joint life annuities at equal ages at the rate of interest at which values are required.

Reverting to the question of the desirability of tabulating values of the function  $e_x$ , a consideration of the results shown in Example 4 points to the conclusion that if it be thought worth while to calculate the values of this function at all, more accurate results would be obtained by obtaining them empirically from known values of  $a_x$  rather than directly from the functions  $Y_x$  and  $e_x$ .

If available the values of  $a_x$  employed for this purpose should be about the centre or between the centre and the end of the range of rates of interest within which isolated values are most likely to be required.

This suggestion has been tested on the basis of the  $H^M$  Tables and gives very good results.

It is, moreover, not without interest to notice that by expanding  $n$  in the equation  $a_x = a_{\bar{n}}$  up to and including the third power of  $i$ , and tabulating the successive coefficients, accurate values of  $a_x$  may be obtained for all practical rates of interest with the aid of a table of annuities-certain only.

## REVIEWS.

*Ministry of National Service, 1917-1919. Report, Vol. I, upon the Physical Examination of men of Military age by National Service Medical Boards from 1 November 1917 to 31 October 1918. Price 6s.*

As stated in the Chairman's letter at the beginning of this volume, the above is the first instalment of the Report of a Committee appointed to consider the best method of utilizing the data obtained from the work of the Recruiting Medical Boards, so as to render it readily available for use, and of placing the information as to the physical fitness of the nation so obtained on record for future use and reference.

Whatever merits the Report may have from the medical aspect, it must be admitted that it is distinctly disappointing from a statistical point of view.

The volume deals with the results of the examinations of all men of military age called up for medical examination under the Military Service Acts from the time of the re-organization under entirely civil administration, which came into operation on 1 November 1917, until 31 October 1918—practically the last year of the war. The claim is that this task afforded the opportunity to make a far-reaching medical survey of the male population of Great Britain and to a limited extent also, of that of Ireland. It is obvious, however, that the medical survey was only a partial one, since it excluded all the men already in the Services.

The statement regarding the work required of the Medical Boards is clearly set out—it was to classify the men into broad groups determined by physical considerations alone and had the satisfactory result that most of the difficulties and anomalies of the previous regime disappeared. The actual groups, or Grades, were as follows:

Grade I.—Those who attain the full normal standard of health and strength and are capable of enduring physical exertion

suitable to their age. Such men must not suffer from progressive organic disease, nor have any serious disability or deformity. Minor defects which can be remedied or adequately compensated by artificial means will not be regarded as disqualifications.

Grade II.—Those who for various causes, such as being subject to partial disabilities, do not reach the standard of Grade I. They must not suffer from progressive organic disease. They must have fair hearing and vision; be of moderate muscular development, and be able to undergo a considerable degree of physical exertion of a nature not involving severe strain.

Grade III.—Those who present marked physical disabilities or such evidence of past disease that they are not considered fit to undergo the degree of physical exertion required for the higher Grades. Examples of men suitable for this Grade are those with badly deformed toes, severe flat foot and some cases of hernia and varicose veins. . . . . The third grade will also include those who are fit only for clerical and other sedentary occupations, such as tailoring and bootmaking.

Grade IV.—All those who are totally and permanently unfit for any form of Military Service.

The first chapter deals with grading as a criterion of health. The comparatively small number of applications to Appeal Tribunals for medical re-examinations indicates that the Boards did their work well. Only 1·4 per-cent of the men examined made such applications; less than half of these were considered by the Tribunals to rest on sufficiently substantial grounds to justify leave for re-examination, and of those re-examined by the Medical Assessors, the grading of 50 per-cent was unchanged; that is, in less than 0·4 per-cent is there reason to believe the grading by the Medical Boards was incorrect. Further than this, we are informed that the general tendency of the Boards was certainly not to grade the men too low; but the small number of appeals would hardly suggest that unduly high grading was causing dissatisfaction.

The results of the year's work were as follows :

		Number.			Percentage.
Grade I	...	871,769	...	...	36·0
Grade II	...	546,276	...	...	22·5
Grade III	...	756,859	...	...	31·2
Grade IV	...	250,280	...	...	10·3
		<hr/> 2,425,184 <hr/>	...	...	<hr/> 100·0 <hr/>

The inferences drawn from these figures are of the most depressing kind, but the arguments by which the conclusions are supported are, to say the least, open to question. The most important point in this respect is to what extent the men medically examined during this fourth year of the war were fully representative of the whole population of military age. It is stated that of the men who flocked to the colours as volunteers a large percentage were accepted in spite

of physical disabilities for which they were invalided after a short period of training, and that later came a better handling of manpower ; that those examined in the period under review included :

- (1) Men from protected industries, such as agriculture, mining, shipbuilding and munitions, which employed an abnormally high proportion of physically fit, who were " combed out."
- (2) Men previously rejected when the need was less urgent, who were re-examined.
- (3) Men now refused exemption, owing to the urgent need for recruits.
- (4) A large proportion of lads who attained the age of 18 during the year.
- (5) A relatively small proportion of men between 41 and 51, who became liable for military service under the Military Service (No. 2) Act (April, 1918).

On these grounds it is assumed probable " that the men examined " during the year under review may be regarded in the aggregate " as fairly representing the manhood of military age in the country " . . . . and that deductions founded upon the observations . . . . " may legitimately be looked upon as a trustworthy criterion of " the national health of this period " (p. 5).

Let us examine this claim. The approximate stages through which recruiting passed were :

Stage 1.—Fit volunteers were taken. In the earliest period an exceptionally high standard was required, this standard gradually becoming lower until it merged into

Stage 2.—At this stage, it may be admitted, many unfit volunteers were taken, unless they were obviously hopeless as recruits.

Stage 3.—Men obtained under the " Derby " scheme. This must generally have provided fit men, leaving those in protected industries and those exempted by tribunals.

Stage 4.—The earlier Military Service Act.

Stage 5.—Men previously protected or rejected, and men between 41 and 51.

It should be remembered that the year under review would comprise the worst part of the fourth stage and the whole of the fifth. The first stage, in spite of mistakes, must have recruited an exceptionally high proportion of fit men. The second would probably have included a good many with physical disabilities, but at this stage recruiting was comparatively slack, and consequently the number of men obtained was probably smaller, whilst the third and fourth stages must again have provided a large number of fit men. The fifth stage obviously would contain men of all physical conditions and all the rejected of the first four stages.

Of the classes which the Report mentions :

- (1) Would admittedly be men of good physical type, but no figures are given to show what proportion these were of the numbers examined.

- (2) Would attain only a poor physical standard.
- (3) Would possibly be slightly above average.
- (4) Lads of eighteen are mentioned as "a large number", but unless a large proportion of men of older ages were already in the forces, they would not be so, comparatively.
- (5) Men of 41 to 51 are described as "a relatively small proportion", but again here, we have ten ages, seven of which were comparatively untouched by previous recruiting.

In the Regional Report for the North-Western Area there is a table (p. 57) setting out the number of men actually graded at each year of age up to 50. For age 18 the total is 4,329; for 19 to 40, both inclusive, the total is 10,260, giving an average of 466 for each age; for 41 to 43, 1,828, an average of 609, and for 44 to 50, 9,218, an average of 1,317. These figures indicate that men of 41 to 51 quite outnumbered the lads of 18, and also show to what extent the intermediate ages had already been drawn on for recruits. It would appear that the figures likely to give the most accurate information regarding the physical health of the nation are those pertaining to lads of 18, and men of 44 to 51. Even here a true aggregate is doubtful in view of voluntary enlistments of those particularly fit below and above the normal military age limits.

In Chapter II is set out a standard, on the basis of the grades, which, in the opinion of Professor Arthur Keith, the health of the general population should attain. This is, in effect, that 70 per-cent should reach Grade I, 20 per-cent Grade II, 7.5 per-cent Grade III, and 2.5 per-cent Grade IV. The standard is justified by comparing the physical fitness of the population to the stature of 1,000 Cambridge students, the mean stature being 5 feet 9 inches. All those 5 feet 8 inches and over are compared to Grade I, 5 feet 7 inches and 5 feet 6 inches to Grade II, 5 feet 5 inches and 5 feet 4 inches to Grade III, and under 5 feet 4 inches to Grade IV. It is obvious that any arbitrary proportions in the grades could be obtained by drawing the limits at requisite points, and that the standard thus set up has no true statistical basis. Further, the grades are weighted, again quite arbitrarily, to obtain a percentage index of fitness. A good criticism of Professor Keith's standard is to be found on pages 107-110 by the Commissioner for the Yorkshire and East Midland Region. We are told here that definite instructions were laid down in respect of 59 common disabilities, probably including 75 per-cent to 80 per-cent of the total disabilities met with in actual practice of the Recruiting Medical Boards. Of these 59, three in each case caused rejection, 27 excluded placing the men in either Grade I or Grade II, whilst three other disabilities excluded placing in Grade I. Many disabilities (*e.g.*, foot malformations) would unfit a man for military service, but not for civilian occupation. It seems improbable that the curve for national fitness would normally follow a symmetrical distribution, and military requirements as indicated above would certainly impart an element of skewness. It is indeed regrettable that such a standard is accepted



as a sound basis for comparisons, instead of being clearly recognized as an empirical one.

Chapter III discusses the relations of occupations and health, and we are given figures setting out the grading of various occupations in the form of percentages (p. 18), but no consideration is given to the bearing which the essentiality of the various trades would have had on the men remaining in them. Those that head the list are all occupations which must by their nature employ a high proportion of fit men, and which were also more or less protected, whilst the unhealthy ones at the bottom of the list were unprotected trades which would already have contributed their fittest members to the Army and Navy.

The last chapter of the general section deals with the causes of low grading and rejection, but again all comparisons are made on the false premises of Keith's Standard. There is ample evidence of a tremendous actual number of men suffering from physical disability; so many as to convince one that no effort should be spared to ameliorate and improve conditions of life and occupations, but we cannot attempt on the data given to judge the relative amount, or the distribution, either as regards age or occupation, with respect to the whole of the male population of the country.

One cannot help feeling that the general section of the Report is too dogmatic in its conclusions: that too much faith has been placed in Keith's Standard. Too little consideration has been given to the disturbing factors introduced by the previous withdrawal to the Forces of fit members of the community and by military requirements with regard to physical fitness.

Part II comprises the Reports of the Commissioners of Medical Services, concerning their respective Regions. Most of them are decidedly interesting, and though one does not expect a high standard of statistical work one occasionally finds the figures set out in good style.

S. J. P.

*The Assessment of Physical Fitness.* By GEORGES DREYER, C.B.E., M.A., M.D., in collaboration with G. F. HANSON.

[Pp. 114. Cassell & Co., Ltd. 1920. 10s. net.]

WITH the question of physical standards so much to the fore, one is interested to note this book, which aims at supplying medical men and others directly interested in the subject, with tables for assessing the physical fitness of any individual, whether man, woman or child.

For this purpose the respective relations to each other of weight, length of trunk (*i.e.*, height sitting), chest measurement and vital capacity of the lungs have been investigated. It is stated that they may be represented by formulæ of the form  $A = kB^n$ ,  $A$  and  $B$  being the weight or measurement, and  $k$  and  $n$  constants.

The explanations and examples are very clear and one is pleased to note the insistence on the precision with which weight and

measurements are to be taken. Eighty-six pages of the book are devoted to tables giving, on the basis of the formulæ :

- (1) Weight from length of trunk.
- (2) Weight from circumference of chest.
- (3) Circumference of chest from length of trunk.
- (4) Vital capacity from weight.
- (5) Vital capacity from length of trunk.
- (6) Vital capacity from circumference of chest.

These are given for males and females, and on the metric and imperial systems, making twenty-four tables in all.

As the tables state merely the relation of one factor to another, by entering the tables with one factor, only the deviation from normal of a second factor with regard to the first is obtained, and for compound assessments the resort, in the examples, is to the expedient of simple averaging.

The book may prove of use to anthropologists for special work where the requisite data is available, and possibly to life offices for confirmatory purposes in assessing the abnormality of lives proposed for insurance. In any event it is an effort towards setting up a definite guide to physical standard.

S. J. P.

## CORRESPONDENCE.

*To the Editors of the Journal of the Institute of Actuaries.*

SIRS,—With reference to Mr. A. T. Traversi's letter (*J. I. A.*, vol. li, p. 304), I beg to make the following observations :

The distinction which Mr. Traversi makes between "rates" and "probabilities" has been recently the subject of new investigations which have shown how the former, which are not exactly the same as differential coefficients or central rates, can be based on the calculus of probabilities instead of being derived from an empirical conception of the so-called "exposed to risk." They can then be identified with Karup's "pure, independent or partial probabilities" in accordance with a definite hypothesis as to the distribution over the years of age of entrants and withdrawals in the experience under consideration and as to the instantaneous rate of mortality or as to the probabilities of death  $q(x+t, x+1)$  for the fractional intervals  $(x+t, x+1)$  on the hypothesis that no other cause of elimination takes place.

If  $\lambda(x)$  denotes the number of individuals in the experience who have attained the exact age  $x$ ,  $I(x+t)$  those who enter on observation at age  $x+t$ , where  $t$  may have any value from 0 to 1,  $U(x+t)$  those who pass out of observation by causes other than death, and  $q(x+t, x+1)$  the above-mentioned probability of death between ages  $x+t$  and  $x+1$ , so that  $\sum_0^1 I(x+t) = I_x$ ,  $\sum_0^1 U(x+t) = U_x$  and

$q(x, x+1) = q_x$ , then  $M_x$ , the probable number of deaths between ages  $x$  and  $x+1$ , is evidently given by the relation

$$M_x = \lambda(x)q_x + \sum_0^1 I(x+t)q(x+t, x+1) - \sum_0^1 U(x+t)q(x+t, x+1).$$

If by way of hypothesis  $q(x+t, x+1)$  is taken as  $=(1-t)q_x$ , then

$$q_x = M_x / [\lambda(x) + \sum_0^1 I(x+t)(1-t) - \sum_0^1 U(x+t)(1-t)].$$

The denominator of this formula coincides exactly with the "exposed to risk" according to the exact-age method, and  $q_x$ —the "rate" of English actuarial literature—is seen to be, from the method by which it has been obtained, Karup's "pure probability."

If the hypothesis of uniform distribution is adopted, then the infinitesimals  $I_x dt$  and  $U_x dt$  are to be substituted for  $I(x+t)$  and  $U(x+t)$ , and the evaluation of the integrals with respect to  $t$  between the limits 0 and 1 gives for the denominator the exposed to risk according to the mean age method.

In addition to showing clearly to what hypothesis the usual methods based on an empirical conception of the "exposed to risk" correspond, the foregoing formulas show also the evident coincidence of "rates" with Karup's "pure probabilities" based on the indicated hypothesis.

On the conception of pure probabilities and on their importance in actuarial mathematics an extensive literature exists. The subject has been exhaustively treated in a few memoirs submitted to the 7th International Actuarial Congress, vol ii, p. 327, *et seq.*

More recently the subject has been fully developed in Italy in connection with the statistical and actuarial application of independent probabilities, especially in the following papers:

F. CANTELLI.—*Genesi e costruzione delle tavole di mutualità* in "Bollettino di notizie sul Credito e sulla Previdenza", no. 3, 1914;

I. MESSINA.—*Le probabilità parziali nella matematica attuariale*, nello stesso periodico no. 4 a 6 del 1915;

F. CANTELLI.—*Sull'applicazione delle probabilità parziali alla statistica*, in "Giornale di Matematica finanziaria", n. 1, 2 e 3 del 1919.

With special reference to the construction of tables of mortality the writer has dealt with the subject in a note entitled *Costruzione e critica delle tavole di mortalità* in the "Giornale degli economisti e rivista di statistica" for December 1917.

If the circulation and study of the papers just mentioned had not been prevented by the war, Mr. Traversi would not perhaps have written his letter, since he would have been able to see clearly how the so-called "rates" of English actuarial literature can only correspond theoretically to independent probabilities in the sense in which Karup uses that term and must be regarded strictly as

referring to independent\* and compatible events so that it is quite legitimate to apply to them Karup's formula

$$p_{ng} = p_n \times p_g \quad (1)$$

just as, in an experience subject to decrement by death and invalidity, the probability of not dying or being invalided is given by the product of the respective independent probabilities (*cf.* H. A. Van den Belt's memoir in the Transactions of the Seventh Congress).

It appears from the above-mentioned investigations that the formulas used by T. B. Sprague, when he considered the two decremental causes "death" and "marriage" in so far as the direct determination of pure probabilities is concerned, involved the employment of hypotheses which cannot be treated as mutually consistent, and this explains the incongruities produced by the simultaneous use of Sprague's formulas for the two decremental causes. Mr. Traversi has adopted these formulas, and yet claims to establish the inapplicability of (1) to the case dealt with by Prof. Cantelli. Such incongruities have been clearly indicated by this author in the first of the two works referred to above.

I have thought it useful to submit these observations to the *Journal* because, having myself devoted some attention to the subject, it seemed to me convenient not to allow the points raised by Mr. Traversi to pass without remark, as this would tend to re-establish conceptions which, as the above-mentioned have shown, can no longer be maintained consistently with the more recent developments of actuarial science, and because it seemed desirable to draw attention to published investigations which, owing to the war, may have been unnoticed outside Italy.

I am, &c.,

GAETANO BALDUCCL.

*Ministry of the Treasury, Rome.*

P.S.—I am late in commenting on Mr. Traversi's letter, but I have only recently seen the numbers of the *Journal* for 1918 and 1919. I may take the opportunity of mentioning that the formula employed by Mr. Savory (*J. I. A.*, vol. li, p. 65) for calculating the excess of mortality due to the war, namely,  $10000(q_{ng} - q_n)$ , in which  $q_{ng}$  and  $q_n$  are central rates, is not exact but is sufficiently approximate in view of the numbers involved.—G. B.

\* And not therefore mutually dependent as Mr. Traversi maintains.

## THE INSTITUTE OF ACTUARIES.

## EXAMINATIONS, DECEMBER 1919.

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EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART I.—SECTION A).*First Paper.*

1. The equations

$$3x^3 - 2x^2 - 11x + 10 = 0$$

$$6x^3 + 11x^2 - 26x - 15 = 0$$

have a common root. Find this and hence solve completely the first equation.

2. Prove that when  $x$  is less than 1 the product of

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots \text{ to infinity,}$$

$$\text{and } 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \text{ to infinity,}$$

is equal to

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{2!}x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{3!}x^3 + \dots \text{ to infinity.}$$

Find by the use of the binomial theorem the value of  $\sqrt[6]{\frac{5}{4}}$  to four places of decimals.

3. A merchant invests £100 in an enterprise which yields a profit, after deducting expenses, of 20 per-cent in the first year and 33 per-cent in each succeeding year. If he invests his profits at the end of each year under the same conditions, find an expression for the total amount invested at the end of  $n$  years.

4. If  $x$  is less than 1, find the expansion of the following expressions in ascending powers of  $x$  as far as the term involving  $x^3$ :

$$(1) \frac{\log(1+x)}{x^2 - 2x + 7}$$

$$(2) \frac{x^2 - 3x}{e^x - 1}$$

5. Find the sum of the first  $n$  coefficients in the expansion of  $(1+x)^n(1-x)^{-2}$ .

$$6. \text{ If } a = x^2 - yz$$

$$b = y^2 - zx$$

$$c = z^2 - xy$$

prove that  $a^3 + b^3 + c^3 - 3abc = (a+b+c)(x+y+z)(ax+by+cz)$ .

7. In how many ways is it possible to place 7 people in 4 rooms, so that not more than 4 are in any room? Any room may be empty.

8. On the average, out of 42 births, 21 are male, 20 female and 1 still; what is the probability that exactly this distribution will occur amongst the next 42 births?

9. There are  $m$  candidates taking an examination paper of  $n$  questions of equal difficulty; assuming that a candidate answers a question correctly or not at all, either being equally likely:

(a) In how many different ways may a paper be answered?

(b) How many different sets of answered papers are possible?

(c) What is the chance that a set of papers is handed in in which a particular question is solved by not more than one candidate?

### *Second Paper.*

1. If  $u_0, u_5, u_{10}, u_{15}$  be 4 values of a function at equidistant points, find expressions true to third differences for  $u_6$  and  $u_8$ , solely in terms of  $u_0, u_5, u_{10}$  and  $u_{15}$ .

$$2. \text{ Given } u_0 = -0.500$$

$$u_1 = -0.484$$

$$u_5 = 0$$

$$u_6 = +0.256$$

find the missing terms.



3. Prove that

$$u_0 + \frac{u_1}{1}x + \frac{u_2}{2}x^2 + \dots \\ = e^x(u_0 + \frac{x}{1}\Delta u_0 + \frac{x^2}{2}\Delta^2 u_0 + \dots),$$

and hence find the sum of the series

$$\frac{1^4}{1} + \frac{2^4}{2} + \frac{3^4}{3} + \dots \text{ to infinity.}$$

4. Prove that, if  $u_x$  be a function the fourth differences of which are constant,

$$\frac{du_x}{dx} = \frac{1}{12}(u_{x-2} - 8u_{x-1} + 8u_{x+1} - u_{x+2})$$

and hence find an approximate value for  $\frac{d}{dx}(\log u_x)$  where

$$\begin{aligned} u_{x-2} &= 42.699 \\ u_{x-1} &= 40.365 \\ u_x &= 37.977 \\ u_{x+1} &= 35.543 \\ u_{x+2} &= 33.075 \end{aligned}$$

5. (i) Find the  $n$ th differential coefficient with respect to  $x$  of

$$\frac{x+1}{2x^2-5x+3}$$

(ii) Differentiate with respect to  $x$

$$\log \sqrt{\frac{a^3 + a^2x + ax^2 + x^3}{a^2 + ax + x^2}}$$

6. Evaluate

$$\begin{aligned} &\int \frac{4x+2}{(x-1)^2(x-7)} dx \\ &\int \log(1-x^2) dx \\ &\int x^n (\log x)^3 dx \end{aligned}$$

7. If  $u_1 = 4157$ ,  $u_2 = 4527$  and  $u_4 = 5435$ , find approximately the value of  $\int_1^4 u_x dx$ .

8. In a certain year A and B were in London for one period only in each case, A for one third of the year, B for one quarter of the year. Assuming that in the case of A any one period of one

third of a year and in the case of B any one period of one quarter of a year is as likely as any other period, find the probabilities that :

- (1) A was in London the whole of the time B was.
- (2) A and B were not in London at any moment together.
- (3) A and B were in London at some moment together.
- (4) A came to London before B.

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART I.—SECTION B).

1. Prove that the total amount paid in interest when a loan is repaid by way of a level annuity is greater than when the loan is repaid by equal instalments of principal, interest being paid on the balances outstanding, but cannot be more than twice as great.

\*2. If it be given that the value of an annuity for  $n$  years at an unknown rate  $i$  is  $a$ , and the value at a known rate  $i'$  is  $a'$ , where  $a'$  does not differ greatly from  $a$ , show that approximately

$$i = i' + i' \frac{a' - a}{a' - nr'^{n-1}}$$

where  $r'^{n+1}$  is calculated at rate  $i'$ .

Apply the formula to find the rate at which  $a_{\overline{18}} = 12$ .

\*3. A loan of £10,000 was made at 5 per-cent per annum, repayable in 20 years by equal yearly annuity payments. With the 10th payment of the annuity, a further sum equal to the principal in the 11th, 12th and 13th annuity payments is paid off, and the balance of the loan outstanding is to be redeemed in 5 years by an annuity calculated to yield interest at 5 per-cent per annum and allow the replacement of capital at 4 per-cent per annum.

Draw up a schedule of repayments, showing principal, interest and balance of loan outstanding.

\*4. A government makes the following advances to a dependent state :

- (1) £500,000 on 1 January 1908, repayable at 6 per-cent per annum by an annuity, to run for 25 years, payable quarterly ;
- (2) £300,000 on 7 April 1910, at interest at the rate of 7 per-cent per annum, payable quarterly, the principal being repayable by annual instalments of £25,000, commencing 7 April 1915.

The borrowing state pays all charges up to and including the annuity payment due 1 January 1918, and then defaults.

On 1 January 1919 it is agreed to consolidate the loans, with all arrears of interest then due, the whole to be repaid by an annuity payable half-yearly for 40 years, calculated on the basis of interest at the rate of 7 per-cent per annum. What will be the amount of the annuity?

5. A sum of money is invested in a stock bearing interest at 4 per-cent per annum, payable half-yearly, and dividends are invested in the same stock. If the price of the stock at successive half-yearly intervals is 95,  $95\frac{1}{2}$ , 96 and so on; and if income tax is deducted at the rate of 6s. in the £ in the first year and 5s. in the second, and the cost of investment or sale is  $\frac{1}{4}$  per-cent, find the net effective rate of interest realized by an investor if he sells out at the end of two years.

\*6. A company issued a leasehold assurance of £1,000 payable at the end of 20 years, at an annual premium calculated on a 3 per-cent basis. Its expenses in connection with the policy were 10 per-cent of the first premium and 3 per-cent of subsequent premiums. The policy was surrendered just before payment of the tenth premium, a surrender value of all the premiums except half the first accumulated at  $2\frac{1}{2}$  per-cent compound interest being allowed. The company made a profit of £40 on the transaction. What rate of interest did it earn on the premiums paid to it?

\*7. A government issues a loan of £1,500,000 in bonds of £1,000 each, bearing interest at 4 per-cent per annum, payable half-yearly, redeemable in 25 years by annual drawings as follows:

20 bonds at the end of each of the first five years			
40	„	second	„
60	„	third	„
80	„	fourth	„
100	„	last	„

If the price of issue is such as to yield the investor, on the average, an effective rate of  $4\frac{1}{2}$  per-cent per annum, find the yield on a bond drawn at the end of the eleventh year.

\* “A Short Collection of Actuarial Tables” will be supplied for use in answering these questions.

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART II).

*First Paper.*

1. What is a table of mortality? A table of mortality was once formed from the number of deaths registered at each age in a certain town during a long period. State briefly what conditions should have existed in order to justify the adoption of such a course.

\*2. A pension society is recruited each year by 100 new entrants at age 20. Each member pays an annual premium in advance of £10, and at age 60 receives a pension of £80 per annum payable by quarterly instalments for 5 years certain, and as much longer as he may live. The first payment of the annuity is made 3 months after attainment of age 60. In the event of death before age 60 the premiums paid are returned without interest.

When the membership of the society reaches a stationary condition,

- (1) How many members will there be under age 60?
- (2) How many pensions will there be being paid?
- (3) What amount will be returned each year in the case of members dying before age 60?
- (4) What amount will be paid each year in pensions?

Give numerical answers assuming each year's entrants are equally distributed throughout the year, that there are no withdrawals, and that the mortality is in accordance with the  $H^M$  Table.

3. In the case of three lives aged 20, 55 and 65, the probability that exactly one survives 10 years is  $\cdot 121875$  and that at least one dies within 10 years is  $\cdot 578125$ .

Find  ${}_{10}p_{20}$ ,  ${}_{10}p_{55}$  and  ${}_{10}p_{65}$ , assuming that they are in geometrical progression.

\*4. A society admits annually 1,000 new members at age 30; if, of the entrants of any one year, there retire at the first and subsequent anniversaries of the date of their admission  $100_1p_{30}$ ,  $100_2p_{30}$ ,  $100_3p_{30}$ , and so on, find the average age at death of those who die while members of the society, assuming  $H^M$  Mortality.

5. The probability that a life aged 20 dies before a life aged 40 is  $\cdot 2697$ , the probability that a life aged 20 dies within 10 years is  $\cdot 7320$  of the probability that a life aged 30 dies within 10 years, and the probability of two lives aged 20 and 30 dying within 10 years of each other is  $\cdot 2942$ . Find the probability of a life aged 20 dying between the ages of 30 and 40.

6. Prove that

$$\mu_x = \frac{1}{\dot{e}_x} + \frac{1}{\dot{e}_x} \frac{d\dot{e}_x}{dx}$$

If  $\dot{e}_x = a(c-x)^n$  find an algebraic expression for  $l_x$ .

7. It has recently been stated that Indian mortality can be represented by the equation  $\mu'_x(\text{Indian}) = \mu_{x+6}(\text{H}^M) + .015$ , and that this relationship enables the  $3\frac{1}{2}$  per-cent annuity-values based on this heavy mortality to be represented by using  $\text{H}^M$  annuity-values at 5 per-cent interest with an addition of 6 years to the age. Taking the assumption as to mortality to be justified, do you agree that the deduction as to the use of the annuity-values is correct? If not, indicate the nature of the approximation.

For the purposes of your answer you may assume that the  $\text{H}^M$  Table follows Makeham's law.

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

### *Second Paper.*

\*1. Assuming an uniform distribution of deaths, find the value of  ${}_t l_x^{(m)}$ .

Find the value of  $\ddot{l}_{[48]+2}^{41}$  on the basis of  $\text{O}^{[NM]}$  Mortality with 3 per-cent interest.

\*2. Find the annual premium, by the  $\text{H}^M$  3 per-cent Table, for a 20-year joint life endowment assurance on two lives aged 30 and 40.

3. Show how the value of an annuity on  $m$  joint lives may be calculated by the substitution of a single life combined with a change in the rate of interest, when the mortality table follows Makeham's law.

If  $\log_{10} c = .03966$  and  $\log_{10} s = -.0026$ , what age and rate of interest would you employ to find the value of  $a_{50:55}$  at 3 per-cent interest?

4. If there are two life tables (A) and (B) such that the force of mortality by one is  $a_1 + b_1 c^x$  and by the other  $a_2 + b_2 c^x$ , show that the present value of a joint life annuity  $a_{xy}$  where  $x$  is an (A) life and  $y$  a (B) life can be found by substituting two lives of equal age one from each table.

If  $y$  be the elder,  $\frac{b_2}{b_1} = 1 + r$  where  $r$  is small,  $d$  = deduction to be made from  $y$  to obtain the equal age, and  $\delta$  = deduction to be made from  $y$  to obtain the equal age if both lives were either (A) or (B) lives, prove that  $d - \delta = \frac{r}{2} (1 - c^\delta) / \log_e c$  approximately.

5. Given a table of values of  $l_x$  how would you calculate and check a table of net single premiums for pure endowments maturing at quinquennial ages for all ages at entry?

6. In two tables, A and B, the rates of mortality are similar for ages  $x$  to  $x+n$ , but in A the rates are higher both below age  $x$  and above age  $x+n$ .

Compare the policy values by these tables for endowment assurances,

(1) At age  $x$  for policies maturing at age  $x+n$  taken out before age  $x$ .

(2) At age  $x+n$  for policies maturing after age  $x+n$  taken out at age  $x$ .

7. Confirm or correct the right-hand side of each of the following equations, giving a reason for your answer from first principles:

$$\text{Lt. } \left\{ \frac{\bar{a}_{x+h:\overline{n}} - \bar{a}_{x:\overline{n}}}{h} \right\}_{h=0} = \delta \cdot \bar{a}_{x:\overline{n}} + \mu_x \bar{a}_{x:\overline{n}} - 1$$

$$\bar{a}_{\overline{yz}|x} = \frac{1}{l_x l_y l_z} \int_0^x e^{t l_x + t l_y + t l_z + t \mu_y + t \mu_z + t \bar{a}_{x+t}} dt.$$

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

### Third Paper.

\*1. A whole life assurance for £500 was taken out at age 25 and has been 20 years in force. There is a reversionary bonus of £180 attaching. The assured desires to surrender a part of the bonus to effect the alteration of the policy to an endowment assurance at age 65. If the annual premium which remains the same before and after alteration be £10. 10s., what portion of the bonus must be surrendered? Use the H<sup>M</sup> 3 per-cent Table.

2. An office which charges an extra premium of 10s. per-cent per annum on the sum assured for a certain climate risk in the case of annual premium policies proposes, in the case of single premium policies, to charge the equivalent of an annual extra premium of 10s. per-cent on the difference between the sum assured and the single premium. Discuss the equity of this arrangement on the basis of net premiums, ignoring questions of loading and bonuses.

3. Under the "extended term" system, if payment of premiums be discontinued after an endowment assurance has been in force any number of years ( $n$ ) the policy remains in force as a term assurance for the full sum assured for such a term



as the value permits, or if the value be more than sufficient to carry the term assurance up to the age when the endowment assurance matures the balance of the value is applied to provide a sum  $E$ , payable at maturity. How would you calculate a table of the values of  $E$  on a net  $H^M$  3 per-cent basis for endowment assurances maturing at age  $M$  for all ages at entry?

\*4. Find the net annual premium diminishing by one-fifth each year for a decreasing term assurance for £100,000 diminishing by one-fifth each year on a life aged 45 by the  $O^{[NM]}$  3 per-cent Table.

\*5. Two partners, A aged 30 and B aged 35, effect a joint life assurance for £1,000 at an annual premium of £33; each partner agrees to pay one-half of the annual premium, the sum assured being payable to the survivor.

At the end of 30 years, B wishes to sell his interest in the assurance to A.

Calculate the sum that A should pay, on the basis of Carlisle mortality with interest at 3 per-cent per annum, and prove the accuracy of your calculation.

\*6. Find the single premium by the Carlisle 3 per-cent Table for an annuity of 1, to commence on the death of a man aged 30, payable for ten years certain and as much longer as his wife now aged 30 shall live. The annuity is to be payable by monthly instalments, the first instalment being paid at death. Assume  $_{10}p_{30} = \cdot 9$  on the Carlisle Table.

7. An office grants whole-life assurances at weekly premiums which terminate when the lives assured attain the age of 65 years, provided the policies have then been 20 years in force or so soon after attaining age 65 as 20 full years' premiums have been paid. The policies provide that 20 per-cent of the premiums paid shall be added to the original sums assured when they become claims. Deduce practicable formulæ for the sums assured to be granted at age  $x$  next birthday per weekly premium of 6*d.*, showing them separately for  $x < 46$  and  $x > 45$ .

You may assume that the office premiums are loaded with a percentage  $k$ , and a constant  $c$ , and that the entries are equally spread over years of life.

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW  
(PART III.—SECTION A).

*First Paper.*

1. Describe a method of obtaining select rates of mortality from the experience of a life assurance company, and show how you would obtain aggregate rates of mortality. State how you would deal with the following cases, mentioning what date you select for termination in case (3).

- (1) Entered 15 January 1917, withdrew 15 November 1917, born 17 July 1877.
- (2) Entered 15 January 1917, died 15 November 1917, born 17 January 1877.
- (3) Entered 15 July 1917, existing on date when observations terminated, born 17 January 1877.

NOTE.—Symbols, if used, must be fully defined.

2. Define the “standard death rate.” Discuss the limitations of the standard death rate as a method of comparison of the mortality of various places, giving a hypothetical example for the four ages 30, 45, 60 and 75 to illustrate your argument.

3. The rules of a friendly society provide (1) for 5 months’ full benefit and reduced benefits thereafter, and (2) that illnesses not separated by a minimum period of 10 months are treated as continuous in interpreting (1).

A man born 8 July 1884 had the following illnesses :

Fell Ill.	Recovered.
12 June 1909	15 November 1909
30 March 1910	30 May 1910
20 November 1910	1 May 1911
15 June 1912	13 July 1912

Use this example to show how illnesses would be treated in getting out rates of sickness from the experience of the friendly society, stating clearly how your assumed ages, durations, &c., would be calculated and how the rates of sickness would be obtained.

4. What mortality tables have been published based on the 1911 Census, and what are their distinctive features? Give a short account of the methods of construction adopted in respect of any one of these tables.

5. The rates of mortality in the following table are to be graduated. Show how to apply a summation formula, giving formula and columns, and obtain one graduated rate. What would

be the effect of successive summing in groups without making any adjustment?

Age	Rate per 10,000	Age	Rate per 10,000
25	62	36	77
26	52	37	66
27	47	38	84
28	40	39	65
29	85	40	101
30	47	41	73
31	33	42	65
32	56	43	84
33	60	44	38
34	43	45	100
35	70	46	114

*Second Paper.*

1. What were the methods employed by Dr. Sprague in constructing his marriage and mortality tables, and what alternative method can be adopted in constructing tables subject to double or treble decrements?

2. Give a brief account of the graphic method of graduation. In what circumstances would you advocate its use, and why?

What are its main disadvantages?

3. Give a short account of the Manchester Unity Sickness and Mortality Experience, 1893-1897, and indicate its distinctive features. Explain why, in calculating the "exposed to risk of sickness", the persons dying in each year of age were assumed to be at risk until the middle of the year.

\*4. The following is a summary of the claim experience of a friendly society in respect of a certain disease:

Ages	Duration of Claims in weeks	Number of Years of Exposure to Risk
20-24	1,112	16,200
25-29	1,908	46,800
30-34	1,175	37,440
35-39	1,125	28,440
40-44	760	27,540
45-49	380	14,040
50-54	40	6,660
55-59	...	2,880
6,800		180,000

Obtain by the graphic method the curve representing the rate of sickness.

†5. How would you make a general comparison between two tables of mortality? Illustrate your method so far as possible from the  $H^M$  and  $O^{[NM]}$  Tables.

6. Which of the graduations given below do you prefer, and why? Can you suggest what methods of graduation were used?

Age Groups	Exposed to Risk	Deaths	RATES OF MORTALITY		
			Ungraduated	Graduated	
				I	II
1st	500	8	·016	·0200	·0240
2nd	1,000	21	·021	·0200	·0190
3rd	4,000	80	·020	·0182	·0150
4th	13,000	143	·011	·0125	·0120
5th	20,000	160	·008	·0096	·0099
6th	30,000	300	·010	·0097	·0098
7th	32,000	320	·010	·0100	·0101
8th	32,000	384	·012	·0108	·0105
9th	30,000	330	·011	·0112	·0112
10th	21,000	252	·012	·0122	·0122
11th	17,000	238	·014	·0130	·0135
12th	7,000	77	·011	·0138	·0151
13th	3,000	45	·015	·0152	·0169
14th	1,500	30	·020	·0195	·0192
15th	500	11	·022	·0230	·0217

7. To what extent may osculatory interpolation be made use of in the construction of life tables? To what class of table and to what function would you apply the method and what advantage does it give?

\* Sheets of cross-ruled paper will be supplied for use in answering this question.

† "A Short Collection of Actuarial Tables" will be supplied for use in answering this question.

## EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW

### PART III.—SECTION B.

#### *First Paper.*

1. It is desired to obtain by means of Lidstone's method the expected claims and strain among endowment assurances in a year. Give the various headings to the columns in the schedule you would employ to obtain this, indicating shortly how the figures in these columns should be used.

2. As a method of valuing endowment assurances it has been suggested that the same age at maturity should be assumed for all policies and that net premiums and other valuation factors should be based on that assumed age at maturity.

What processes would you employ in applying the suggested method to the valuation of the with profit business of a company, so as to eliminate, as far as possible, likely errors?

What do you consider the advantages of and the objections to the use of the method?

3. The investigation into the incidence of the premium income of a life assurance office shows the undernoted results. What annuity-values would you use in consequence of these results in valuing (a) whole-life and endowment assurance policies classified in groups, and (b) policies which have to be valued individually?

*Distribution of renewal premium income.*

Premiums due	Amount	Premiums due	Amount
	£		£
15 January ... ..	50	15 July... ..	60
15 February ... ..	50	15 August ... ..	60
15 March ... ..	60	15 September ... ..	90
15 April ... ..	50	15 October ... ..	100
15 May ... ..	50	15 November ... ..	130
15 June ... ..	70	15 December ... ..	230
			£1,000

4. A life office with a discounted bonus scheme declares a bonus less than the rate of bonus discounted. How should policies under that scheme be treated, and how would you calculate the requisite increase in premium if the policyholders have an option of paying an increased premium in place of having the sum assured reduced? Differentiate if necessary between whole-life and endowment assurance policies and discuss what difference it would make if the bonus discounted is a compound reversionary bonus in place of a simple reversionary bonus.

5. Describe the various methods commonly employed to prevent forfeiture of policies on the non-payment of the premiums, and state how you would deal with such policies at a valuation.

6. Draft the headings for a class book and explain in detail the method of valuation you would adopt in dealing with a large number of immediate annuities, some payable half-yearly and some quarterly, most of the annuities being apportionable but some being non-apportionable.

7. An office has issued a number of endowment assurance policies providing for cessation of premiums during invalidity. How would you deal with these in a valuation?

*Second Paper.*

1. An office has for a number of years past been valuing on the O<sup>M</sup> 3 per-cent Table and has been distributing profits by a modified contribution method, surplus interest being divided in proportion to the reserves held at beginning of quinquennium and the balance of profit being divided in proportion to ordinary premiums paid during quinquennium. In order to meet a heavy depreciation in securities the valuation basis is changed to O<sup>M</sup> 3½ per-cent. Losses have been sustained owing to war mortality, but apart from depreciation the trading profits of the quinquennium have been normal, the loss on mortality having been made up from increased interest and profit on lapses and surrenders.

Explain how would you deal with the surplus disclosed on valuation, giving reasons for your decisions.

2. In valuing the liabilities for non-profit policies, an office charging premiums that are lower at most ages than the valuation premiums takes credit for full office premiums. It does a large amount of non-profit business for several years and then its new business falls off, the decrease being entirely on the non-profit business. What will be the effect on the surpluses?

3. What objects should be kept in view in devising a continuous method of classification?

Describe in detail the working of the method which you recommend, and state your views as to the desirability of abandoning valuation class-books in favour of the card system of classification.

4. Explain concisely how you would value the following special classes of policies :

(a) Double endowment assurances :

(b) Endowment assurances carrying a guaranteed bonus of  $x$  per-cent of the office premiums paid, accumulated at compound interest at the valuation rate and payable as a reversionary addition to the sum assured.

If alternative methods are suggested, state which of the methods you prefer, and on what grounds.

5. An old-established industrial company has on its books a large number of policies effected at weekly premiums on the lives of infants, the sum assured increasing on the usual sliding scale both with the age at entry and the duration, until at age 11 the full sum assured under the ordinary adult table is payable. On what bases would you calculate the reserves for such policies in practice before, and after, the attainment of age 11 by the child?



6. An office making annual valuations wishes to compare the actual with the expected claims and strain.

Describe the method you would adopt in practice in making such an investigation and indicate to what extent your results would be approximate and the direction of the error involved.

EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW  
(PART IV.—SECTION A).

*First Paper.*

1. What are the special provisions in the Assurance Companies Act, 1909, as to policies issued or to be issued by collecting societies or industrial assurance companies?

2. Describe briefly the returns under Schedules I, II and III of the Assurance Companies Act, 1909, which have to be prepared by a life assurance company.

3. What particulars have now to be furnished to the Board of Trade in respect of employers' liability business?

State in what manner the liabilities of a company are affected by the recent change in the Government rates of life annuities.

4. It is desired to transfer to a proprietary fire and life office the business of a proprietary company transacting life assurance and employers' liability insurance.

State clearly the procedure to be adopted :

- (i) Before the transfer ; and
- (ii) After completion of the transfer.

Is it now possible, and if so under what conditions, for the business of one insurance company to be acquired by another without the Court's consent?

5. State the present position of a registered friendly society with reference to :

- (a) Liability for income tax ;
- (b) The limiting amount for which it may issue assurance and annuity contracts ;
- (c) The selection of new investments.

In what circumstances may a friendly society be converted into a mutual assurance company?

6. In the case of policies with limited premiums, it is generally a condition that in lieu of surrender value a proportionate paid-up policy may be granted, depending on the relation between the number of premiums paid and the maximum number of premiums payable. Discuss, from an actuarial standpoint, whether this practice is justified. What considerations have in practice to be borne in mind when quotations for surrender value are given?

7. In what respects do you consider that the Assurance Companies Act, 1909, so far as it is concerned with life assurance companies, might advantageously be amended? Deal specially with the provisions of the Act in reference to:

- (a) Foreign companies transacting business in this country;
- (b) Industrial assurance companies;
- (c) Amalgamations and transfers;
- (d) Annual and valuation returns as prescribed in the schedules.

8. State the provisions of the Friendly Societies Act, 1896, with reference to the preparation and filing of

- (1) Annual returns, and (2) quinquennial valuation reports or returns.

Give an outline of the information required to be stated by the valuer in his valuation report.

### *Second Paper.*

\*1. Under the will of his grandfather, A, aged 40, is entitled as tenant in tail, expectant on the death of his father B, aged 65, to a certain estate. A has created a base fee and covenanted to enlarge into a fee simple on coming into the estate. He wishes to raise £3,000 to be secured by a reversionary charge.

The estate consists of 3,000 acres, with a rental of £2,600 and outgo for tithe, rates and land tax of £280, and was valued 10 years ago at £60,000. B has appointed a jointure in favour of his wife, aged 62, of £700 a year, and A has already granted reversionary charges amounting to £18,000.

Value A's interest, and state whether and on what conditions you would grant him the £3,000 he requires.

\*2. Apportion the following fund between the life tenant, a male aged 63, and the reversioner.

What shares should be allotted to each:

- (a) On the assumption that both desire the partition;
- (b) If the partition is at the request of the life tenant?

*Fund* : £2,000 5 per-cent War Bonds, 1927 ;

£150 Madras Railway B Annuity, 1956 :

£2,000 Canada 3 per-cent Stock, 1938 ;

Leasehold premises, having 60 years to run, let at  
£150 per annum, with ground rent £20 per annum.

\*3. A, a widow aged 65, is entitled to a life interest in one-fourth of the following funds, determinable on remarriage :

£5,000 Great Western Railway 5 per-cent Irredeemable  
Debenture Stock ;

£2,000 Midland Railway  $2\frac{1}{2}$  per-cent Debenture Stock ;

£3,000 Great Eastern Railway 4 per-cent Debenture Stock.

B is entitled to an interest in another one-fourth during A's life or until her remarriage, and an absolute reversion expectant on A's death or remarriage to one-half of the funds.

Value A's and B's interests separately, and say how much you would advise lending to them jointly ?

State clearly what policies, if any, you would consider it necessary to effect.

4. What tables are generally used in the practical valuation of reversions and life interests ? Say which you would employ at the present time and give reasons. What circumstances or special features would influence you in any particular transaction in arriving at the rate of interest to be employed ?

Give a formula for valuing a contingent life interest to X after the death of Y, and say what class of policy would be required.

\*5. A sum of money is required on security of a reversionary life interest commencing at the death of a female aged 71, the funds consisting of trustee securities, valued at the present time at £50,000 and producing an income of £2,250 per annum gross.

The reversioner, a male aged 46, is now uninsurable, but offers existing whole life without profit policies, already in force, for £25,000 at annual premiums amounting to £1,100 as part security.

Say whether you would deal with such an application, and if so, whether by way of loan or otherwise, and why. Discuss the limits of such an advance and state in what directions you would require to be satisfied before completing the transaction.

\*6. A.B., aged 25, is entitled to :

- (i) The life interest in the income from the undermentioned fund during the joint lifetime of his mother, aged 60, and himself ; and
- (ii) The contingent reversion to the fund expectant on his mother's death, if he be then living.

What is the market value of A.B.'s interests, if offered for sale together, and what amount and description of assurance should be effected ?

*Fund* : £5,000 5 per-cent War Loan, 1929-47 ;

£3,000 Mortgage of Freeholds in the City of London, carrying interest at  $5\frac{1}{2}$  per-cent per annum.

\* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

## EXAMINATIONS, JUNE 1920.

### BOARD OF EXAMINERS.

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### EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE

#### (PART I.—SECTION A).

#### *First Paper.*

1. During a railway strike three men, living in the same street, set out for the Bank. A walks at 4 miles an hour, B gets a lift in a cart for part of the distance and walks the rest at 2 miles an hour, and C bicycles at 10 miles an hour. They start at equal intervals and arrive together. B passes A when he has gone two-thirds of the way, 5 minutes after C has left home.

Find the distance they have to go, the speed of the cart, and how far it took B.

2. Solve the equations

$$(1) \quad x^2 + y^2 = 5$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$$

$$(2) \quad x^2 - yz = \frac{b-c}{x}$$

$$y^2 - xz = \frac{c-a}{y}$$

$$z^2 - xy = \frac{a-b}{z}$$

3. Prove that

$$\left[ 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \text{ to } \infty \right]^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots \text{ to } \infty$$

for all values of  $x$ .

$$\text{Find the sum of } \frac{1^3}{1} + \frac{2^3}{2} + \frac{3^3}{3} + \dots \text{ to } \infty$$

4. Prove that  $(N^3 + x)^{\frac{1}{3}} = N + \frac{xN}{3N^3 + x}$  approximately, if  $x$  is small compared with  $N^3$ .

In what place of decimals will  $(1001)^{\frac{1}{3}}$  differ from  $10 + \frac{10}{3001}$ ?

5. If

$$f(x) \equiv x^5 - \frac{a^5(x-b)(x-c)}{(a-b)(a-c)} - \frac{b^5(x-c)(x-a)}{(b-c)(b-a)} - \frac{c^5(x-a)(x-b)}{(c-a)(c-b)}$$

show that  $(x-a)(x-b)(x-c)$  are factors of  $f(x)$  and find the quotient when  $f(x)$  is divided by  $(x-a)(x-b)(x-c)$ .

6. Find whether the series whose general terms are as follows are convergent or divergent:

$$(1) \quad \sqrt[n^2+1]{n^2+1} - \sqrt[n^2-1]{n^2-1}$$

$$(2) \quad \frac{nx^n}{n^2+1}$$

$$(3) \quad \frac{x^n}{n^2-y}$$

7. A purse contains four half-crowns, three pennies and two shillings. Four coins are drawn at random. How many different sums can these amount to, and what is the most probable sum?

8. A and B play a match of 5 games. A's chances of winning, drawing and losing any game are in proportion to 3, 2 and 1 respectively; 2 points are scored for a win and 1 for a draw. What is the chance that the match is drawn?

9. A, B, C and D each throw two dice for a prize. The highest throw wins. If equal highest throws are thrown by two or more players those players continue the contest. A throws 10. What is his chance of winning the prize?

### Second Paper.

1. Given  $u_0 = 1,876$ ,  $u_1 = 777$ ,  $u_3 = 19$ , and  $u_6 = -218$ , interpolate the values of  $u_2$ ,  $u_4$ , and  $u_5$ , and find the values of  $x$  for which  $u_x$  is a maximum or minimum.

2. Prove that

$$u_{x:y} = u_{0:0} + x\Delta_x u_{0:0} + y\Delta_y u_{0:0} + \frac{x(x-1)}{2} \Delta_x^2 u_{0:0} + \frac{y(y-1)}{2} \Delta_y^2 u_{0:0} + xy\Delta_x\Delta_y u_{0:0} \text{ approximately.}$$

Given  $u_{0:0}$ ,  $u_{0:1}$ ,  $u_{0:2}$ ,  $u_{1:0}$ ,  $u_{1:1}$ ,  $u_{2:0}$ , can you suggest another method of approximating to  $u_{x:y}$ , not necessarily using all the data?

3. Establish the relation between the operators  $\frac{d}{dx}$  and  $\Delta$ .

$$\text{Show that } \frac{du_x}{dx} = \frac{1}{h}(u_{x+h} - u_{x-h}) - \frac{1}{2h}(u_{x+2h} - u_{x-2h}) + \frac{1}{3h}(u_{x+3h} - u_{x-3h}) - \dots$$

4. Find  $\frac{d^2u}{dx^2}$  in terms of differentials with regard to  $x$ ;  $u$  and  $v$  being functions of  $x$ . Find the second differential coefficient of  $x + \sqrt{1+x^2}$  with regard to  $x - \sqrt{1+x^2}$ .

5. Prove that the first three terms of the expansion of  $(1+x)^{\frac{1}{x}}$  in powers of  $x$  are  $e - \frac{ex}{2} + \frac{11}{24}ex^2$ .



6. Evaluate (1)  $\int \sqrt{x^2 + 2ax} dx$

(2)  $\int \frac{x^3 dx}{(a^2 - x^2)^{\frac{3}{2}}}$

(3)  $\int \frac{\log x dx}{(1-x)^2}$

7. Find the value of  $\frac{d}{dt} \int_0^{\log t} e^{tx} dx$ , where  $t$  is independent of  $x$ .

8. Explain the meaning of the expression  $\int_a^b f(x) dx$ .

In a certain industry, the number of men employed decreases at a rate proportional at any moment to the number employed at that moment, and the output per man increases at a constant rate. Find an expression for the total output in ten years, if the numbers employed at the beginning and end of the period are  $N_0$  and  $N_{10}$  respectively, and the output per man during the first and tenth years  $A_1$  and  $A_{10}$  respectively.

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART I.—SECTION B).

“A Short Collection of Actuarial Tables” will be supplied for use in answering this Paper.

1. Explain what is meant by an effective rate of interest and a force of interest.

The amount of £100,000 at the end of two years at a certain rate of interest convertible momentarily exceeds the amount at the same rate convertible yearly by £96. Find the rate of interest.

2. If the rate of interest varies from year to year so that the rate  $i_n$  for the  $n$ th year ( $n > 1$ ) is given by  $\frac{1}{1+i_n} = \frac{1}{1+i_1} \cdot \frac{n-1}{n+1}$  find the value of a perpetuity in terms of  $i_1$ .

3. A loan of £1,000 is granted for a term of five years, to be repaid by an annuity increasing £10 per annum, of such amount that the lender will receive interest at the rate of 6 per-cent per annum on his whole capital for the term and be able to accumulate a sinking fund at 3 per-cent per annum.

Find the amount of the payment for the first year and draw up a schedule dividing the payments into principal and interest.

4. A corporation issues a loan bearing interest at 3 per-cent per annum for the first ten years, 4 per-cent per annum for the second ten years, and 5 per-cent per annum thereafter, payable annually. The loan is redeemable at par by means of a cumulative sinking fund of 2 per-cent per annum. Find the price of issue to yield  $4\frac{1}{2}$  per-cent per annum to an investor taking up the whole loan.

5. A sum of £10,000 is being repaid in 20 years by means of a sinking fund calculated on a  $3\frac{1}{2}$  per-cent basis.

At the end of the 10th year it is found that owing to depreciation the fund produced by the investment of the annual sinking fund payments is only worth as much as would have been produced if the payments had been accumulated at 3 per-cent per annum.

Find the half-yearly payment which must be paid into the fund for the remainder of the term on the assumption that the future rate of interest earned will be 4 per-cent per annum.

6. An investor bought an annuity of £100 per annum payable yearly for 20 years for £1,100. To replace his capital he effected at the same time a sinking fund policy maturing at the end of the term. If the premium for this policy was calculated on the basis of interest at the rate of  $3\frac{1}{2}$  per-cent per annum for the first 10 years and 3 per-cent per annum thereafter, what rate of interest did he earn on his invested capital?

7. The following two issues of stock are made :

- (a) At 78 bearing interest at 3 per-cent per annum payable yearly, redeemable at the end of 20 years at par ;
- (b) At 98 bearing interest at  $4\frac{1}{2}$  per-cent per annum payable half-yearly, redeemable at the end of  $n$  years at par.

If the yield to an investor under the two issues is the same find the value of  $n$ .

## EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE

### (PART II).

#### *First Paper*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Define "Rate of Mortality" and "Force of Mortality."

Under what conditions would the "Rate of Mortality" be equal to the "Force of Mortality"?

2. If  $L_0 = 4,460$

$$L_1 = 4,064$$

$$L_2 = 3,892$$

find  $l_0$ ,  $l_1$ , and  $l_2$ , as accurately as possible.

3. A stationary community containing 20,000,000 males supports an army in which the total number of regulars and reservists is 600,000. Recruits enter at exact age 20 and are passed into the reserve on attaining age 25. There is no discharge from the army except discharge from the reserve on attaining age 35. On the basis of the  $H^M$  Table of Mortality, what proportion of the men who attain age 20 are called up each year, and what is the strength of the regulars and reservists respectively?

4. An office grants pensions of two-thirds of the final salary to those of its staff who reach the age of 65. The pensions are payable annually and the first payment is due at age 66.

It is proposed to introduce a new scheme under which the pension granted at 65 will be reduced to one-half the final salary; a gratuity of one and a half year's final salary will be paid in addition at retirement; while to the representatives of those who die in service a year's salary will be paid.

Assuming that all members of the staff enter at age 21, that promotion from grade to grade is automatic, that the scale of salary is

£100 by £10 per annum to £200

£200 „ £15 „ £500

£500 „ £20 „ £600

and remains at £600 till retirement, that there are no withdrawals, and that the mortality follows the  $H^M$  Table, find the difference in the ultimate annual charge on the funds of the office in the two cases for a staff of 1,000.

5. Find an expression for the probability that  $x$  will survive to the end of the  $n$ th year, and that the joint lives of  $y$  and  $z$  will have failed at least  $t$  years previously, the failure being caused by the death of  $y(n > t)$ .

If  $x$ ,  $y$  and  $z$  are aged 60, 86 and 86 respectively, what is the probability by the  $H^M$  Table when  $n = 20$  and  $t = 5$ .

6. Define  $Q_{x:y:\overline{t}|}^1$ , and show that it is equal to

$$1 - {}_t p_x (1 - Q_{x+t:y}^1).$$

7. If  $\mu_x = A \log x$ , find an expression for  $l_x$ , and calculate the value of  $\mu_{20}$ , being given that  $\log_{10}(5p_{20}) = 1.9862$  and that  $\log_{10} e = .4343$ .

## Second Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Write down the symbol for the value of a complete annuity, payable  $m$  times a year, on the life of  $x$ , to begin to accrue immediately on the death of  $y$ , and prove that the value of such an annuity is equal to

$$\frac{1}{v^{2m}}(a_x^{(m)} - a_{xy}^{(m)}).$$

2. If Makeham's law holds, show that

$$\bar{A}_{xy}^1 = -\log s \cdot \bar{a}_{xy} + (\mu_x + \log s) \bar{a}'_{xy}$$

where  $\bar{A}_{xy}^1$  and  $\bar{a}_{xy}$  are calculated at a rate of interest  $i$ , and  $\bar{a}'_{xy}$  is calculated at a rate  $j$  such that  $1+j = \frac{1+i}{c}$ .

3. Give a formula for the value of  $A_{x,y,z}^1$ . If an annual premium were required for this assurance, state how you would calculate it. During what status would you make it payable, and why?

4. Prove that

$${}_nV_x^{(m)} = {}_nV_x \left\{ 1 + \frac{m-1}{2m} P_x^{(m)} \right\}.$$

5. Given a complete table of values of net annual premiums for whole-life assurances and the rate of interest, how would you calculate a table of values of  $l_x$ ?

6. Express in the form of integrals

(i) the probability that  $x$ ,  $y$  and  $z$  will all die within  $n$  years in the order named,  $x$  within 1 year of  $y$  and  $y$  within 1 year of  $z$ ;

(ii) the probability that  $x$ ,  $y$  and  $z$  all die within  $n$  years, at least  $m$  years separating each death ( $n > 2m$ ).

7. A, B, C and D all aged 80 have an annuity of 1 per annum payable until the death of the last survivor of them. They agree to toss every two years, the winner to receive the annuity payments for the ensuing two years if he lives, or, if he dies during the period the survivors toss again in respect of the one or two remaining payments. A has just won the toss for the first two years. What are the values by the H<sup>M</sup> 3 per-cent Table of his expectation under the annuity and that of any one of the others?

*Third Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. A man now aged 40 effected 5 years ago a whole-life assurance for £500, the number of annual premiums being limited to a maximum of 20. He now desires to alter it to an endowment assurance maturing at age 55 for a reduced amount at the same annual premium. Find the amount of the endowment assurance on the basis of the  $H^M$  3 per-cent Table and net premiums.

2. Find by the  $O^{(NM)}$  3 per-cent Table the net annual premium limited to a maximum of 10 payments for a whole-life assurance for £1,000 on a life aged 30 with the proviso that interest at the rate of 4 per-cent per annum on the total amount of the premiums paid is to be paid at the end of each year he survives.

3. A company grants an annuity of £100 per annum payable by quarterly instalments to a life aged 60 with the special condition that if the life dies within 4 years from the granting of the annuity the company will on payment of £200 grant a similar annuity to another life aged 60 at the date of death of the annuitant with the same special condition and so on indefinitely. What should be the purchase money for the annuity by the  $H^M$  3 per-cent Table?

4. Find the value by the  $O^{(NM)}$  3 per-cent Table of an annuity of £400 per annum payable half-yearly for 20 years certain, the half-yearly payment to be reduced to £150 after the first death and £100 after the second death of two lives aged 30 and 40 if such deaths occur during the term of the annuity.

5. If in a special mortality table the number living at age  $40+x$  is less by  $100x$  than the number living at the same age in the  $H^M$  Table for all values of  $x$  from 0 to 10, find the net annual premium by the special table at 3 per-cent interest for a 10-year endowment assurance effected at age 40.

6. Find the single premium with a loading of  $7\frac{1}{2}$  per-cent for a contingent survivorship policy payable if a life aged 30 dies before a life aged 60, or within 5 years of the death of the latter, with the return of the single premium without interest 5 years after the death of 60 if no claim under the policy arises by the death of 30.

Use the Carlisle 3 per-cent Table.

EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW  
(PART III.—SECTION A).

*First Paper.*

Write an essay on *each* of the two following subjects :

(1) The Institute of Actuaries proposes to collect a new mortality experience of life offices. Draft a report to the Council advising :—

- (a) What data should be collected.
- (b) In what form the offices should be asked to supply the data.
- (c) What groups should be investigated and what life tables should be prepared from the data.
- (d) By what methods the data should be analyzed and collated in order to obtain rates of mortality.

[Graduation need not be discussed.]

(2) A National census is about to be taken.

- (a) What information should be obtained for use in connection with subsequent investigations into mortality, marriage and fertility?
- (b) What statistical tables would you deduce from the data so collected?

*Second Paper.*

1. Calculate from the following data, which represent the experience of a calendar year, the ungraduated rate of mortality  $(bdq)_x$ , and the probability of marrying in a year  $(bm q)_x$  among bachelors, at age 25 :

Age $x$	New Entrants $n_x$	Surviving Entrants $s_x$	Withdrawals $w_x$	Existing $e_x$	Bachelors' Deaths $d_x$	Marriages $m_x$
18	2	1	...	...	...	...
19	10	3	...	3	...	...
20	24	6	1	8	...	1
21	32	11	1	14	...	2
22	34	16	3	20	1	7
23	38	22	2	25	...	5
24	31	26	...	29	...	10
25	22	30	2	34	2	12
26	14	32	4	37	...	12
27	15	34	5	44	1	8

The ages in each case are "nearest" ages.



2. Given the numbers living at each age according to the Census and also the births and deaths at each age for a number of years previous to the Census date, show how to find the rates of mortality for ages 0-4 inclusive.

What difficulties may arise in connection with your method and how would you deal with them?

3. What second difference error is involved in the following summation formula :

$$\frac{[3][5][5]}{150}(-u_2 + 4u_0 - u_{-2})$$

Retaining the same number of summations, find a formula correct to third differences and express it in terms of the values of the function to be graduated.

4. It is required to graduate the following experience of a special class of lives :

Age Group	Exposed to Risk	Deaths	$q_x$
20-24	2,847	10	·0035
25-29	2,913	14	·0036
30-34	4,760	20	·0042
35-39	5,427	39	·0072
40-44	5,331	42	·0079
45-49	4,476	43	·0096
50-54	2,964	48	·0162
55-59	1,404	26	·0185
60-64	765	22	·0288
65-69	307	16	·0521

Give your reasons for and against adopting :

- The graphic method ;
- A summation formula ;
- Makeham's Law ;
- Mr. King's osculatory method.

State what method of graduation you would employ and give reasons for your decision, indicating the function or functions upon which you would operate.

You may assume that the data for individual ages are available.

5. An old-established provident society consisting of over 50,000 members grants the following benefits, subject to six months' membership :

- A weekly allowance during temporary disablement from work, not exceeding 52 weeks ;

- (b) A lump sum payment on death of member ;
- (c) A weekly pension on retirement after attainment of age 65, or after age 60 if incapacitated for further duty ;
- (d) A weekly pension of one-half the amount payable under (c) after 20 years' membership if incapacitated for duty before age 60.

Membership of the society is compulsory for all employees.

It is desired to deduce from the experience of the society over a period of 5 years the following unadjusted central rates as functions of the members living at each age :

- (1) Mortality ;
- (2) Secession ;
- (3) Weeks of temporary disablement per member per annum ;
- (4) „ „ full pension per member per annum ;
- (5) „ „ half „ „ „

State precisely the particulars you would require to enable you to carry out the investigation and describe in detail, giving formula, how you would obtain the various rates.

How would you construct the combined mortality and secession table from the adjusted central rates ?

6. Into what groups were the data divided for the purpose of investigating the mortality and sickness respectively of the Manchester Unity Experience, 1893/7 ?

State how the "exposed to risk" of sickness used in that experience differed from that used in previous investigations, and show how values of  $K_x$  can be calculated from the rates of sickness given by the 1893/7 experience.

What important point had to be borne in mind in connection with sickness current at the beginning of the experience ?

## EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW (PART III.—SECTION B.)

### *First Paper.*

Write an essay on *each* of the two following subjects :

- (1) The net premium method of valuation.

*Its origin, development, advantages and disadvantages.*

- (2) The various methods of distributing surplus.

*Their advantages and disadvantages and their relation to the valuation basis.*

*Second Paper.*

1. From the following facts, complete the Revenue Account, ascertain the surplus at the end of the year, and analyze the profit earned during the year into its component parts, showing how much is derived from mortality (including minor sources of profit), interest, loading and surrenders. What are the usual minor sources of profit included under the heading "Mortality" ?

	£
Reserves at beginning of year (Old 3 per-cent valuation) ...	8,629,275
Reserves at end of year ... ..	8,814,629
Surplus brought forward from previous year's valuation ...	27,290

*Revenue Account—Items.*

	£
*Premiums ... ..	792,481
Con sideration for Annuities ... ..	2,409
Claims—by Death ... ..	532,357
„ by Maturity ... ..	207,404
Annuities paid ... ..	8,185
Surrenders ... ..	28,314
Commission and expenses ... ..	131,939

\* The corresponding net premiums may be assumed to be 80 per-cent of the office premiums.

NOTE. (1) The surrender values may be assumed to be 70 per-cent of the reserves ; (2) the average rate of interest earned during the year was £3. 19s. per-cent, after deduction of income tax.

2. During and since the war conditions of employers' liability insurance business have changed in the following respects :—

- (1) By arrangement with the Government insurance companies have agreed to cover ex-soldiers at ordinary rates whatever the state of health or disability may be on resumption of employment ;
- (2) The rates of compensation have been materially increased.

Would you make any allowance for these factors, and if so, in what manner, in valuing the liabilities of an old-established business ?

3. An office transacting ordinary business finds it necessary to make some allowance in its valuation for the heavy initial cost of new business. What methods have been proposed for doing this ? Which method do you prefer, and why ?

4. In 1915 a large mutual life office valuing on an  $OM\ 2\frac{1}{2}$  per-cent basis notwithstanding the war paid its customary reversionary bonus, being assisted by the large amount of undivided profit carried forward at the preceding valuation. Since 1915 it has done a large and rapidly increasing new business. It reduced its non-profit rates in 1916 and has done about 50 per-cent of its new business under non-profit tables. Discuss the probable position at the time of the next valuation in 1920 with special reference to the surplus and the source from which it will be derived.

5. How would you value the following policies, assuming in each case that the class is a large one? Point out any important objections to the method suggested.

- (1) Survivorship assurances;
- (2) Deferred annuities without return of premium on death before the deferred age (the answer need only refer to valuation during the period of deferment);
- (3) Double endowment assurances.

6. Describe in detail how you would classify and value the joint life policies at weekly premiums of a large industrial insurance company which issues a great number of such policies.

#### EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW

##### (PART IV.—SECTION A).

##### *First Paper.*

Write an essay on any *two* of the three following subjects:

- (1) The Life Assurance Companies Act, 1870, and the Assurance Companies Act, 1909, as affecting Life Assurance Companies, including:—
  - (a) *The reasons for the introduction of special legislation affecting the business of life assurance.*
  - (b) *The main provisions of and the reasons for the 1909 Act.*
  - (c) *The principal alterations in the then existing law made by this Act.*
  - (d) *What further amendments, if any, should now be made?*

## (2) The Friendly Societies Acts, including :—

- (a) *The duties and privileges of registered societies.*
- (b) *Annual and quinquennial returns, including a resumé of the official forms and instructions to public valuers.*
- (c) *Investment of funds.*
- (d) *Payments on death ; nomination and revocation.*
- (e) *Limitation of benefits.*
- (f) *Amalgamation, conversion and transfer.*

## (3) Reversionary securities as investments :—

*Their advantages and disadvantages.*

*Second Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper. If the Candidate considers that the Tables furnished are not suitable for the purpose of valuation, he should indicate the basis he would adopt and use the Tables furnished for the purpose of approximating thereto.

1. Discuss the general principles on which surrender values should be determined, and suggest formulæ for the calculation of surrender values of with and without profit policies in the following classes :

- (a) Whole life assurances. (Premiums payable throughout life.)
- (b) Whole life assurances. (Limited premiums.)
- (c) Endowment assurances.

(NOTE.—In answering this question you may make your own assumptions as to any special features of the life office in question which may influence your answer.)

2. A, the vendor, aged 26, whose life is assurable subject to an addition of 10 years to the age at a non-profit whole life annual premium of £2. 7s. 0d. per cent, is entitled to a reversionary life interest of £30,000 per annum and to a capital sum of £100,000 (contingent) upon the death of B, aged 30 last birthday. B's health is very much impaired and on medical evidence his age should be rated up 25 years.

A has agreed to sell his interests to a private individual for the following consideration :

- (a) An annuity of £1,000 per annum, free of income tax, for the remainder of the vendor's lifetime ; and

- (b) A cash payment of £60,000, A having the right to repurchase at any time within 10 years for the sum of £450,000.

A subsequently endeavoured to set aside the agreement on the grounds that the terms were harsh, and in particular that the repurchase price was exorbitant.

You are consulted by the purchaser with a view to justifying the transaction. State fully your opinion and mention what practical considerations arise in view of the impairment of health of the life tenant and reversioner.

3. A lady, aged 52, having a husband, aged 56, is entitled to a life interest, forfeitable upon remarriage, in the following funds :

£80 G. I. P. Railway Annuity, Class " A " (1948) ;  
 £100 East Indian Railway Annuity, Class " A " (1953) ;  
 £100 Madras Railway Annuity, Class " A " (1956) ;  
 £200 Scinde Punjab Railway Annuity, Class " A " (1958) ;  
 £5,000 H. H. the Nizam's State Railway 4 per-cent Mortgage Debenture Stock (irredeemable).

State (1) what loan you consider could safely be made, giving full particulars of any policies required, and (2) What larger sum, if any, could be advanced, if two children of the life tenant, aged respectively 23 and 25, entitled to the absolute reversion to one-half each at death or remarriage of their mother, joined in the transaction. What observations have you to make on the nature of the funds ?

4. At her death A was absolutely entitled in reversion to a sum of £6,000 Consols expectant on the death of her mother, B, aged 65. The value is required for probate and the list shows the price of Consols as  $50\frac{1}{2}$ -51 on the date of A's death. Value this reversion for probate and explain what courses are open to A's executors in regard to the payment of Estate Duty thereon.

5. If A, a male aged 60, should predecease B, a lady aged 80, C's estate will be entitled to one-fourth of the following fund, namely :

£2,500  $3\frac{1}{2}$  per-cent War Loan (1925-8) ;  
 £3,000 4 per-cent *net* War Loan (1929-42) ;  
 £4,500 Union of South Africa 4 per-cent stock (1943-63).

Then, upon the subsequent death of B, C's estate will become entitled to a further one-eighth. If B predecease A, C's estate will immediately be entitled to the whole three-eighths of the fund.

Find the market value of C's interest.



6. A, a lady aged 60, has a life interest in the income from :

400 shares of £5 (£1 paid) in Lloyds Bank now producing £80 per annum and £800 Gas Light and Coke Company Ordinary Stock now producing £24 per annum.

B, who is entitled to the absolute reversion, has these investments standing in his name as sole trustee under a deed of gift, but desires to realize his interest. What difficulties confront a prospective purchaser, and how can they be overcome ?

7. Explain the various ways in which reversionary investments can be treated in the books of a life office. Give your views as to the advisability of setting up covering policies for contingent reversions and also annuities in the books to provide for payment of premiums and interest during the joint lives of the life tenant and reversioner.

## EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW.

### (PART IV.—SECTION C).

#### *First Paper.*

1. State what alterations were made by the National Health Insurance Act, 1918, in the financial basis of National Health Insurance and give a short account of the reasons for these alterations.

What changes were introduced in the pooling arrangements of small societies and why were these changes necessary ?

2. State your opinion generally upon the use of a rate of withdrawal in the valuation of :

(a) A Friendly Collecting Society ;

(b) A Railway Staff Pension Fund.

3. The superannuation fund of a certain authority provides (*inter alia*) the following benefits for the employees :

*Pensions.*—On retirement at age 60 subject to 40 years' service, or at age 65, or earlier in the event of permanent incapacity, one-sixtieth of average salary of the three years prior to retirement for each year of service (maximum, forty-sixtieths).

*On death after retirement.*—A cash payment equal to 3 months' pension.

*On death in the service.*—A cash payment of 1 year's salary.

On the basis of the subjoined extract from the "Service Table" and interest at  $3\frac{1}{2}$  per-cent per annum, calculate the Reserve in respect of an employee now aged 63 who has completed 35 years' service and is in receipt of a salary of £300 per annum. The annual rate of contribution payable to the Fund is 10 per-cent of the salary.

Age $x$	Existing in Service ( $l_x$ )	Deaths ( $d_x$ )	Retirements ( $r_x$ )	Average Salary at Age $x$ ( $s_x$ )	Annuity- value for Pensions ( $a_x$ )	Age $x$
60	4,495	45	854	260	9,557	60
61	3,596	39	1,079	261	9,303	61
62	2,478	31	1,115	262	9,051	62
63	1,332	20	666	263	8,799	63
64	646	13	323	264	8,545	64
65	310	8	302	261	8,291	65
66	...	...	...	...	8,033	66

### Second Paper.

1. A friendly society grants the following benefits :

*Sickness benefits.*—9s. per week for the first 52 weeks ;

5s. per week after 52 weeks for remainder  
of sickness.

*Funeral benefits.*—£10 on death of member ;

£5 on death of member's wife in member's  
lifetime.

The following data are supplied to you for a quinquennial period :

Age Group	MEMBERSHIP		Total Annual Contributions to Benefit Fund at end of quinquennial period	Age Group
	Beginning of period	End of period		
16-30	102	90	£ 78	16-30
31-45	91	97	85	31-45
46-60	50	53	47	46-60
61-75	27	32	28	61-75
76-84	4	5	5	76-84

TOTAL BENEFIT FUNDS		ACTUAL NET INTEREST INCOME		ACTUAL SICKNESS COST		
Beginning of period	£	Year		Year	Full	Reduced
End of			£		£	£
1st year	3,579	1	142	1	115	113
2nd "	3,679	2	150	2	140	119
3rd "	3,713	3	144	3	181	124
4th "	3,791	4	148	4	141	125
5th "	3,876	5	161	5	150	98

Total number of members' deaths in the quinquennium 17.

On the basis of the Manchester Unity, Whole Society, Experience calculate approximately the expected sickness cost and the expected number of deaths for the quinquennial period. Make a valuation as at the end of the period, draw up the valuation balance sheet and show the degree of solvency.

State whether as a result of the approximate investigation into the sickness and mortality experience you would modify the standard valuation factors in any way, and give your reasons.

2. Give a short account of any official investigations into occupational mortality.

To what extent can valid conclusions be drawn from such investigations, and what points have to be borne in mind in interpreting the results?

3. Determine from the following data the approximate values of the mean, median, mode, upper and lower quartiles, and standard deviation.

How can the relative values of mean, median and mode be checked approximately?

Value	Frequency	Value	Frequency
0-1	·011	6-7	·120
1-2	·017	7-8	·163
2-3	·026	8-9	·196
3-4	·038	9-10	·181
4-5	·056	10-11	·093
5-6	·083	11-12	·016

1·000

## PROCEEDINGS OF THE INSTITUTE.—SESSION 1919-1920.

*First Ordinary Meeting, 24 November 1919.*

The President (MR. GEOFFREY MARKS, O.B.E.) in the Chair.

A paper entitled "The Reversionary (or Prospective) and Collective "methods of valuing Widows' Funds, with some notes on the valuation of "the Church of Ireland Widows' and Orphans' Fund", was submitted by Dr. C. E. Howell.

The following gentlemen took part in the discussion:—Messrs. T. Tinner, W. Penman, J. Bacon, A. D. Besant, E. C. Thomas, and the President.

*Second Ordinary Meeting, 2 February 1920.*

The President (MR. GEOFFREY MARKS, O.B.E.) in the Chair.

Resolved:—(1) That the Messenger Legacy Fund be permanently appropriated for educational purposes; (2) That a Gold Medal of the Institute be given to commemorate work of outstanding originality in actuarial science on such occasions as the Council may deem fit.

Messrs. James Tait Campbell, F.F.A., George Hugh Recknell, F.F.A., Walter Carlow Reid, F.F.A., and John Stark Benson Wilson, F.F.A., were elected Associates of the Institute.

A discussion took place on Approximate Group Valuation, introduced by the following papers:—(1) Some further suggestions on the subject of Approximate Valuation, by Mr. Alfred Henry, and (2) A new method of valuing Policies in Groups, by Mr. H. L. Trachtenberg.

The following gentlemen took part in the discussion:—Messrs. R. D. Anderson, R. C. Simmonds, Sir Alfred Watson, W. Palin Elderton, and the President.

*Third Ordinary Meeting, 10 May 1920.*

The President (MR. GEOFFREY MARKS, C.B.E.) in the Chair.

Sir Joseph Barn, K.B.E., opened a discussion on the Actuarial Aspects of Industrial Assurance, with special reference to the Report of the Departmental Committee on the Business of Industrial Assurance Companies and Collecting Societies.

The following gentlemen took part in the discussion:—Messrs. A. D. Besant, C. W. Kenchington, S. G. Warner, J. Murray Laing, C. S. Kelham, H. H. Austin, and the President.

*The Seventy-third Annual General Meeting, 7 June 1920.*

The President (MR. GEOFFREY MARKS, C.B.E.) in the Chair.

The proceedings at the Annual General Meeting will be found on page 230.

## REPORT, 1919-1920.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1919-1920, the seventy-second year of its existence.

There has been an *increase* of 21 in the total number of members, as compared with the previous year. At the end of the official year in

which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-four years later, at 31 March 1909, it was 998. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1910	259	335	348	21	963
1911	267	339	303	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896
1918	295	332	215	18	860
1919	288	330	205	18	841
1920	305	345	197	18	865

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

*Schedule of Membership, 31 March 1920.*

	Fellows	Associates	Students	Corres- ponding Members	Total
i. Number of Members in each class on 31 March 1919 . . .	288	330	205	18	841
ii. Withdrawals by					
(1) Death . . . . .	4	3	3	..	22
(2) Resignation or otherwise . . . . .	...	5	7	..	...
	284	322	195	18	819
iii. Additions to Membership					
(1) By Election . . . . .	...	5	..	..	16
(2) By Examination . . . . .	...	...	32	..	...
(3) By Re-instatement . . . . .	...	3	6	..	...
	284	330	233	18	865
iv. Transfers					
(1) By Examination: from Associates to Fellows . . . . .	18	18	..	..	...
	302	312	233	18	865
(2) By Examination: from Students to Fellows . . . . .	3	...	3	..	...
	305	312	230	18	865
(3) By Examination: from Students to Associates . . . . .	...	33	..	..	...
v. Number of Members in each class on 31 March 1920 . . .	305	345	197	18	865

[Continued on page 226.]





ending 31 March 1920.

1919.				1920.					
£	s.	d.	Journal—	£	s.	d.	£	s.	d.
399	10	5	Printing of Nos. 271, 272 . . . . .	518	19	2			
43	2	6	Editorial Expenses . . . . .	43	2	6			
442	12	11		562	1	8			
128	8	6	Less Sales during the year . . . . .	208	2	3			
314	4	5					355	19	5
49	3	2	Library—Binding, Purchases, &c. . . . .	43	18	8			
...			Moving and refitting Bookcases . . . . .	93	15	7			
							137	14	3
21	12	0	Meetings . . . . .				49	10	1
40	16	0	Legal Charges . . . . .				...		
...			Examination Charges . . . . .	698	1	2			
...			Less Fees received from Candidates (1919) . . . . .	295	1	0			
							403	0	2
...			Tutors for Classes in Parts I and II . . . . .	750	3	0			
...			Less Fees received from Students . . . . .	591	3	0			
							159	0	0
600	0	0	Office Expenditure—Rent . . . . .	600	0	0			
534	6	2	Salaries . . . . .	601	16	6			
55	5	1	House Expenses . . . . .	100	13	3			
32	14	10	Fire and other Insurance . . . . .	32	17	10			
...			Furniture and Fittings . . . . .	10	17	4			
149	7	1	Stationery and Printing . . . . .	220	6	2			
38	2	11	Postage and Telegrams . . . . .	49	4	11			
4	4	8	Sundries . . . . .	6	6	11			
1,414	0	9					1,622	2	11
...			Lectures on Finance and Statistics . . . . .				138	2	6
...			Valuations for the Royal Patriotic Fund Corporation (out of pocket expenses) . . . . .				70	8	0
0,086	0	3	Amount of Funds at the end of the year as per Balance Sheet . . . . .				9,933	4	

Examined and found correct, 28 April 1920.

1,925	16	7	E. W. HUMPHRY, STANLEY HAZELL, D. M. CARMENT,	} Auditors.	£12,869	1	8

ending 31 March 1920.

£	s.	d.		£	s.	d.
277	5	8	Sales (excluding <i>Journal</i> ) . . . . .	422	1	2
249	6	4	Stock (excluding <i>Journal</i> ) at the end of the year . . . . .	144	12	0

Examined and found correct, 28 April 1920.

£526	12	0	E. W. HUMPHRY, STANLEY HAZELL, D. M. CARMENT,	} Auditors.	£506	13	2

March 1920.

£	s.	d.	ASSETS.	£	s.	d.
1,875	0	0	£3,000 Natal 3 per-cent Consolidated Stock, 1929–49 . . . . .	1,875	0	0
726	5	0	£1,000 Dominion of Canada 3½ per-cent Registered 1930–50 Stock . . . . .	726	5	0
700	0	0	£1,000 New South Wales 3½ per-cent Inscribed 1930–50 Stock . . . . .	700	0	0
357	15	0	£600 Belgian Government 3 per-cent Sterling Loan of 1914 . . . . .	357	15	0
4,275	0	0	£4,500 5 per-cent War Stock, 1929–47 . . . . .	4,275	0	0
400	0	0	£400 National War Bonds, 4 per-cent, 1927 . . . . .	400	0	0
249	6	4	Stock of Publications (excluding <i>Journal</i> ) in hand . . . . .	144	12	0
250	0	0	Cash on Deposit Account . . . . .	...		
480	18	2	Cash on Current Account and in hand . . . . .	623	10	6
63	0	0	Subscriptions in Arrear . . . . .	57	15	0
903	7	0	£954. 5s. 6d. 5 per-cent War Stock, 1929–47 (G. F. Hardy Fund) . . . . .	903	7	0

The Stock Exchange Securities are taken at the values at which they stood on 31 March 1916, and at cost price in the case of any acquired since that date.

Examined and found correct, 28 April 1920.

10,280	11	6	E. W. HUMPHRY, STANLEY HAZELL, D. M. CARMENT,	} Auditors.	£10,063	4	6

There are also 230 candidates admitted as Probationers, and 79 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes at 31 March 1914 was 200 and 67. Since that date the numbers have been as follows :

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1915	188	72	1918	156	70
1916	172	73	1919	169	73
1917	173	67	1920	230	79

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of four Fellows, Messrs. A. B. Adlard, J. Coles, N. C. M. Home, and A. Pearson; three Associates, Messrs. S. G. Martin, E. H. Panton, and T. Peele; and three Students, Messrs. H. A. Bolton, V. M. Kime, and H. Watson. Lieutenant Watson, and two Probationers, Captain G. Y. Heald and Lieutenant A. V. Clare, were killed in the War.

The Council regret that it has not yet been possible to place in the Hall of the Institute a memorial of the Members and Probationers who fell in the War. Proposals have been under consideration, and it is hoped that the desired object may soon be achieved.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £2.201. 6s. 6d., as compared with £1.547. 3s. 6d. in the previous year. The Income and Expenditure for the year were £2.783. 1s. 5d. and £2.935. 17s. 4d. respectively.

The Examinations were held in June and December last year with the following results :

Part	Entered	Sat	Passed
I, Sec. A . . . .	167	142	63
I, Sec. B . . . .	132	113	52
II . . . . .	75	67	36
III, Sec. A . . . .	54	52	24
III, Sec. B . . . .	49	47	23
IV, Sec. A . . . .	47	45	22

The names of the successful Candidates have already been published in the *Journal*.\*

Lectures on Finance, by Prof. H. S. Foxwell, M.A., and on Modern Statistical Methods, by Miss Ethel M. Elderton, were delivered during the Session. The Council hope that it may be possible to arrange in the future for further courses of Lectures on subjects of general interest to the profession.

An announcement has already been made of the Council's intention to present a Gold Medal to commemorate work of outstanding originality in actuarial science and to give prizes out of the Prize Funds for any work of exceptional merit. The Council desire to express the hope that the opportunity may soon arise to make such awards.

Women are now eligible for membership of the Institute on the same conditions as men. Twelve women have been admitted to the Class of Probationer since the new regulation came into force.

\* See Vol. li, p. 394; Vol. lii, p. 114.

The stock in hand of the Institute publications on 31 March was as follows :

No. of Copies	Description of Work
30,248 . . . . .	Parts of <i>Journal</i> .
724 . . . . .	Index to Vols. 1 to 40.
1,321 . . . . .	<i>Text-Book</i> , Part I (Revised Edition).
225 . . . . .	<i>Text-Book</i> , Part II (Second Edition).
628 . . . . .	Government Joint-Life Annuity Tables
726 . . . . .	Select Life Tables.
307 . . . . .	A Short Collection of Actuarial Tables (New Edition).
731 . . . . .	Frequency-Curves and Correlation, with Addendum and Errata (W. P. Elderton).
36 in cloth 2,290 in paper	Lectures on Finance and Law (Clare and Wood Hill).
1,524 . . . . .	Lectures on the Companies Acts (A. C. Clauson).
1,169 . . . . .	Lectures on the Law of Mortgage (W. G. Hayter).
681 . . . . .	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,315 . . . . .	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
791 . . . . .	Lectures on Stock Exchange Investments (Sir J. Burn, K.B.E.).
482 . . . . .	Messenger Prize Essay on Friendly Societies (Sir G. F. Hardy, K.C.B.).
1,472 . . . . .	Lectures on Friendly Society Finance (Sir A. W. Watson).
313 . . . . .	South African War Mortality (F. Schooling and E. A. Rusher).
232 . . . . .	Life Assurance Law (A. R. Barrand).
70 . . . . .	Newton's Interpolation Formulæ (D. C. Fraser).
619 . . . . .	British Offices' Valuation Tables.
620 . . . . .	British Offices' $2\frac{3}{4}$ per-cent Temporary Annuity Values.
132 . . . . .	Transactions of the Second International Congress of Actuaries.
772 . . . . .	Index to Transactions of Seven International Actuarial Congresses.
129 . . . . .	Examination Questions, 1912-15, and 1919, separately bound.

## EXAMINATIONS, JUNE 1920.

Examinations were held on the 21st, 22nd and 23rd June 1920, at London, Liverpool, Cardiff, Norwich, Edinburgh, Dublin, Melbourne, Sydney, Perth, Montreal, Toronto, Ottawa, Winnipeg, Singapore, Shanghai, Bombay, and Calcutta, with the following results:

## PART I.—SECTION A.

One hundred and eleven candidates sent in their names, of whom one hundred and one presented themselves, and sixteen passed, namely:

Cale, C. E.	Gregory, Miss G. C.	Pal, A. T.
Care, H. C.	Grimshaw, W. E.	Perks, W.
Cooper, F. R.	Jones, H. G.	Scott, Miss H. M.
Davis, C. M.	Keady, P. J.	Williams, G. T.
Dhar, S. C.	Kops, C. W.	
Gill, D. H.	Maggs, C. G.	

## PART I.—SECTION B.

One hundred and five candidates sent in their names, of whom eighty-eight presented themselves, and twenty-seven passed, namely:

Ascott, F. E.	Davis, Miss D. B.	Lynch, L.
Beckworth, C.	Dodwell, Miss M. E.	Parker, R. O.
Bowrey, Miss M. L.	Dowsett, R. E.	Read, E. K.
Cale, C. E.	Grimshaw, W. E.	Rowan, H. R.
Care, H. C.	Guest, C. F.	Saunders, F. W.
Christie, V. G.	Hallett, H. J.	Schwehr, W. E. P.
Clough, W. H.	Johnston, A. W.	Scott, A. W.
Creese, H. R.	Keady, P. J.	Scott, Miss H. M.
Entler, D. G. S.	Kops, C. W.	Williams, G. T.

## PART II.

Thirty-six candidates sent in their names, of whom thirty-two presented themselves, and nineteen passed, namely:

Cope, A.	Martin, F. C.	Shaw, E. H.
Creese, H. R.	Miller, I. A.	Starke, L. G. K.
Curtis, A. T. G.	Milnes, H. L.	Taylor, F. R. S.
Eames, G. S.	Pedoe, A.	Thurston, O. F. a.
Finch, G.	Reynolds, P. C.	Walker, J. B.
Forster, W. A.	Sanger, C. W.	
Goodfellow, P.	Schwehr, W. E. P.	

## PART III.—SECTION A.

Thirty-six candidates sent in their names, all of whom presented themselves, and ten passed, namely :

Atkins, F. C.	Reid, W. C.	Thorpe, A. H.
Innes, F. F.	Rowell, A. H.	Welsh, W.
Jones, R. McNair	Rutherford, C. D.	
Lochhead, R. K.	Stockman, G. D.	

## PART III.—SECTION B.

Twenty-four candidates sent in their names, of whom twenty-three presented themselves, and thirteen passed, namely :

Brown, E. A. L.	Johnson, A. N.	Shrewsbury, A. H.
Clegg, C.	Johnston, W. N.	Stockman, G. D.
Freeman, H.	Knowles, M. B.	Wilkins, C. H.
Hocking, W. S.	Mabon, J. B.	
Holgate, B.	Marriott, A.	

## PART IV.—SECTION A.

Twenty-four candidates sent in their names, of whom twenty-three presented themselves, and eighteen passed, namely :

†Barrett, C. C. C.	†Gopp, J. I.	†Owen, D. J.
†Blake, W. T. C.	†Granger, C. K.	†Savory, D. S.
†Brown, B. G. H.	†Lafford, H. G.	†Tayler, H. H.
†Davidson, A. R.	†Maddex, G. H.	†Thompson, F. A.
†Edwards, A. J. C.	†Marshall, A. W.	†Weyer, D.
†Gawler, O.	†Moore, W. R.	†Woffindin, R. H.

## PART IV.—SECTION C.

One candidate presented himself for this Section and passed, namely :

†Maddex, G. H.

Those marked (†) have now completed the Examination for the Class of Fellow.

By Order of the Council,

C. R. V. COUTTS.

*Chairman of Board of Examiners.*

H. M. TROUNCER,

A. C. THORNE,

*Joint Honorary Secretaries.*

7 September 1920.

## PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventy-Third Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 7 June 1920, Mr. Geoffrey Marks, C.B.E. (President), in the Chair.

The PRESIDENT, in moving the adoption of the report and accounts, was glad to be able to call attention to the fact that during the year under review the fall in the membership had been arrested. He hoped that the upward movement in their numbers would now continue, and this hope was strengthened by the very large number of probationers who had entered during the past year, amongst them being twelve women. The number of women who had entered was not so large as he had expected, but he thought it was a sufficient justification of the action of the Council and of the members in throwing open the doors of the Institute to the other sex. He would refer very briefly to the deaths of a few members of the Institute, amongst them being three of the older members, Mr. Adlard, Mr. Pearson and Mr. Coles. Mr. Adlard was well known to many of the members of the Institute, and he, like Mr. Pearson, was one of those older actuaries who had always been regarded with great respect, and in many cases with affection, largely on account of their personal qualities and their general kindness to and interest in the welfare of the younger members. Mr. Pearson was almost a tradition in the Institute; he became a member of the old Institute of Actuaries' Club as far back as 1864. Mr. Adlard became a member of the Actuaries' Club about 1890, some thirty years or more ago. He had retired for some time, but his two sons would carry on his name and his memory. Mr. Coles was an actuary who left the profession to take up financial work, with great success, but he had always maintained his interest in the Institute and had devoted his financial experience to the benefit of his office. Mr. Home was a barrister as well as an actuary and was not well known to the members personally, because he practised in Shanghai. Quite recently the Council had received the news of Mr. Townley's sudden death, which seemed especially lamentable in view of the short interval which had elapsed since the equally sudden deaths of Sir Thomas Whittaker and of Mr. Vyvyan Marr. He had known Mr. Townley from a long association personally, and he had had a very high respect for his capacity: he was one of the sound men from whom he hoped much, not only because he was thoroughly qualified as an actuary but because he took a personal interest in and studied questions outside the immediate range of actuarial science, questions with which actuaries in their practical work were constantly brought into contact. His death was a loss to the profession both on the professional and on the business side, and it was greatly to be regretted.

The accounts showed an excess of expenditure over income of about £150, a deficit which was mainly due to the expenditure on the special lectures and to the fact that last year, as on some previous occasions, the Institute had undertaken to revalue the funds of the Royal Patriotic Fund Corporation. In connection with the expenditure on the lectures he would like to say that he thought the attendance was somewhat disproportionate to the expense which they entailed. He mentioned last year that whether the lectures would be continued or not would depend on whether the Council found that sufficient interest was shown in them by the members to warrant their continuance. Having regard to the character of the lectures and the subjects with which they dealt, it was doubtful whether the interest shown in them afforded justification for their continuance. The Council was



giving serious attention to the question of the annual expenditure, particularly under the item of the printing and publications of the Institute, a subject which had formed a topic of discussion at several Council meetings recently, and would come up again at the first Council meeting held after that Annual Meeting. It was a serious matter and one to which very close attention must be given. He had nothing to say with regard to the balance sheet except to call attention to the fact that the assets which were there enumerated had, since the Report was printed, been realized, and the proceeds re-invested in short dated securities. That was a course which commended itself to the Finance Committee and the Council as being in some measure a protection against the constant depreciation from which the funds of the Institute had suffered during the last few years.

Quite recently, on the 15th May, the Belgian Institute of Actuaries celebrated the twenty-fifth anniversary of their foundation, and the Council, feeling sure that it would be the wish of the members that a cordial message of goodwill should be sent to them, appointed a delegation which consisted of Mr. Woods (past-president), Mr. Besant (vice-president) and Mr. Bearman—who was in Paris—to attend the meeting and represent the Institute and transmit to them the following resolution:—"That the Council of the Institute of Actuaries desire, through their delegates, to express their lively sense of the great services which the Association of Belgian actuaries has rendered to actuarial science during the past quarter of a century and to wish the Association long endurance and every success in the future."

The public work of the Institute continued to extend, and during the past year the Council had been invited by the Lords of the Treasury to give assistance on actuarial questions in connection with the reparation clauses of the Treaty of Peace with Germany. The Council had appointed a Committee to deal with the matter. The Council had presented to the governing body of the Royal Patriotic Fund Corporation the valuation of their funds as at the end of 1918, and he thought that one result of that valuation had been the amalgamation of certain of the funds, a necessity which had been impressed upon them almost every time that the Council had undertaken the valuation. As was well known, the President of the Institute was *ex officio* a member of the governing body of the Corporation, and his predecessors in the past had made a point of the consolidation of the smaller funds, but only now had they been able to influence the Corporation to adopt the suggestion. A Committee of the Council had submitted its views with regard to improvements in the forms to be used for obtaining information in connection with the next census. They were asked to do that for the Census Department, and he thought that the Royal Statistical Society also had appointed a Committee which made a report very much on the same lines. It was hoped, therefore, that in the forthcoming Census there might be certain improvements, both in the form and in the presentation of statistics, which would make them even more valuable than in the past.

He would like to commend the Students' Society, which had resumed its useful and manifold activities, to the notice of the younger members, with a view to securing for it all possible support, not only because it was a good thing that young men should be active in the discussion of problems that came before them, but because he believed it to be a very valuable training ground for them, not only in debate but also in the practical views which were very necessary to success in their future careers.

Many members of the Institute had taken part in various forms of public work during the past year; he did not propose to refer to them individually, but would call attention to the appearance in the Honours List of the names

of several members, particularly Mr. Elderton, Mr. Falk, and Sir Alfred Watson. He was sure there was no one in that Hall to whom it was not as gratifying as it was to him to be able to congratulate Sir Alfred Watson on the signal honour he had earned, not only by his personal capacity but by his devotion to one of the noblest aims of any man, namely, the welfare of the State. Sir Alfred had been called into consultation in a great many matters which had not, perhaps, any intimate connection with the work of the actuary, but in which his knowledge and tact had been of very great assistance to the Government.

Mr. A. D. BESANT, in seconding the motion, said the President had covered almost the whole field in his survey of the work of the Institute during the past year, but there were just one or two points on which he might touch. The present report was in effect the first peace time report, but it still included deaths on war service, and he found, on examining the past reports, that the Institute had lost on war service 43 of its members and 36 probationers. The visit of the delegation to the Belgian Institute was exceedingly interesting. There were delegates from Holland, France, Sweden and other countries. He thought a deep debt of gratitude was owing to Mr. Todhunter and Mr. Spencer, the joint Editors of the *Journal*, for having throughout the long period of the war produced the *Journal* at intervals of six months, without fail.

On the motion of Mr. MAUNDER, seconded by Mr. TAYLER, Messrs. Hazell, Carment, and A. S. Sellar were elected Auditors for the ensuing year.

Mr. TARN, in proposing a hearty vote of thanks to the President, the Vice-Presidents, the Council and the Officers, including the Assistant Secretary, for their services during the past year, congratulated Mr. Marks on the fact that his term of office had been marked by the declaration of peace, and referred to the two important events that had occurred during the past year, namely, the admission of women to membership and the resumption of the examinations. He was glad to find that the Assistant Secretary was included in the resolution, because everyone knew of his courtesy and the great help he gave to all members of the Institute, especially the younger ones.

Mr. MORAN seconded the motion, which was carried unanimously.

The PRESIDENT, in acknowledging the vote, thanked the members in his name and in that of the Vice-Presidents, the Council and the Officers. He would like to add to what Mr. Tarn had said his very hearty thanks to all the officers, members of Council and Mr. Jarvis for all they had done to help him during his term of office. He laid that office down with great regret, but thankful to all for their election of him and for giving him their support during the somewhat trying time of reconstruction. He hoped he had done nothing to dim the traditions of the office and he trusted that the same support and confidence which had been given to him would be extended to his distinguished successor.

A vote of thanks to the Auditors, Messrs. Humphry, Hazell and Carment, for their services during the past year was carried unanimously.

#### ELECTION OF OFFICERS.

The report having been unanimously adopted, a ballot was taken for the election of new members of the Council. The President subsequently announced that the Officers and Council for the ensuing year would be as follows:

#### *President.*

SIR ALFRED WILLIAM WATSON, K.C.B.

#### *Vice-Presidents.*

SIR JOSEPH BURN, K.B.E.  
JAMES DOUGLAS WATSON.

ABRAHAM LEVINE, M.A.  
WILLIAM PALIN ELDERTON, C.B.E.

*Council.*

SAMUEL JOHN HENRY WALLIS ALLIN, C.B.E.	HENRY EDWARD MELVILLE.
HERBERT HENRY AUSTIN.	*HENRY JOHN PERCY OAKLEY, M.C.
HENRY JAMES BAKER.	*WILLIAM PENMAN.
*ARTHUR RHYS BARRAND, M.P.	WILLIAM PEYTON PHELPS, M.A.
SIR JOSEPH BURN, K.B.E.	WILLIAM CHARLES SHARMAN.
LOUIS ERNEST CLINTON.	*FRANK PERCY SYMMONS.
WILLIAM PALIN ELDERTON, C.B.E.	ALFRED CHARLES THORNE.
*ALFRED HENRY.	HAROLD MOLTKE TROUNCER, M.A.
LEWIS FREDERICK HOVIL.	SAMUEL GEORGE WARNER.
CHARLES WILLIAM KENCHINGTON.	SIR ALFRED WILLIAM WATSON.
ABRAHAM LEVINE, M.A.	K.C.B.
GEORGE JAMES LIDSTONE, F.R.S.E.	JAMES DOUGLAS WATSON.
HAROLD EDWARD WILLIAM LUTT.	ARTHUR THOMAS WINTER.
GEOFFREY MARKS, C.B.E.	ERNEST WOODS.
REGINALD GEORGE MAUDLING.	WILLIAM ARTHUR WORKMAN.

\* Not Members of the last Council.

*Treasurer.*

LOUIS FREDERICK HOVIL.

*Honorary Secretaries.*

HAROLD MOLTKE TROUNCER, M.A. | ALFRED CHARLES THORNE.

Mr. MARKS then vacated the chair, which was taken, amidst great applause, by SIR ALFRED WATSON, who said it was impossible for him to find other words than those which had been used by a long succession of his distinguished predecessors in thanking the Institute for the honour which it had conferred upon him. It was a proud moment in the life of any professional man when he reached the headship of his professional body, and he apprehended that if he was properly constituted his accession to office must be accompanied, not only by a deep sense of humility in surveying the tasks that lay before him and the responsibilities which he had undertaken, but also by a high sense of resolve that he would do that which in him lay to justify the choice which his professional associates had made. That was his profound and sincere feeling, and it was with those thoughts uppermost in his mind that he thanked the members most sincerely for the endorsement they had just given of the choice made by Council in electing him to the presidential chair. He would like further to say one word with regard to a matter to which Mr. Marks had been good enough to refer. He did feel a sense of profound pleasure and satisfaction in the honour which it had pleased His Majesty to confer upon him, and with no sense of false modesty would he attempt to decry the satisfaction which he felt. But he also felt—and those who knew him best would realise how sincere he was—deep satisfaction in that it had fallen to his lot to raise to a certain status in the Civil Service the office which he held, and which he held in trust, as the first occupant, for those who would follow him. It was very largely from that point of view and from the point of view of the profession that he felt such profound satisfaction in the honour that had just been given to him. Such else as he had to say—and there might be a good deal—he would reserve until November, except one point. Remarkable figures had been read out by Mr. Besant as to the number of members killed in action. Seventy-nine members or probationers of the Institute had died. Out of a body of about twelve hundred, many of whom were above military age and large numbers of whom were kept on their civilian duties by the exigencies of the time, that was a remarkable proportion, and testified to the manner

in which such a small professional body had done its share in bringing about the position of security in which we could all conduct our daily business. Those men who went to the war did their duty nobly, and the price that the Institute had paid had been very heavy. A high ideal of duty had been set, and he thought he could promise for the incoming Council that no steps should be left untaken on their part to live up to that ideal and to do their best to discharge the trust committed into their hands. In conclusion, he would remind the members of what Mr. Marks could not say himself, but what certainly had to be said—that during Mr. Marks's term of office he had upheld the high traditions of the presidency in a very conspicuous manner. He had been called upon to discharge many public duties and had created what was a record for the Institute and probably for the head of any profession, in that during his period of service as President he had served with great industry and ability on two Royal Commissions. That was a fact of which the Institute was entitled to be very proud.

### *Additions to the Library.*

The following works have been added to the Library since the publication of the *Journal* for October 1919 :

*By whom presented  
(when not purchased).*

#### Accountants and Auditors, Society of.

List of Members, etc., 1920.

*The Society.*

#### Actuarial Society of America.

Transactions, 1919.

*The Society.*

Containing, *inter alia*—

“Taxation—especially in its bearing on Life Assurance”, by H. Moir.

“Should the ‘American Men’ Mortality Table (A.M.) be the basis for Premiums and Reserves”, by A. Hunter.

“Heart Murmurs. Their influence on Mortality”, by Dr. O. H. Rogers and A. Hunter.

“Gain and Loss on Annuity business”, by J. W. Tebbetts.

“Formulae used by various companies in computing the expected mortality on net amount at risk”, by E. G. Fassel.

“The Influenza Epidemic of 1918”, by J. D. Craig and Dr. L. I. Dublin.

“Principles involved in comparing Mortality Tables, with application to certain modern tables and to historical survey of Connecticut Mutual Ultimate Mortality”, by H. J. B. Rice.

#### Actuaries, Faculty of.

Transactions, 1919–1920.

*The Faculty.*

Containing, *inter alia*—

“An analysis of the Profit from Endowment Assurance and Whole Life Policies”, by W. A. Robertson and A. G. R. Brown.

“Inaugural Address”, by Dr. A. E. Sprague.

“The Selection of Lives”, by L. P. Orr.

*By whom presented  
(when not purchased).*

- Actuarial Society of New South Wales.**  
 Proceedings, 1918. *The Society.*
- Actuarial Society of Scandinavia.**  
 Transactions, 1919-20. *The Society.*
- American Mathematical Society.**  
 Transactions, 1919-20. *The Society.*
- American Statistical Association.**  
 Transactions, 1919-20. *The Association.*
- Archer (J. A.).**  
 Tables for the repayment of Loans, etc. 5th edit. }  
 La. Svo. 1919. } *The Publishers.*
- Association des Actuaire Belges.**  
 Bulletin of the. *The Association.*
- Association of Life Insurance Medical Directors of America.**  
 Abstract of Proceedings, 1917-18, 1919-20. *The Association.*
- "Biometrika."**  
 Vol. XII, Parts III and IV. *Purchased.*  
 Containing, *inter alia*--  
 "On the mathematical expectation of the  
 moments of Frequency Distribution", by  
 A. A. Tchouproff.  
 "On generalised Tchebycheff theorems in the  
 mathematical theory of Statistics", by  
 Prof. Karl Pearson.  
 "On the degree of perfection of Hierarchical  
 Order among correlation coefficients", by  
 Dr. G. H. Thomson.
- Blaschke (Dr. E.).**  
 Die Ausgleichung von Absterbeordnungen aus der  
 Bevölkerungsstatistik. 4to. Wien. 1917. *The Author.*
- Brussels.**  
 Compte Rendu des Opérations et de la Situation de la }  
 Caisse Générale d'Épargne et de Retraite, 1913, } *The Belgian*  
 1914. } *Government.*
- Cannan (E.).**  
 The Paper Pound of 1797-1821. A reprint of the ,  
 Bullion Report. Svo. 1919. } *Purchased.*
- Carnegie Foundation for the Advancement of Teaching.**  
 Fourteenth Annual Report of the President and of ,  
 the Treasurer. 4to. Boston, Mass. 1919. } *G. J. Lidstone.*
- Casualty Actuarial and Statistical Society of America.**  
 Proceedings, 1919. *The Society.*  
 Containing, *inter alia*--  
 "On the graduation of Frequency Distributions",  
 by H. C. Carver.
- Chalmers (R.).**  
 A History of Currency in the British Colonies. Svo. }  
 1893. } *Purchased.*



*By whom presented  
(when not purchased).*

Chartered Insurance Institute, Journal of the.

Vol. XXII. 8vo. 1919.

*The Institute.*

Denmark.

Beretning fra Forsikringsraadet for aaret 1919.

*The Danish  
Government.*

Deutscher Verein für Versicherungs-Wissenschaft.

Zeitschrift für die gesamte Versicherungs-Wissenschaft.  
1914-1920.*The Society.*

Döring (C.).

Die Bevölkerungsbewegung in Weltkreis. 8vo.  
Copenhagen. 1919-20.*Purchased.*

Economic Society (Royal).

Journal of the, 1919-20.

*Purchased.*

Eugenics Lecture Series.

XI. National Life from the standpoint of Science, by  
Prof. Karl Pearson. Sm. 8vo. 1905.XII. The Function of Science in the modern State, by  
Prof. Karl Pearson. Sm. 8vo. 1919.*Purchased.*

France.

Recueil de Documents relatifs aux Assurances sur la  
vie. Sixième Rapport sur le fonctionnement du  
contrôle des Sociétés d'assurance sur la vie et des  
Sociétés de Capitalisation pendant les années 1912  
and 1913. 8vo. Paris, 1920.Recueil de Documents relatifs aux Assurances sur la  
vie réunis par le Ministère du Travail et de la  
Prévoyance Sociale. No. 2. Réglementation du  
Contrôle. 8vo. Paris, 1919.*The French  
Government.*

Frankel (Dr. L. K.) and Dr. L. I. Dublin.

Influenza Mortality among wage earners and their  
families. New York, 1919.*The Authors.*

Fraternal Actuarial Association.

Proceedings and Papers, 1918-1920. 8vo. Boston,  
Mass.*The Association.*

Gephart (W. F.).

Effects of the War upon Insurance, with special  
reference to the substitution of insurance for  
pensions. 8vo. New York, 1918.*The Author.*

Hendriks (F.).

Contributions to the History of Insurance and of the  
Theory of Life Contingencies. With a restoration  
of the Grand Pensionary De Wit's Treatise on  
Life Annuities. And other Papers. 8vo. 1850,  
1851, 1854.Papers on Annuities, Insurances, and Statistics. 8vo.  
1856-1863.*Purchased.*

Holland.

Nationale Levensverzekering-Bank. Verslag over het  
jaar 1917.*The "Nationale"  
Company.*Vereeniging voor de Verzekerings-Wetenschap. Het  
Verzekerings-archief.*The Society.*



*By whom presented  
(when not purchased).*

**Insolera (F.).**

Sull' assicurazione obbligatoria contro l'invalidità e la  
verchiata. 1919. } *The Author.*

**Institut des Actnaires Français.**

Bulletin of the. *The Institute.*

**Institute of Bankers.**

Journal of the, 1919-20. *The Institute.*

**International Actuarial Congresses.**

Bulletin du Comité Permanent des Congrès Inter-  
nationaux d'actnaires. Svo. Brussels, 1919. } *The Committee.*

**Italy.**

Giornale di Matematica Finanziaria rivista tecnica }  
del Credito e della Previdenza. Svo. 1919. } *F. Insolera.*

**Japan.**

Life Table, No. 2, based on the Registrations of Deaths  
and Births during the period 1898-1903. Compiled  
by T. Yano. 4to. Tokio. } *The Japanese  
Government.*

**Joffe (S. A.).**

Parallel proofs of Everett's, Gauss's and Newton's  
Central Difference Interpolation formulæ. New  
York. 1919. } *The Author.*

**Kemmerer (Dr. E. W.).**

The A B C of the Federal Reserve System. 3rd edit.  
Svo. Princeton, N. J., 1919. } *Purchased.*

**Keynes (J. M.).**

The Economic Consequences of the Peace. Svo. 1919. *Purchased.*

**League of Nations Secretariat.**

Currencies after the War. A survey of conditions in  
various countries. Svo. 1920. } *Purchased.*

**Life Offices' Association.**

Minutes, etc., of the Meetings of the Standing  
Committee, 1919-20. }  
Notes prepared by E. J. MacGillivray, LL.B., on British  
Life Policies and Alien Enemies. Fol. 1920. } *The Association.*

**Liverpool (Charles, Earl of).**

Treatise on the Coin of the Realm. 4to. 1805. *Purchased.*

**Loewry (Dr. A.).**

Zur Theorie und Anwendung der Intensitäten in der  
Versicherungsmathematik. Heidelberg. 1917. } *The Author.*

**London Mathematical Society.**

Proceedings, 1919-20. *The Society*

**Manes (Dr. A.).**

Versicherungs-Staatsbetrieb im Ausland. Ein Beitrag }  
zur Frage der Sozialisierung. Svo. Berlin, 1919. } *The Author.*

**Meade (E. S.).**

Trust Finance. Svo. 1914. *Purchased.*

*By whom presented  
(when not purchased).*

**Morgan (William).**

Facts addressed to the serious attention of the people  
of Great Britain respecting the expense of the  
War and the state of the National Debt. And } *Purchased.*  
other papers. 8vo. 1796-1801.

**National Health Insurance.**

Reports of Medical Research Committee. 8vo. } *Purchased.*  
1919-20.

**Newsholme (Sir Arthur, K.C.B., M.D.).**

Public Health and Insurance: American Addresses. } *Purchased.*  
8vo. Baltimore. 1920.

**Northcote (Sir Stafford H., Bart.).**

Twenty years of financial policy. A summary of the  
chief financial measures passed between 1842 and } *Purchased.*  
1861, with a table of Budgets. 8vo. 1862.

**Norway.**

Forsikringsselskaper. Beretning fra Forsikringsraadet, } *The Norwegian*  
1912-1917. 8vo. Kristiania. 1915-19. } *Government.*

**Overstone (Lord).**

Collection of Tracts and other publications on Paper } *Purchased.*  
Currency and Banking. 8vo. 1857.

**Palgrave (R. H. Inglis).**

Dictionary of Political Economy. 3 vols. 8vo. 1917- } *Purchased.*  
1919.

**Parliamentary Papers.**

Assurance Companies. Returns to the Board of } *The Board*  
Trade. 1919. } *of Trade.*

**Colonies.**

**Canada.**

Report of the Superintendent of Insurance for  
the year 1918. } *The Government*  
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for the year 1919.

**Commonwealth of Australia.**

Census, 1911. Vol. I. Statistician's Report, } *Commonwealth*  
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1917.

**New South Wales.**

Friendly Societies, &c. Report of the Registrar  
for 1919. } *The Government*  
Official Year Book, 1919. } *of N.S.W.*  
Statistical Register for 1918-19 and previous  
years. }  
Vital Statistics. Report for 1919 and previous  
years. }

**New Zealand.**

Government Insurance Department. Annual  
Report, 1918. } *The Government*  
Friendly Societies Annual Report of the } *of N.Z.*  
Registrar, 1918. }  
National Provident Fund. Report for 1918.  
Actuarial Examination for Triennium  
ended 31 December 1916. }

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(when not purchased).*

**Parliamentary Papers—continued.**

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Thomas Bond Sprague, M.A., LL.D.



# JOURNAL

OF THE

## INSTITUTE OF ACTUARIES.

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### MEMOIR

OF

### Thomas Bond Sprague, M.A., LL.D.,

*Fellow and Past President of the Institute of Actuaries;  
Honorary Fellow and Past President of the Faculty of Actuaries in Scotland;  
Fellow of the Royal Society of Edinburgh, &c., &c.  
Born 29 March 1830. Died 29 November 1920.*

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DR. SPRAGUE remained in complete retirement from all matters actuarial for fully twenty years before his death, and it is therefore difficult for the present generation to realize how unrivalled was his supremacy forty to sixty years ago. The young men of those distant days stood afar off and worshipped; and yet he was very approachable and considerate, and helpful to any who sought his assistance or advice. Once a friend he was always a friend, and generous in all his thoughts, and words, and deeds. In his private affairs it was the same, and for many years he was liberal with his means in giving assistance to deserving cases which came under his notice. There are not a few now in comfortable circumstances who have to thank him for setting them on their feet in times of financial stress, and thus enabling them to make a success instead of a failure of their lives.

Dr. Sprague was a man of action, whose actions were

governed by theories formed after viewing the questions impartially, from various sides, and which were then tested practically, and proved to be sound. He possessed a great intellect, almost irresistible force of character, and unwearied industry, all combined with straightforward honesty of purpose, and candour in everything he said, or wrote, or did; and he had complete self-confidence, without self-conceit or undue self-esteem. He formed strong opinions, which he maintained with outspoken vigour against all opponents; but he was always willing to reconsider these opinions when necessity arose as a result of further investigations; and, when convinced that to change an opinion was the right thing to do, he frankly acknowledged the change, and was ready to state the reason why. It must be admitted, however, that he was sometimes rugged, and a little aggressive, and not always easy to get on with; and his earnestness led him on rare occasions to be perhaps over-vehement in his expressions. He had no thought of being personally discourteous, or of hurting the susceptibilities of those to whom he was for the moment in opposition. The opinions were attacked, and not the persons. These characteristics come out clearly in his voluminous writings, and more particularly in his Presidential Addresses.

During his long connection with the Institute of Actuaries he was to a large extent its guiding spirit, and he did much to mould its destinies, and to bring it up to its present position of world-wide influence. In saying this, however, we cannot forget the services in the same direction of many other eminent men, whom we have always been proud to honour; and, if any one of these should be individually mentioned, he is Charles Jellicoe, the second President, and the first Editor of the *Journal*, of whom Dr. Sprague himself said in his first Presidential Address, that the Institute and the profession owed more to him than to any other single person.

In preparing this memoir, an effort has been made, not so much to review Dr. Sprague's writings, as to educe from them his character, and to make THE MAN stand forth as he was, so that even those may see him who never had the privilege of meeting him, and of associating with him: but the task has not been an easy one.

Thomas Bond Sprague was born on 29 March 1830, and was the eldest son of Thomas Sprague of London, wholesale stationer. He was educated at Tarvin Hall, a private school

near Chester, under the Head Mastership of Dr. John Brindley. While at school, he distinguished himself so much in mathematics that Dr. Brindley advised his father to send him to Cambridge, and he accordingly entered St. John's College as a Sizar. At the end of his first year he became a Proper Sizar, and in consequence of the excellent places he took in the College examinations, he became in due course a Scholar.

In 1853 he achieved the highest academic honours possible in mathematics, being Senior Wrangler and First Smith's Prizeman; and in the same year he was elected a Fellow of St. John's College, and appointed one of the College Lecturers. About this time he also qualified in law and was called to the bar, having in view the law as his future profession. But he became engaged to be married; and, progress in the law being slow, he sought a career which would be more immediately remunerative. He selected Life Assurance; and entered the Eagle Insurance Company in 1855 as a pupil of the late Charles Jellicoe, and thus became one of the distinguished group who, in after life, achieved great success.

After holding some minor appointments, Dr. Sprague was, for a short time, Actuary of the then Liverpool and London Assurance Company, until in 1861 he was appointed, at the age of 31, Actuary and Secretary of the Equity and Law Life Assurance Society, with which society he remained for twelve years. In 1873 the management of the Scottish Equitable Life Office became vacant, and the Directors sent a deputation to London to choose a manager; and, after careful consideration, they offered the post to Dr. Sprague, who accepted it. This involved his removal to Edinburgh, but it did not diminish his interest in the Institute of Actuaries, and he was very frequently in London to give his time and attention to Institute matters. In 1900, after being manager of the Scottish Equitable for twenty-seven years, he finally retired from active business life.

In 1893 the University of Aberdeen conferred on him the degree of LL.D., the diploma being dated the 6th of April of that year. Hence, in this memoir, the name and title "Dr. Sprague" are used throughout, for the sake of uniformity, and also because that designation is more familiar to the present generation of actuaries.

Dr. Sprague was elected Associate of the Institute of Actuaries on 25 February 1856, and was not long in giving evidence of his energies. His first paper, under date October

1856, is to be found in the *Institute Journal*, vol. vi, and is a short one on a certain method of distributing surplus. He was elected a Fellow of the Institute on 30 November 1857, and within a few months he contributed his second paper, dated July 1858, on the Terms upon which the Business of one Insurance Company may be equitably transferred to another. That paper appears in vol. vii of the *Journal*, and on subsequent occasions, when the failures of the Albert and of the European, in 1869 and 1872 respectively, had brought the question into prominence, he wrote other papers on analogous subjects.

Dr. Sprague was elected to the Council of the Institute for the first time in 1863, and he remained a member without break until his final retirement from business in 1900, a total period of thirty-eight years, which is a record. He became Editor of the *Journal* of the Institute in 1867, succeeding his former chief, Mr. Charles Jellicoe, who was the first Editor; and he retired from the editorship in 1883, having occupied the post for over sixteen years. The volumes of the *Journal* published during his editorship bear witness to his activity and zeal. Before retiring from the editorship, he prepared the index of the first twenty volumes of the *Journal*.

Dr. Sprague was elected President of the Institute, the eighth President, in 1882, and occupied the presidential chair for four eventful years, as, during his Presidency, the Institute received its Charter. In 1874 he was elected Fellow of the Faculty of Actuaries in Scotland, and was President from 1894 to 1896. No one else has occupied the Presidential chairs of both the Institute and the Faculty. He was also President of the Actuarial Society of Edinburgh on three occasions, 1874, 1882 and 1891. That Society was dissolved in 1901, its functions being taken over by the Faculty.

Dr. Sprague's unwearying devotion to the Institute of Actuaries, as embodying his own high ideals, comes out clearly in his first Presidential Address, delivered on 27 November 1882 (*J.I.A.*, vol. xxiv, p. 1). There is nothing selfish in it, but, on the contrary, an earnest striving for the raising of the status of the profession, and, through the profession, for the good of the world at large. The address consists, for the most part, of a brief but comprehensive history of the Institute from the preliminary discussions and meetings, which took place early in 1848, and its establishment by formal resolution adopted on 8 July 1848, until the time that Dr. Sprague was speaking, on

the eve of the granting of the Charter. That history is well worthy of study even at the present day.

There were great difficulties in starting the Institute on account of the opposition of a small minority of influential and respected men, who seem to have thought that professional matters should be kept private, and that publicity would be injurious to the members of the profession, by stimulating undue competition from outside quarters. It was through these men that the long delay arose in the granting of the Charter, but on this point Dr. Sprague says that, on the whole, in his opinion the delay had been beneficial, because the Institute had thus been left for a period of thirty-four years to develop itself by its own resources, without any external aid or encouragement, and had thus attained to a high and influential position.

Dr. Sprague looked upon the educational facilities provided by the Institute for its young members as one of its most important functions. He thought that all the young men connected with life assurance should be trained in the scientific department, although they could not all secure high appointments. Even without such high appointments they might be very useful members of the profession ; and, should they drift away from it to obtain posts elsewhere, their actuarial training would always be a great asset. It was, however, to the encouragement of publicity that Dr. Sprague attached the most importance, and he had no fear of competition. On this point he said :

“ Very much in consequence, I believe, of the teaching and example of the leading members of the Institute, it has become much more the practice for actuaries to work in the light of day, and to explain fully their processes, and the reasons which influence them. In the olden time I believe it was the custom for actuaries to conceal their methods, or only to reveal them to their most favoured pupils ; and, if an actuary had constructed a private mortality table, this was perhaps the most valued part of his stock-in-trade. These ideas, I think, may be fairly said to be now quite out of date.”

He adopted with high approval the opinions given by others in the very early days of the Institute. For instance, he quoted the late Mr. Peter Hardy, the father of the late Mr. R. P. Hardy who was a friend of so many of us, as saying :

“ The Institute was not originated in any selfish feeling of professional vanity. Its founders took no counsel together how to narrow the access to our profession, or how to exclude other competitors than ourselves from a fair share of its honours



or emoluments. It was originated with a far higher, a far nobler motive, in the honest hope of educating the future actuary, of improving among ourselves our present theories, and of enlarging the bounds of that science on which the safety of so many millions sterling depends, and in which the social happiness of so many thousand families is involved."

Also, he welcomed appreciation from actuaries of other countries, and quoted, as follows, the late Mons. E. Maas, Manager of the "Union" Company of Paris, a Corresponding Member :

"A few years ago men engaged in the same pursuits were envious of each other, and endeavoured to conceal all that could afford any improvement to their competitors. The English people, moved by a nobler spirit, did not fear to exhibit to all mankind the treasures of their industry, so that everyone might profit by their example, and take advantage in imitating them. The same feeling induced the Institute of Actuaries to convene the actuaries of all nations on the native ground of insurance, to participate in the discoveries of their science and the results of their experience."

Dr. Sprague himself followed the practice which he inculcated, and many volumes of the *Journal* contain the papers in which, whenever he discovered or worked out anything that he thought would be useful to others, he gave full details, and kept nothing back for his own private advantage.

The first great work of the Institute in which Dr. Sprague took part was the preparation of what are now known as the Institute of Actuaries' Life Tables. On 14 January 1862, the Council of the Institute of Actuaries appointed the following six gentlemen a Committee to carry through the investigations: Messrs. A. H. Bailey, Archibald Day, W. B. Hodge, Charles Jellicoe, T. B. Sprague, and Robert Tucker; and, after correspondence, the Managers and Actuaries in Scotland appointed six to co-operate with the London Committee. They were Messrs. David Chisholm, John M. McCandlish, James Meikle, Samuel Raleigh, Alexander G. Ramsay, and William Smith. To the name of each of these twelve gentlemen we must now prefix the words "the late", Dr. Sprague being the last of them to pass away.

The experience was that of twenty Offices, and was published in 1869, and at the Annual Meeting of the Institute on 5 June of that year, Mr. Samuel Brown, the then President, submitted an early copy of the volume of Experience, and, in doing so, specially referred to Dr. Sprague, and to Dr. Sprague only, for



his efficient help in preparing the experience. At the same time he announced that a scheme of notation was being prepared by the Council for general use, and again he mentioned Dr. Sprague, and Dr. Sprague only, as having given exceptionally valuable help. That scheme of notation was published in the volume of monetary tables based upon the experience, and has since, with extensions, been adopted by resolutions of International Actuarial Congresses as the notation to be used throughout the world. The volume of subsidiary and monetary tables was published in 1872, the life tables having been graduated by Mr. Woolhouse, and the subsidiary and monetary tables having been calculated by Mr. Peter Gray, whose name at that time was a household word among Actuaries.

The Albert failed on 13 August 1869, and the European on 12 January 1872; but, for some time before even the failure of the Albert, there had been ominous premonitions that the catastrophes were impending, and bills were introduced into Parliament whereby the promoters hoped to make such disasters impossible for the future. The Council of the Institute took a very active interest, and put forth strenuous efforts to guide the legislation in the right direction. The first bill was crude, and would have done more harm than good. The watchword of the Institute was "Liberty with Publicity": no interference by the State, but, instead, the publication in respect of each Company of summaries of its business, of the particulars for valuation, and of the principles followed, sufficient to enable outside actuaries to judge as to its real position. The result was the Act, passed on 9 August 1870, which, by general consent, is held to have worked well, and which has now been superseded by the Act of 1909 on very similar but somewhat extended lines as regards Life Insurance, but which also provides for Insurance business of all other classes.

Dr. Sprague was one of the foremost advocates of Liberty with Publicity, and on 25 April 1870, when the Bill was before Parliament, he read a paper (*J.I.A.*, vol. xv, p. 411) on "The Proper Method of Estimating the Liability of a Life Insurance Company." At that time there were influential advocates of legislation to impose on the Companies a strict net premium valuation, and to set up a standard of solvency; but against both of these suggestions Dr. Sprague took a firm stand, and in the paper above referred to he said regarding valuations:

“In valuing by the net-premium method, the supposition is tacitly made that the expenses chargeable to a policy are equally spread over its existence. But this is very far indeed from the truth, as the expenses incurred in the first year of the policy’s existence far exceed those of subsequent years. Indeed, by far the greatest part of the expenses of every Life Insurance Company are chargeable to the new business—in particular, all advertising, inspection of agencies, and medical fees, and a considerable proportion of the postage, directors’ fees, salaries, and office expenses. For if a company decided, as some have done, on working out the existing business, on taking no new business, and closed its doors with the intention of working out the existing business as economically as possible, it is obvious that the expenses, exclusive of the commission, might be reduced to a very trifling sum. This reduced sum, or at all events such a sum as an established company would undertake to conduct the business for, is all that should be charged to the old business.

“If we keep these considerations steadily in view, we shall see that the only case in which the net-premium method of valuation can be expected to lead to satisfactory results is when, from the magnitude of the business transacted, or other causes, the total expense of conducting the business, irrespective of commission, amounts only to a small percentage of the premiums. In all other cases we must seek for some other rule.”

He returned to this point, after fifteen years’ experience of the Act, in his fourth Presidential Address, delivered on 30 November 1885 (*J.I.A.*, vol. xxv, p. 293). There he said :

“The theoretical expression for the value of a policy takes no account of expenses of management ; and in adopting this formula as the basis of the valuation of a company, it is assumed that the expenses are in every case less than the uniform loading added to the uniform annual net premium. This is an assumption, however, which is not justified by the facts, both as regards the total business of most young companies, and as regards the new business of most old companies. It is therefore necessary, in order to get an appropriate formula to introduce into our calculations the amount of expense at which the new business is obtained ; and thus, for old and young companies alike, we conclude that the proper reserve to be made for the liabilities is less than that given by the ordinary net-premium method of valuation.”

There is nothing in the Act of 1870 about a standard of solvency, but Dr. Sprague took up this question in a paper on Legislation as to Life Insurance and Life Insurance Companies, read on 28 November 1870, just after the Act had been passed (*J.I.A.*, vol. xvi, p. 77). He there maintained :

“Firstly: That no legislative enactments whatever can succeed in absolutely securing the solvency of Life Insurance Companies.

“Secondly: That even if this could be done, it would be highly undesirable that it should be attempted.”

He supported these two propositions by lengthy and cogent reasoning, and summed up by saying:

“It is impossible to say that one particular estimate of the value of the liabilities is right, and all others wrong. It is of course easy to find limits above and below which difference of opinion ceases, so that if a Company with known liabilities possesses a certain amount of assets, all actuaries will agree in saying that it is solvent, while on the other hand if it possesses less than a certain other amount of assets, all actuaries will agree in saying that it is insolvent. But, these limits would be very wide; and if the Company possesses an intermediate amount of assets, probably some actuaries will say that in their opinion it is solvent, while others will say that in their opinion it is insolvent. The conclusion we arrive at therefore is, that the solvency of Life Insurance Companies being so much a matter of opinion, the Government cannot with propriety attempt to decide the question by laying down a legal standard of solvency.”

This may be thought to be ancient history, but even ancient history has sometimes a tendency to repeat itself: and we can see from what Dr. Sprague did in 1870, and from what he wrote on the subject in the *Journal* at the time, that he would have ever been on the alert, and that he would at all times have put forth all his powers to oppose legislation involving government control of Insurance Companies, any standard of valuation, and more particularly any standard of solvency.

He would have maintained that vigilance is imperative, but he would not have resisted all legislation. From his writings we know that he thought the Act of 1870 might have been carried further with advantage, and at the present day it is very likely he would have said, for instance, that it is quite proper to give wider powers to the Board of Trade than it possesses, but that the powers to be granted must be minutely scrutinized.

It has already been mentioned that, on account of the opposition of influential men, who had formed themselves into the then Actuaries' Club, there had been interminable delays in securing a Charter for the Institute. The question was, however, taken up again seriously, and at the Annual Meeting held on 3 June 1882, the meeting at which Dr. Sprague was elected to

his first Presidency, a resolution was passed on the recommendation of the Council to apply for a Charter, and thereupon an application was presented to the Privy Council on 15 September 1882. The application was signed by Dr. Sprague as President, and by the then four Vice-Presidents, Messrs. M. N. Adler, G. W. Berridge, C. J. Bunyan, and R. P. Hardy, who have all now passed away. It was found that the Actuaries' Club was still in opposition, and had lodged an objection to the granting of the Charter; but negotiations took place in a friendly spirit between the Council of the Institute and the Actuaries' Club, and the objections of the Actuaries' Club were overcome, and all the members of the Club, fourteen in number, were brought into the Institute as Fellows under the Charter. Not one of these fourteen is now with us, all having passed away. The Charter was granted on 29 July 1884, and bye-laws had to be prepared, and under the Charter the then Council was to remain in office until the bye-laws had been approved by the Privy Council. Hence the then Council remained in office unchanged for four years, 1882 to 1885 inclusive. The bye-laws were allowed by the Privy Council on 26 February 1886, and this was reported at the Annual Meeting held on 5 June 1886, under the Presidency for the last time of Dr. Sprague, who then retired from the chair.

A great deal is due to Dr. Sprague for the success of this application for the Charter, but he himself at the Annual Meeting in 1886 offered a very generous and just acknowledgment to the late Mr. William Sutton, who was himself subsequently President of the Institute. Dr. Sprague said that he was sure all would pay a tribute to the services of Mr. Sutton, who had done most to bring about the change in the constitution of the Institute. Mr. Sutton had drafted the Charter and bye-laws, which had then been carefully considered by the Council; and also his influence as Actuary to two Government Departments had been invaluable. Mr. Sutton at that time was Actuary to the Registrar of Friendly Societies, and also Actuarial Adviser to the Board of Trade; and he had found, under the Friendly Societies Act more especially, that it was very important that the position of actuaries should be defined and settled, so that unauthorized persons might not be permitted to practise. We may therefore say that the final success in obtaining the Charter was due in great part to Dr. Sprague and Mr. Sutton.

The reconciliation between the members of the actuarial

profession was complete, and later on the old Actuaries' Club and the Institute of Actuaries' Club, which had been formed from the leading members of the Institute on much the same lines as the Actuaries' Club, were fused, and continued under the old name of The Actuaries' Club.

Dr. Sprague was a great mathematician, and yet in comparatively few of his actuarial writings does much of the higher mathematics appear. He kept that in reserve, and preferred, whenever possible, to restrict himself to more simple ordinary algebra. Nevertheless he had an essentially mathematical mind, which it was his delight to apply to practical work. He possessed in rare degree what may be called mathematical instinct. He thought mathematically, and he spoke mathematically, and he wrote mathematically, even when there was but little appearance of mathematics on the surface. All this comes out clearly in his Presidential Address, delivered to the Actuarial Society of Edinburgh on 13 November 1874 (*J.I.A.*, vol. xviii, p. 403). The address was on the Usefulness of Mathematical Studies to the Actuary; and, on a first perusal, it seems to be almost absurdly elementary; but, on further acquaintance, it is found to contain deep wisdom. He says that the mathematician takes nothing, not even the simplest thing, for granted; and that by him language is used in a strict and precise manner. Words are, as it were, symbols, and sentences formulas; and hence the mathematician thinks clearly and accurately, and speaks and writes without ambiguity. All this presupposes what has been called above the mathematical instinct, which Dr. Sprague combined with profound mathematical knowledge; but here it may be remarked that, with mathematical instinct, combined with even only moderate attainments in the higher mathematics, a man will not be liable to fall into error. He will think for himself, and will avoid pitfalls by which the unwary may be entrapped. He will travel safely; whereas a man who has read a great deal of mathematical literature, and who has learned and remembered a great deal, but who is not endowed with the mathematical instinct, and who has not thought for himself, will be in danger; and may try, as it were, to force a round formula, very good in itself, into a square hole. He will sometimes fail to perceive when the abstruse methods which are at his finger ends are not really applicable to the case in hand; and he will be apt to use them recklessly, and under conditions where they cannot produce satisfactory results.



This is more particularly so in the case of actuaries, who have, as Dr. Sprague says, "to deal even more with figures than with formulas", and who "must submit every formula to the test of actual numerical calculation": and on this point he adds that:

"In many branches of applied mathematics, it has been too much the custom in treatises designed for the English student—especially, perhaps, for the student who is reading for mathematical honours at the universities—to disregard the numerical application of the formulas proved."

At the time of the Address in 1874, there was not much in the way of higher mathematics required of actuarial students for their examinations; but Dr. Sprague looked forward to a change in that respect, and he wrote:

"It can scarcely be said that the extraction of the facts (for a mortality experience) is a process for which much mathematical knowledge is required; but when the facts have been extracted, the process of adjusting them so as to deduce a satisfactory mortality table, that shall proceed without any abrupt changes, and shall yet not deviate too far from the original facts, is a problem requiring for its complete solution the very highest mathematical attainments. Not only is an intimate knowledge of the Differential and Integral Calculus useful for the purpose, but a knowledge of the Method of Least Squares may also be extremely serviceable. Until we have come to this point, we have been able only to speak of a knowledge of the differential calculus as serviceable in a general way to the actuarial student; but when we come to the present part of our subject, and examine the methods that have been applied with the most success to the graduation of mortality tables, we are compelled to say that no actuary can be considered to be fully qualified for the discharge of all the duties he may be called upon to undertake, who has not a competent knowledge of the differential and integral calculus. It is not to be forgotten, moreover, that an extensive use has of late years been made of the calculus, to obtain more exact values of annuities payable by half-yearly, quarterly, or other instalments; also of complete annuities, and in a variety of other questions, especially in the Theory of Complete Annuities. It is not yet essential for an actuary to be familiar with these investigations, but it is impossible to say how soon it may become so: and in the meantime a study of them cannot fail to give the actuarial student a greater command over the simpler parts of his subject."

Only two papers, illustrative of Dr. Sprague's mathematical genius, can be referred to here: the first on "A new Formula for Interpolation" (*J.I.A.*, vol. xxii, p. 270, July 1880), and the



second, on "The Graphic Method of Adjusting Mortality Tables" (*J.I.A.*, vol. xxvi, p. 77, October 1886). They differ much from each other, but each possesses its own well marked character.

Dr. Sprague's method of interpolation is that which is now known as the Osculatory method, a name first given by Dr. Karup. Dr. Sprague foreshadowed it in an earlier paper in 1867 (*J.I.A.*, vol. xiii, p. 305) on the Value of Annuities payable half-yearly, &c., on page 322 of which he indicated its principles, without developing them. The problem is, from equidistant, usually quinquennial, pivotal values of a function,  $u_0$ ,  $u_1$ , &c., to  $u_5$ , to interpolate the values intervening between  $u_2$  and  $u_3$  in such a way that when interpolation is similarly effected between the values  $u_1$  and  $u_2$ , and the values  $u_3$  and  $u_4$ , there shall be no break at the points of meeting of the adjacent partial curves at  $u_2$  and at  $u_3$ .

Dr. Sprague secured the necessary smoothness by arranging that, at the point of meeting of each pair of adjacent partial curves, these curves shall have the same tangent, and the same radius of curvature, that is, that they shall have the same first differential coefficient, and the same second differential coefficient. The values of the differential coefficients at the points  $u_2$  and  $u_3$  have to be determined by means of the given pivotal values of the function; and that is done by supposing parabolic curves of the fourth order to be passed through the points  $u_0$  to  $u_4$ , and through the points  $u_1$  to  $u_5$  respectively. Starting with the equations to these curves, and working by an abstruse mathematical process, the five leading sub-divided differences of  $u_2$  are determined, and the terms intervening between  $u_2$  and  $u_3$  are then inserted by addition.

The process followed by Dr. Sprague in deducing the formula may be simplified and shortened a little, but it must always be abstruse. Nevertheless, the arithmetical application of the formula is easy, and the resulting completed table is free from all roughness. Dr. Sprague used the formula for the five select columns of his Select Mortality Tables, the pivotal values having been calculated by a graphic method; and the columns are very smooth.

In the earlier paper of 1867 Dr. Sprague mentions a much shorter formula of the same kind, with only three differences. At the point of meeting he gives to the two partial curves only the same tangent, and not necessarily the same radius of curvature; that is, he gives them the same first differential

coefficient, and ignores the second. It is found from experience that, for the purpose of constructing or of graduating mortality and other tables used in life contingencies, and for pension funds, &c., the shorter formula is all that is required, and that it gives excellent results.

This, the first of the papers chosen as illustrative of Dr. Sprague's methods, bristles with the higher mathematics. It is on interpolation; but that also necessarily includes graduation; and Dr. Sprague himself used the formula for the purpose of graduating his Select Tables.

The second illustrative paper is on graphic graduation; and in it Dr. Sprague condemns in language without restraint all mathematical formulas for graduation. He does not, however, condemn mathematics, but, on the contrary, he shows how there must be a sound knowledge of the mathematical properties of curves if a graphic graduation is to be really scientifically successful. On this point he says, "In order to understand and successfully apply the graphic method of graduation, it is necessary to study carefully the relations that exist between the progression of the numbers and their differences, and the form of the curve." In fact, according to Dr. Sprague, when graphically drawing a curve we must not, as some have said, be guided solely by the eye, but we must allow mathematical instinct to have full play.

Dr. Sprague, with various diagrams, gives a beautiful analysis of the nature of curves, and then supplies copious numerical examples of graphic graduation. He furnishes the figures all complete; and one cannot but be struck by his conscientious industry. He spares no labour to make his final results perfect.

Dr. Sprague's paper on graphic graduation led to a heated controversy. In it he had examined the results of several graduations by mathematical formulas, including that of Mr. Woolhouse, and compared them with his own graphic graduation; and he summed up by saying "We must include in one general condemnation all such graduation formulas as Mr. Woolhouse's, Mr. Higham's and Mr. Ansell's."

Naturally, Mr. Woolhouse replied, and Dr. Sprague replied again, and much dust was raised; but, in the end, something like peace was restored. Dr. Sprague made his "final remarks" in *J.I.A.*, vol. xxix, p. 232. To test the matter thoroughly, he made a graduation by Woolhouse's formula of the Text Book table and said: "Taking a general view of the figures, we infer

“that, although Mr. Woolhouse’s method slightly disturbs the law of the series ( $l_x$ ) to which it is applied, the practical effect of this disturbance on the probability of dying is so slight as to be of no practical importance except at extreme old ages. I feel therefore bound to admit that my objection to the method, on the ground of its tendency to disturb the law of the series, although theoretically well founded, may be disregarded in practice.”

Besides being an actuary of unusual erudition, Dr. Sprague was a first-class business man; and of this the prosperity during his management, extending over thirty-nine years, of the two great Insurance Societies of which he was in succession Chief Officer, is sufficient evidence. The business of life assurance, using the words “life assurance” in their broadest sense, was the business of his life and in all his investigations he kept in view their immediate practical application, and refrained from acting on any theories he might have formed until he had put them to practical test. His writings all bear witness to this, his prevailing characteristic; and one would like to refer to many of them in some detail; but, in such a memoir as this, to do so is impossible. They are too voluminous. The mere list of his communications to the *Journal* of the Institute alone occupies nearly five and a half pages of the index. Then, there are actuarial publications of his to be found elsewhere. In April 1874 he brought out a volume entitled “Life Insurance Accounts”, in which he analysed minutely all the returns to be made by companies under the Act of 1870, and gave careful explanations of their meaning, and of how the schedules should be filled in—all of which was very useful at the time, when the new form of the accounts was but little understood except by the few.

In 1876 he contributed to the 9th edition of the *Encyclopædia Britannica* an exhaustive article on “Annuities”, in some parts highly mathematical, which superseded the article by Joshua Milne written many years before, and which was reviewed by the late Mr. Sutton in the *Journal*, vol. xx, p. 112. There are also to be found in the Transactions of the Actuarial Society of Edinburgh sundry contributions from Dr. Sprague, including some of his Presidential Addresses to that Society and his too little-known paper “On Probability and Chance, and their connection with the Business of Insurance.” All of these we must pass over, and confine ourselves to touching lightly on only two or three of the most salient points.

Dr. Sprague's monumental Select Tables have already been mentioned. In two elaborate papers in the *Journal*, vol. xxi, p. 229, dated November 1878, and vol. xxii, p. 391, dated January 1881, their construction and use were fully explained, and monetary tables at 4 per-cent interest were given; and in 1896 they were published separately in book form, with extensive monetary tables at four rates of interest.

Early in his managerial career, Dr. Sprague took up with his usual earnestness the question of investments in reversionary securities of all kinds, and he investigated them from every point of view, and examined them in all their phases. The result is a long series of papers, developing the mathematical theory of, and the principles to be followed when carrying out transactions in, reversions, reversionary interests, &c., whether of the more simple descriptions, or whether complicated by probabilities of marriage, of the birth of issue, &c., &c. The titles of a few of these papers may be quoted, just to show how comprehensive were Dr. Sprague's researches:

*J.I.A.*, vol. xiv, p. 417, On the Valuation of Reversionary Life Interests.

*J.I.A.*, vol. xviii, p. 77. On the Apportionment by mutual consent of a Fund between the Life Tenant and the Reversioner.

*J.I.A.*, vol. xxiv, p. 327, On the Calculation of the values of Benefits that depend on the Death without Issue of One or More Persons.

*J.I.A.*, vol. xxv, p. 160, On the Probability that a Marriage entered into at any age will be fruitful; and that, if a Marriage has been childless for several years, it will afterwards become fruitful.

Perhaps the most important paper on this class of subjects is that in *J.I.A.*, vol. xxi, p. 406, which was read before the Institute on 31 March 1879. It has the comprehensive title, "On the Construction of a Combined Marriage and Mortality Table, from the Rates of Marriage and Mortality among any body of Men; and on the Calculation of the Values of Annuities and Assurances that depend on the Contingency of Marriage, as well as Death; and their Application to determine the Rate of Premium for an Assurance against the Contingency of a Bachelor of a given age leaving Issue."

This paper was illustrated mainly with statistics derived from the British Peerage families, but also with some derived from other sources; and it gave the graduated tables of Marriage



and Mortality, which follow the usual Mortality Table form ; and it gave also the monetary values of Marriage and Mortality benefits.

These tables of Dr. Sprague are really the precursors of the Service Tables now prepared for Pension Funds, which are used for measuring the values of benefits falling in on death, or on withdrawal, or on retirement on pension, and sometimes even on marriage, but which also include varying salaries as an additional complication.

At the present day the statistics used by Dr. Sprague are, for the most part, obsolete on account of changed social conditions ; but the principles remain for our guidance and help.

The only other portion of Dr. Sprague's work which can be referred to here is his papers on legal subjects, on which, as barrister, he was an authority. Besides law being involved in many of his other papers, he wrote on the Policies of Assurance Act 1867, on the Sales of Reversions Act 1867, and on the Married Women's Property Acts 1870 and 1882 ; and his very last contribution on assurance matters was that on Lost Policies, and was read before the Institute on 26 April 1897, being followed by a most interesting discussion. It appears in the *Journal*, vol. xxxiii. p. 373, and is valuable at the present day, although naturally circumstances are not all the same.

Dr. Sprague's first paper appeared in the number of the Institute *Journal* for October 1856, and his last in the number for October 1897, there being thus exactly forty-one years between them.

One feature of many of Dr. Sprague's contributions to the *Journal*, which must be mentioned, although not without regret, is the peculiar spelling which he adopted. He was an ardent advocate of spelling reform, and thought he was doing good service to the cause by using, in season and out of season, spelling of a phonetic character. The Council objected, holding that, although some of them were in favour of reformed spelling, yet a scientific journal was not the place to introduce it. But Dr. Sprague was obdurate, and refused to contribute more papers unless his spelling were admitted ; and the Council gave way. In later years, however, Dr. Sprague relented, and, for some time before he retired, he allowed his papers to appear in orthodox garb. The papers in the phonetic spelling are irksome to read, and thus lose some of their value.

Dr. Sprague did not confine his energies to matters actuarial, but took an active interest in other branches of science. He

was Fellow of the Royal Society of Edinburgh, and of the Edinburgh Mathematical Society, and he was a member of the Edinburgh Field Naturalists and Microscopical Society, to all of which Societies he made communications; and on one occasion at any rate he also contributed to the proceedings of the British Association.

In the publications of the Royal Society of Edinburgh there are papers by him :

On the Nature of the Curves whose Intersections give the Imaginary Roots of an Algebraic Equation. (1883.)

On a New Algebra, by means of which Permutations can be Transformed in a variety of ways, and their Properties investigated. (1893.)

He also contributed to that Society papers on actuarial subjects, but these have been either reprinted in the *Journal* of the Institute of Actuaries, or their subject matter repeated by Dr. Sprague in other papers in that *Journal*.

To the Edinburgh Mathematical Society he contributed the following papers :

Note on the evaluation of Functions of the Form  $0^0$ . (1884-85.)

On the Different Possible non-linear arrangements of Eight Men on a Chess Board. (1889-90.)

On the Transformation and Classification of Permutations. (1890-91.)

On the Geometrical Interpretation of  $i^i$ . (1893-94.)

On the Eight Queens' Problem. (1898-99.)

On the Singular Points of Plane Curves. (1902-03.)

His outdoor recreations supplied him with material for communications to the Edinburgh Field Naturalists and Microscopical Society, of which the following is a list :

Bones and shells taken from a kitchen midden on Inchkeith during 1881.

The "Green Balls" of Loch Kildonan.

On the Growth of Leaves.

Fibre Balls.

Ripple Marks on Sand.

On the occurrence of the Fresh Water Mussel.

Notes on the Bournemouth Cliffs.

Notes on the Entomostraca (Water-fleas) of Midlothian.

The last of these papers was prepared conjointly with his daughter Miss B. Sprague, and it gained a prize for a beautifully mounted collection of the minute crustaceans treated of.



Dr. Sprague was twice married, first in 1859, to Miss Margaret Vaughan Steains, younger daughter of Mr. James Steains of Liverpool, by whom he had eleven children, eight of whom survive ; and, second, in 1908, to Miss Jean Elizabeth Stuart, third daughter of Mr. Morris Forsyth Stuart, Navy and Army Contractor of Edinburgh, who also survives him.

It is worthy of note that his eldest surviving son, Dr. A. E. Sprague, is, at the time of writing, President of the Faculty of Actuaries in Scotland. This is the first occasion of both father and son having achieved that great distinction.

GEORGE KING.

*Address by the President, SIR ALFRED WILLIAM WATSON, K.C.B.*

[Delivered 29 November 1920.]

IN commencing the address with which custom ordains that the recently elected President shall inaugurate the session, I am constrained again to express my deep appreciation of the honour conferred upon me in my elevation to this Chair. I realize the dignity, and not less the responsibility of the Presidential Office, and I cherish the hope that a profound sense of duty to the Institute, to which I owe so much, will enable me to meet the obligations which my fellow members have seen fit to impose upon me.

It is usual, and fitting, on these occasions to refer to those who have recently passed from our midst. Meeting under the shadow of the fierce and protracted struggle from which we have thankfully now emerged, the remarks of my immediate predecessors in reference to this subject necessarily took a new character, and with the expression of the sorrow that has animated us all in contemplating the losses that have fallen upon our little community, there rang out a note of pride in the response of our younger colleagues to their Country's urgent call. Our membership of all classes reached its highest point in 1915 when we numbered just over 1,200, yet from among us no fewer than 428 men are known to have gone to the Forces, of whom 79 gave their lives. These figures are surely eloquent enough to need no addition. It is only that the full tale shall be told that I refer again to a subject which has already been so feelingly dealt with from the Chair.

We have to regret the death of four Fellows since our last meeting in this Hall. Mr. W. P. Clirehugh, who became a Fellow in 1863, had long ceased to be closely associated with our profession but had attained prominence as an insurance manager. He will be warmly remembered by certain of us who in recent years enjoyed his society in another circle.

Mr. Marco Besso of Rome was elected a Fellow in 1879 and, in the prefatory remarks to an interesting compilation of statistics for which he was responsible, was described in vol. xxvi of the *Journal* as an insurance expert of wide experience and great repute throughout the continent. We regret the loss of one who was a personal link between ourselves and an allied nation for which Englishmen cherish a warm regard, and whose present difficulties command their sympathy.

The death of Mr. Oscar Nash in tragic circumstances was lately reported to us. Mr. Nash had served on the Council and several times had acted as an Examiner. I myself retain grateful recollections of him as a tutor and a friend. We have also to regret the loss of Mr. L. G. Atkins, who died at an early age after a prolonged period of ill-health, which had cut short a promising career.

At the time when the last Presidential address was delivered, we still seemed to hear the reverberation of the guns which heralded the Armistice, the greater number of our younger members were still wearing His Majesty's uniform, and of the older men and others to whom military duty had not been allotted, a considerable proportion were performing tasks to which the national emergency had called them. In reference to some of these I am emboldened to make a similar claim for the professional training given by the Institute, and for the type of career of which that training is the prelude, to that which is so frequently made and conceded in respect of the educational system of our great Universities. The range of duties to which many of our members who were retained in civil life were suddenly called, was remarkably varied and entailed real, and, in some cases, even grave responsibilities. It was naturally to be expected that those who had attained success in the practice of their profession and in the conduct of the great organizations with which it is so closely linked, would find their new responsibilities well within their trained and disciplined capacity, but in regard to the younger men, in whose case experience had not ripened into maturity the powers implanted by an actuarial

education, the position was different, and the experiment might pardonably have been watched with some misgivings. If such were felt they were superfluous. To those of us who had the opportunity of seeing large and serious problems handled by junior members of the Institute the manner in which the work was done, and the capacity which it revealed, came as a striking proof of the value of the training which our curriculum affords. We always knew that our course provided a solid foundation of actuarial knowledge, the first necessity of professional success. We now discovered that like other sound educational systems, it made for the production of capable administrators, competent to grapple with novel and varied problems and to bring to their solution a calm and disciplined judgment, without which enthusiasm and industry would have availed but little. The first place in our regard, when we look back on the protracted agony of the War, must always be given to those of our members who responded to the call to Arms, so many of them never to return to us, but we may properly spare some feeling of pride for the competence and resourcefulness shown by those who served their country in other capacities.

Two years have elapsed and it is interesting to trace so far as we can the extent to which the course of actuarial life has been brought back to its normal channel. Disturbing and disquieting reflections are induced by a wide survey of national conditions, but a due sense of perspective will serve to keep these in check and, with some knowledge of history, will stimulate the faith we steadfastly hold in the destiny of our country and in its powers of recovery from the heavy blows under which the nations of Europe are still reeling. But whatever mood of despondency from time to time may be induced by the present social turmoil, no such emotion can obtain even a fleeting mastery over us when we confine our attention to our own professional sphere.

We naturally turn first in our survey to the work of the Institute itself and especially to its educational activities, which were entirely suspended during the war. The official classes for our students were reconstituted at the earliest possible date following the Armistice and have attracted a phenomenal attendance of students.

Our examinations were resumed in June 1919, and, in accordance with the special arrangements made for the post-war period, have subsequently been held at half-yearly intervals.

Bearing in mind the long interruption in the studies of our younger members, and the nature and the completeness of that interruption, the extent to which our students have availed themselves of the facilities for examination recently afforded to them is remarkable. On each occasion candidates have presented themselves in large numbers and passes have been quite as numerous as in the circumstances could have been expected. Whatever the truth of the general complaint of indifference and slackness in the resumption of the normal round, no charge of the kind can be made in regard to the students of the Institute of Actuaries.

As you are aware, certain modifications of the syllabus were made by the Council to ease the path of our students to the goal of the Fellowship during the transitional period. These modifications were framed with full regard to the guarantee of professional competence which the grant of the Institute's diploma is rightly held to imply; and in this respect, we found ourselves, in common with other professional bodies, severely limited by considerations touching the public interest. It is gratifying to know that the only protest which has reached us from students is that the tests we now impose upon them are not sufficiently exacting. That complaint, if such I may call it, is supported by arguments which merit our full consideration.

Having before us the somewhat imminent need for settlement of the permanent syllabus, we shall watch with interest the results of the options now given to candidates for Part IV of the Examination. Proficiency in certain technical subjects is an essential condition of the grant of the Fellowship. But with the constant expansion of knowledge, and the increase of efficiency demanded in every profession, specialization of work, upon more or less definite lines, has become the general practice. The actuarial profession has certainly not been unaffected by this development, and if in framing the syllabus of our examinations we refuse to take it into account we are confronted by a dilemma. We can either exact from our students proof of competency to deal with a far wider range of subjects than individually is ever likely to come before them in practical life, or, aghast at the task thus imposed upon them, we can restrict our tests to the branches of actuarial science in which the majority are likely to find their subsequent career. A middle course, under which to a sound knowledge of actuarial science and of its more general applications we might add a limited acquaintance with the problems involved



in certain branches of specialized practice, may have its advocates, but is obviously open to serious criticism.

Faced with this difficulty the Council have decided, as an experiment, to give the student preparing for his final examination some opportunity to follow his inclination in regard to subjects which can fairly be regarded as falling within special groups. The experiment is distinctly limited, the choice lying between the subjects proficiency in which is essential, on the one hand, to the actuary to a life office, and on the other to an actuary chiefly advising friendly societies or pension funds, or to one who looks for an official career in a State Department. Time is needed to show whether the new system is justified, and whether it can be further developed with advantage.

In connection with our educational work it has been decided by the Council, after prolonged consideration, that the revision of Part II of the Text Book has become necessary. This decision calls, I think, for something more than passing mention.

Part II of the Text Book was published in 1887 and a second edition, incorporating the few changes that time had shown to be desirable, was issued in 1902. For 33 years, therefore, the present Part II of the Text Book has maintained its ground as the standard work on the science of life contingencies. It is not too much to say that every actuary now in practice has cause to be deeply grateful to Mr. George King for the Text Book on which his knowledge of the technical side of our craft is based. It must rarely happen that any educational work maintains its authority, and that over the whole world, for a third of a century. Such, however, is the record of Part II of the Text Book, a work which testifies at once to the learning of its author and to the literary skill which has clothed with living interest a subject that, under other treatment, might so easily have become dry and repellent.

The changes of the last 30 years have certainly not, however, left our educational system unaltered. The young men who begin their actuarial studies to-day are equipped, as a rule, with a knowledge of advanced mathematics which was rarely available in the school education of 30 years ago. They approach their new studies from a point of view different from that of the generation whose needs inspired the Text Book of 1887. Much that was then relegated to the category of optional study for those of mathematical taste has now become the commonplace of our science. While the essentials of the Text Book must

remain, their order must be changed and their relative stress must be varied. To meet the altered conditions it has been decided to issue the new Text Book in two volumes, one relating to the mathematical theory of which a sound knowledge has become essential, the other comprising a full treatise upon the theory of life contingencies. The fact that Mr. A. Henry and Mr. E. F. Spurgeon have agreed, respectively, to become responsible for the new volumes will inspire full confidence in the success of the undertaking.

Turning to wider subjects we find much cause for gratification in the events of the last two years. If we seek to test the worth of our professional work to the community, the first criterion which we apply is naturally the progress of that great business of life assurance which called the profession into existence and without which our science, if it existed at all, would be little more than a shadow. The faith of the British public in the system of life assurance might have been highly tried by the consequences of the universal depreciation of securities which came immediately with the clouds of approaching war and was intensified as the struggle developed and men began dimly to realize the destruction of material capital and the diversion of human energy from productive work, to which, with the loss of valuable lives, their generation was called upon to submit. I am not sure that we fully appreciated the courage and inflexibility of purpose of those actuaries who were first required by the accident of fate to put into practice the doctrines which in days of peace, when discussion was largely theoretical, were laid down without much difficulty as appropriate to certain eventualities. It was a different matter to apply those doctrines to the test under the conditions evoked by the European War, and honour is surely due to those who, resisting an easier way and one for which a plausible defence might doubtless have been found, determined to stand by their principles and to rely upon the good sense of their policyholders to support them. As case followed case in which the distribution of profits was suspended, it was not difficult to discern that public opinion was supporting the Companies in the action upon which they had decided and in regard to which the views of the responsible actuaries must generally have been the deciding factor. Speaking with the freedom of one whose work has lain outside this particular field, I make bold to say that at no time in the long history of the system of life assurance did that system



stand higher in public opinion than at the termination of the war, when the suspension of bonus distributions had tended to become universal. In support of this view, which has no such unworthy basis as a mere desire to say pleasant things to my professional friends, I would instance the quite remarkable progress in life assurance during the year 1919. This gives us an unmistakable test, for with a profound derangement of currency values the effectiveness of the great mass of policies of life assurance which represented provident transactions, had been diminished to a menacing extent. If the system was as valuable, and as highly prized by the people, as those immediately concerned would fain believe, they were entitled to expect a large expansion in the volume of new assurances even before the community had settled down to its normal life, and despite the urgency of the need for capital which was already deflecting great sums to new trading and industrial enterprises. This expectation was realized. Comparing the year 1919 with 1913, the last year before the war, the new policies issued (not taking account of industrial assurances) rose by 70 per-cent, the sums assured under such policies by over 100 per-cent and the premiums payable in relation thereto by 150 per-cent. Our Institute is a scientific community, but to affect a lofty indifference to figures such as these would be absurd. Life assurance, if a business, is nevertheless a great social service. Having regard to its intimate connection with our profession and to the mutual dependence of the two, we have every reason to be satisfied with the position that the system of life assurance holds in public regard and to express that satisfaction in any review of the progress of actuarial activities.

But it is not only in the sphere of life assurance that the influence of actuarial science begins to flow in a wider channel. In his consultative work the actuary of established repute has frequently found himself in a position of high responsibility and has taken a gratifying share in the guidance of undertakings of great importance. This class of work in the nature of the case is generally of a confidential character, and it is possible neither to survey it in detail nor to discuss its developments. But there is reason to think that it is increasing in volume, and I believe that, with the growth of public and private spirit in regard to those essential services of which the need has hitherto been inadequately acknowledged, the actuary's prospects of useful work will further expand. It is at once a peculiarity and

a gratifying incident of our professional work that the matters in which the actuary is called upon to exercise his skill relate as a general rule to undertakings in which large numbers of people are deeply concerned. It is no matter for surprise that a calling round which a veil of mystery was long supposed to hang—with a measure of gratification, I have sometimes thought, to some among those who practised it—is now becoming, at least in name, familiar to the general public. It follows that there is a growing recognition of the circumstances in which the help of the actuary should be sought. This, I think, will continue, to the advantage of the community and certainly to the satisfaction of those who believe in the capacity of their profession to minister to the public welfare.

The appointment of a member of our profession to the Departmental Committee on Railway Superannuation Funds was some years ago the subject of pleasurable comment among us, and, when the Report of the Committee was published, it was not difficult for us to discern how large a part in its work Mr. D. C. Fraser had taken and how considerable was the influence which had been conceded to his technical knowledge and personal judgment. More recently we were gratified by the appointment of Sir Gerald Ryan to the Chairmanship and of Mr. Ernest Woods, then President of the Institute, to membership of a Departmental Committee entrusted with the onerous task of revising the National Insurance Act as a basis for legislation in regard to finance and administration. Remembering the satisfaction with which these appointments were regarded I feel that my address would be incomplete were it not to refer to the part which the actuarial profession has taken during the past two years in the work of Royal Commissions and Departmental Committees. The arduous labours of our late President, Mr. Geoffrey Marks, on the Royal Commission on the Income Tax are well known to those of us who were in close association with Mr. Marks during the period, long in itself but short for a Royal Commission, over which the deliberations of the Committee extended. The Country has cause to be grateful to the eminent body of Commissioners who undertook the heavy burden of revising our Income Tax system. Their Report is a great piece of constructive work, and the promptitude with which their chief recommendations were incorporated in our fiscal system is the best testimony to the success which they achieved. It cannot be expected that taxation in itself will

ever be otherwise than disagreeable to its victims, but it is a great thing to remove those inequalities in the incidence of taxation which create real grievances. I am not bold enough to predict, and certainly Mr. Marks would not desire it of me, that inequalities will not reveal themselves in the new system. But I venture the opinion that the Royal Commission has found a system which, to take a criterion which they adopted, satisfies "the common sense and instinctive judgment of the people." I cannot forbear a reference to the elegance of the various curves of taxation which result from the application of essentially practical methods of deductions and allowances from income in differing circumstances. It must have entertained Mr. Marks when working at the scheme with his colleagues, to discover how effective a graduation could be secured, in the particular case before him, without resort to those scientific processes which it has been our delight to explore in Staple Inn Hall.

I ought, perhaps, as the President of a scientific body, to express a conventional sympathy with the views of those who regret that no use has been made of mathematical methods. I am afraid I cannot do so. It may be true, as an eminent professor suggests, that the general public are satisfied to read the dial of a town clock without going behind to inspect the works,\* but I cannot join in the inference that taxpayers "are concerned only with the amount that each has to pay" and are uninterested by the mechanics of the scheme. With the practical wisdom of men of affairs the Commissioners have chosen the better way!

When I come to other enquiries upon which our profession has been represented I approach something of a dilemma. In the circumstances of official life it has fallen to me to take some part in these activities, and under all ordinary conditions I should be constrained to avoid the topic. But the opportunities which come to our profession to contribute to the solution of problems of public administration must always be of interest to us and, if only as a matter of historical record, the Presidential address should include some reference to them. It is from this point of view that I propose to deal with certain recent cases.

The Royal Commission on Decimal Coinage† included three actuaries among its 22 members, and the importance not less than the variety of the actuarial considerations which arose in the course of its work, and are dealt with in the Report, emboldens

\* Edgeworth, *Economic Journal*, Sept. 1920, p. 400.

† Report, Parl. Paper, Cmd. 628.

me to say that the profession was not over-represented. The active share taken by Mr. Marks and Sir Joseph Burn in the proceedings of the Commission is indicated by the published evidence. The history of the movement for placing our currency upon a decimal basis is a long one and few controversies make more interesting reading. It is scarcely too much to say that all that man could do to promote a project of which he was enamoured was done by Professor De Morgan for decimal coinage sixty years ago, and it is evident from the early volumes of our *Journal* that his views, and those of his associates, exercised a powerful influence upon the actuaries of their day. In the long period which has since elapsed, the penny has gained in importance in many directions, and immeasurably so, for example, in the case of industrial assurance, in which it stands, undoubtedly, for a definite fraction of the gold sovereign and in that capacity governs fifty million contracts. It would be useless to speculate as to what the attitude would have been to-day of those who promoted the decimal system in the middle of the nineteenth century. I do not claim greater wisdom for ourselves than for the able men in our profession, or associated with it, who threw themselves with such ardour into the battle then raging. But it is permissible to express a measure of relief in the fact that on the present occasion the actuarial profession was uncommitted to particular theories and that those among us who were called to serve on the Royal Commission were unembarrassed, in the exercise of their judgment on the evidence, by any prior declaration of supposed corporate views. We shall do well to keep the significance of this fact before us. The Institute has rightly attained to great influence, and that this shall be upheld it is incumbent on us to see that the weight of its authority is never lightly accorded. I have known debatable projects other than decimal coinage in regard to which it has been too easily assumed, outside these walls, that actuarial opinion must needs support particular views.

The Departmental Committee on the Superannuation of Local Government Officers\* was called upon to investigate a question of public importance on which we speak with special authority, and it was fitting that the Institute should be directly represented on the Committee as it was in the person of Mr. Elderton. It is seldom, I imagine, that a Departmental Committee can have found itself so frequently in contact with

\* Report, Parl. Paper, Cmd. 329.



actuarial problems in both their theoretical and practical aspects, and I think I can say for Mr. Elderton and myself that we shall not easily lose the recollection of the tasks imposed upon us by the zeal of our colleagues. In its masterly survey of a subject of singular complexity the Report owes much to the chairman, Mr. R. C. Norman, then Chairman of the London County Council, but as the matter is to be dealt with by legislation and has not yet wholly emerged from the controversial stage, it would be unfitting for me to comment upon the many points of actuarial interest which the Report explores. I should like, however, to suggest that having regard to the great number of authorities and of their employees whom the project affects, we should be rendering some service to the public interest were we to debate the general subject here. It would be egotistical to imagine that there is no room for differences of actuarial opinion on some of the questions treated by the Report and a debate initiated by one of those actuaries who have acquired special experience in the finance of Municipal Superannuation Funds could not fail to be profitable to ourselves as well as beneficial to the community.

The possibility of forthcoming legislation also precludes any detailed reference to the Report of the Departmental Committee on Workmen's Compensation.\* But allusion may be permitted to the part which actuarial considerations must play in the settlement of future premiums if the proposals of the Committee are adopted, and especially if, as the Committee recommend, the business of insurance against Workmen's Compensation risks is given a compulsory foundation. I hope I shall offend no susceptibilities when I say that in the course of my work upon the Committee I was impressed by the limited part that actuarial science had heretofore taken in the conduct of this branch of insurance. To mention a single point on which our science might be helpful it would be of great advantage, I am convinced, to all concerned if the collection and analysis of the statistics on which the companies base their rates were actuarially supervised. Lest I should be misunderstood by any of our friends of the accident offices who may chance to read my remarks, I hasten to say that I am not inviting them to submit to such elaborations as in the case of the State of California have produced a schedule of disablement allowances providing for 12,711,240 different cases. We may doubtless accept the statement that

\* Report, Parl. Paper, Cmd. 816.

these disablement ratings "have been carefully computed and "made readily understandable by a person mentally competent "to find in a railroad time-table when a train leaves." But given a series of events, the probability of the occurrence of each of them is a not unimportant element in the premium to cover the risk undertaken by the insurer. Information unfortunately stops short at this point and one is left wondering whether the whole transaction, from the calculation of the premium to the assessment of the emerging claims, is as severely scientific as we might be tempted to imagine.

The subject of Workmen's Compensation has not, I think, been before us since Mr. Penman gave us his valuable paper—which led to an excellent discussion—in 1910. Subject always to the disturbance caused by the war, I cannot but feel that many questions then indeterminate should now be nearer solution by actuarial methods, and that, especially, the principles governing the assessment of reserves for incumbent liabilities could again be profitably explored. With this I would add, as a fitting subject for actuarial discussion, the proposals of the Committee as to re-insurance of the liability for children's allowances in cases of fatal accident.

The Report of the Departmental Committee on Old Age Pensions\* which has been followed by legislation adding substantially to the rates of pension and to the charge upon the National Exchequer, will probably be of chief interest to actuaries, in their professional capacity, by reason of the recommendation of the Committee that enquiry should be instituted into the possibility of so developing and extending the present system of National Insurance as to make adequate provision, under that system, for all cases of invalidity and disability arising before the age of 70. In this the Committee found a difficult subject and prudently left it for extended examination at hands untrammelled by the general question which was their immediate care. It is clear that action in the direction indicated by the Committee would have to be preceded by a good deal of research work, and the question seems to be one in which the initiative might well be taken by private effort, prepared equally, in a scientific spirit, for negative as for positive conclusions. I commend the subject to any courageous young actuary who may be disposed to survey the growing ramifications of our system of public assistance. In the course of his studies he should by

\* Parl. Paper, Cmd. 410.



no means neglect the bold experiments which have been made by various of the Dominions.

The usually quiet surface of our professional life has been somewhat ruffled by the Report of the Departmental Committee on the system of Industrial Assurance.\* We have discussed the subject in general meeting, and very definite interest has been displayed in those points of actuarial practice which the Report has raised. The Institute is not concerned, in my opinion, in the questions of business policy and administration which the Departmental Committee have examined, and no reason other than this is required to lead to the exclusion of these embarrassing topics from our debates. But on matters affecting actuarial practice the position is otherwise, and in my judgment the Institute should be prepared to continue the discussions already initiated, with the purpose of eliciting the trend of professional opinion.

It is many years since a President surveying the opportunities of useful work that lay open to the Institute included "the interesting question of industrial insurance so far as concerns the actuarial principles and processes obtaining in its administration, about which very little is known."†

What at that day was "interesting" has, with the subsequent expansion of the business of industrial assurance, become of extreme importance, and the professional responsibilities of those of our body who are associated with the great industrial companies are not to be distinguished from those of any other class of our members. Can it be said that we have devoted a due share of the time of the Institute to the special problems and the difficulties that some of them have to meet? If the answer is in the negative those immediately concerned must take a share of the blame, since in the ordinary course we can only discuss what is submitted to us, and papers upon industrial assurance problems have been rare. But wherever the fault lies the fact remains that upon many questions affecting the valuation of industrial assurance companies, as for instance, the growing divergency of the rates of mortality from the favourite English Life No. 3 Table, the allowance to be made for future expenses, and the proper treatment of "negative values", or of dubious assets appearing in the balance sheets of certain companies, there is no authoritative body of professional

\* Parl. Paper, Cmd. 614.

† B. Newbatt, *J.I.A.*, vol. xxix, p. 14.

opinion to which an actuary confronted by difficulties can appeal for guidance. Our transactions are equally silent on matters such as the action properly to be taken in the various circumstances in which a deficiency may appear on the valuation of an industrial office, or the points to which an actuary should address himself when professionally consulted with regard to a proposed transfer of the engagements of such an office. These *lacunæ* in the volume of recorded actuarial opinion are regrettable, and in my opinion react seriously upon the applicability to industrial assurance of the principles of State supervision which are familiarly summarized in the maxim "freedom with publicity." Publicity is only a safeguard when translated into terms of active and competent criticism based upon knowledge. It is our task, I submit, to encourage such debates in this Hall upon the actuarial problems of industrial assurance as, by the interchange of ideas and the subjection of personal views to the discipline of examination, will yield a body of professional doctrine for the guidance of the practitioner and the education of the critic.

Having resumed our sessional meetings we look to our members to contribute papers for discussion. Apart from contributions on the theory of our science, the scope for which is possibly diminishing—though looking back on actuarial history the prudent man will hesitate to commit himself on the point—there are many interesting and important practical questions worthy of the attention of the actuary and I propose to avail myself of the license usually taken by the President to suggest a few topics for others to work upon.

I approach the subject of life assurance with the diffidence becoming an onlooker, but consideration of the Board of Trade returns suggests that the special conditions to which the business of life assurance has been subjected during the last few years might well be brought up for discussion, and debate enriched by the reflections of those who, in positions of high responsibility, have been immediately in contact with weighty problems.

We recollect that after the South African War we had the benefit of a valuable paper by Messrs. Schooling and Rusher upon the mortality experience of the Imperial Forces during the war. The recent struggle, with its appalling list of casualties, will furnish material on a far greater scale, and in relation to modern warfare, of a more suitable character, for the guidance of the actuary desirous of investigating liabilities which he may be invited or, in some circumstances, may be constrained to

undertake in the course of business. Recognizing the probabilities of further and still more revolutionary developments in the methods of conducting hostilities, it is nevertheless to be desired that our statistics as to the recent war should be as complete as possible. In the records of life offices, both ordinary and industrial, there must be information on the subject of great importance. Doubtless this would require to be supplemented, in order to secure its maximum statistical utility, by data secluded at present in the archives of Government Departments. That fact should not deter such work as is possible. The task of exploring all is doubtless beyond the powers of any one man, as the problems presenting themselves for investigation are beyond the scope of any one contribution to our discussions. May I hope that those who have access to statistical material on the subject and are free to do so, will give the Institute the benefit of their studies? In this connection I would express the hope that, at no distant date, there may become available for examination some digest of the mass of important data which the Ministry of Pensions must be accumulating. The statistics as to widows and children which the Ministry must possess, to take one branch only of its activities, would prove of great value in the solution of various problems in regard to which it is always difficult to find relevant data. I speak from experience when I say that, in this connection, the remarriage experience of the Crimean War widows, deduced by our colleague, Sir Joseph Burn, from material provided by the records of the Patriotic Fund, has been invaluable for actuarial purposes. While I do not overlook the consideration of expense, I do not hesitate to say that the community in general and actuaries, as the expert advisers of the community on many financial problems, would derive great advantage from the systematic publication of statistics such as the records of the Ministry of Pensions would yield.

To revert to my main theme I would make the further suggestion that in the financial convulsions which have produced so great a depreciation of securities and a correlative increase in the rate of interest (tempered, however, in the latter case by the demands of the taxing authorities) and in the resulting effects upon reserves and valuation bases, there is much to invite investigation of old problems from a new point of view. The violent derangement of the foreign exchanges presents another problem which, especially in relation to business abroad must have

given concern to some of us and ought to be worth examination, while the many indications of anxiety to do full justice to classes of policyholders affected in different ways by the suspension of bonus distributions suggest a topic which is not only worth discussion by the Institute, but is the easier to discuss because so many of our colleagues have had to face it in practice.

Questions relative to the rate of mortality among annuitants have assumed a new prominence with the continued increase in the net rate of interest obtainable in the case of annuity funds. The recent revision of the terms upon which Government annuities are sold, following the further decline in the market value of Consols, has not passed unremarked but has done no more than stimulate an interest which was beginning to find expression. There seems to be no doubt that the general experience is to the effect that no existing standard fully measures the vitality of annuitants at the present time, and the marked rise in the rate of interest, which normally might suggest a reduction in the price of annuities, certainly cannot be considered apart from this fact. When the experience of annuitants is again tabulated, I confess that I should like to see specific attention given to a feature which has always seemed to me to be important, namely, the expectation of further advancement in the duration of life. Past experience must necessarily be the basis of our work, but that work relates, as a rule, to the future. In dealing with annuities or in the valuation of pension funds have we always paid sufficient attention from this point of view to the course of past experience? The disturbance of accepted methods is admittedly difficult, but it is worthy of remark that Parliament has lately supplied us with a precedent for action within wider limits of assumption than, commonly, we have hitherto permitted to ourselves. I refer to the Welsh Church Act, 1914, Schedule 4 of which lays down the terms on which the existing life interests of incumbents and others shall, in certain circumstances, be commuted. After fixing the rate of interest and the basic tables of mortality, the schedule goes on to direct that allowance shall be made not only for the greater longevity of the clergy as compared with other classes of the community, but also for any prospective decrease in the death rate. This last requirement has presented an interesting actuarial problem to those who have had to deal with the Schedule and in its general aspect I should like to see it explored at our Institute, as a subordinate issue certainly, but rather more than a side issue, of some larger investigation.



In the consideration of subjects for discussion we cannot overlook those developments of social insurance which in recent years have done so much to widen the boundaries of our professional influence. It is now nearly ten years since the scheme of National Health Insurance was submitted to the House of Commons and for eight years that scheme has been in full operation associated with a modest plan of insurance against unemployment, which has just been extended to include some twelve millions of people within its scope. Recalling the acute political controversies in which the Insurance Act was born and through which in its early days it had to struggle, it is interesting to note how completely it appears to have been accepted, with the passage of time, as an established part of our social system. That the recent enlargement of the contributions and benefits under both schemes, to give effect to the altered value of the currency, should have been carried through Parliament without exciting a vestige of the old controversies is significant. It indicates, I take it, that in discussion the subject may now be approached without uneasy feelings of political undercurrents.

Nothing perhaps, in the early years of National Health Insurance, was so freely canvassed as the financial soundness of the scheme. It is a matter for regret that conditions due to the war have delayed the valuation of approved societies, but the answers given to recent Parliamentary Questions indicate that progress is being made and we may not have to wait long for information on a subject in which actuaries must be keenly interested. Having regard to the strength of the original work of our distinguished brethren, Sir George Hardy and Mr. Wyatt, the stabilising amendments for which Sir Gerald Ryan's Committee was responsible and the fall of the claims in recent years, as indicated by the published reports of the Department, we shall be justified in awaiting the valuation results with some complacency. The future of the scheme cannot but be influenced, moreover, by the high rate of interest, free of income tax, obtainable on the investments of the large accruing funds. These already amounted on 31 March 1920 to £78,000,000. After taking into account depreciation, an element of possibly limited importance in the special circumstances of the case, profit from interest should become a substantial factor.

Some general questions connected with social insurances

appear to have escaped full discussion, and I think, are worthy of our attention, as persons who should be interested in financial problems. We have in each of the two existing plans of State Insurance a scheme under which the contributions are provided jointly by employers, by employees and by the State. The distribution of the burden among the three parties concerned is not the same in both the schemes nor is the form which the State contribution takes, though in the long run, doubtless, the substance is the same. It is an interesting speculation as to what the contributions of the several parties ought, theoretically, to be for different forms of insurance effected under compulsion, and whether there is a guiding principle by which Parliament can direct itself in such matters. I realize that the question involves economic doctrine, tempered by the psychology of taxation, and that problems of the kind are seldom of the order for which positive solutions can be found. But I express the opinion that a discussion of the economics of social insurance with special reference to the distribution of the burden would at the least afford an attractive intellectual exercise.

There are other points, more certainly falling within our professional sphere, upon which some of our members are in a position to enlighten us. Statistics as to industrial life are far from complete. We can get from the census an approximation to the numbers of employed people in successive age groups. We find a heavy decrease in those numbers as age advances. We know, of course, that mortality, emigration, and, in the case of women, marriage, are operating to reduce the numbers employed, while errors of age, whether wilful or otherwise, obscure the clarity of all censal figures. But with all these factors properly discounted, there still remain discrepancies, puzzling to the student and a distracting influence in practical estimates. Approved societies are in a position to add to our knowledge in these matters. Their records, if tabulated and analyzed, should throw light on the movement of the population between industrial employment and other vocations; equally they should yield valuable information as to the rates of sickness prevalent from time to time in different classes of the community, while statistics as to the contributions paid might enlighten us as to the course of employment and the effect of age upon unemployment, a subject to which much rhetoric has been devoted but as to which little is known in fact. It is possible no doubt that official attention will from time to time



be directed to some of these points, but official investigations are subject to limitations which do not apply to private research, especially when the sources of information are less directly available to the official than to the private investigator. Those of our members who have access to facts that can tell us much of the social conditions of the people should be eager to use their opportunity.

The problem of unemployment has received little attention from our profession and when a beginning was made with unemployment insurance as a national responsibility in 1911, the reports of Mr. Ackland practically broke new ground so far as actuaries were concerned. May I express the hope that the Bill of 1920 (now an Act), with the call which it made for further actuarial work, will stimulate professional interest in the subject? To those who are inclined to take up the problems of unemployment with special reference to their financial incidence, I would recommend a preliminary perusal of Sir William Beveridge's valuable work entitled "Unemployment: A Problem of Industry." Apart from its absorbing general interest this volume indicates in some degree the nature of certain difficulties which beset the eminent actuaries who were asked, some dozen years ago, to advise the Royal Commission on the Poor Law.

Apart from the practical troubles in which an actuary finds himself involved when called upon to consider a subject without reliable data, a point upon which unemployment insurance is certainly distinguished from most contingencies with which we are required to deal, there appears to me to be one great difference between the subjects of our daily work and unemployment insurance. Whereas in the former we have types of risk that increase more or less steadily with age and demand the constant growth of reserves over long periods, we have, in unemployment a risk which moves in cycles dependent on the course of trade. The theory of insurance in regard to this contingency predicates the accumulation of reserves in times of active trade and their absorption in periods of stress. In the practical operation of a national enterprise this may have awkward reactions.

If the new scheme of insurance against unemployment should operate in accordance with the theoretical premises—and I see no reason to expect the contrary—we must be prepared for cyclical movements of the fund of a kind to which we are unaccustomed in any other form of insurance. This may be accompanied

by some degree of popular misapprehension, leading at one time to demands for extension of benefits, and at another to vaticinations as to a coming collapse, both equally unreasonable. Actuaries will do good service to the public if they will interest themselves in unemployment insurance from their own special point of view, and assist in the dissemination of sound knowledge as to its financial characteristics.

The Ministry of Labour is another department which, if only through its administration of the Unemployment Insurance Act, should be able to contribute statistics relative to social conditions. The rule which limits unemployment benefit during the insured life-time of the contributor to one week for every six contributions he has paid makes it necessary to keep a running account in respect of the contributions and benefits of each person among the millions brought under the scheme, and with the record available of the industry, and the particular occupation in that industry, in which each person is engaged, the Ministry will find itself, in the course of time, in the possession of a volume of facts rivalling that presented by a census, while affording a wealth of details as to industrial conditions to which no census-taker could aspire. We actuaries have more than a general interest in statistics such as these records would yield. They would give us the material for the solution of some of the new problems which we must expect in the gradual expansion of our professional work.

It is perhaps worth mention that the Unemployment Insurance Act opens out the prospect of some measure of interesting employment for our profession. I refer to the provisions of the Statute under which in lieu of insurance through the general scheme any industry may, under certain conditions, set up a scheme of unemployment insurance for itself or may adopt a supplementary scheme. The benefits provided by a special scheme, though not less favourable, may be widely different from those of the Act and may include provision for partial unemployment, such as "short-time" and loss of wages through exceptional conditions. I imagine that sooner or later the provisions as to special and supplementary schemes will enjoy a certain amount of popularity, but I do not think that in their financial details they can proceed far without assistance such as, given suitable data, the actuarial profession would be prepared to render.

Thus far I have dealt with schemes that are now on the

Statute book, but we have precedents for the discussion of projects of social welfare before the stage of legislation is reached. I recall particularly the valuable papers which were presented to us by the late Mr. R. P. Hardy "On the formulæ for determining the value of benefits according to the principle of 'collective assurance'", and by Mr. T. E. Young on "The German law of insurance against invalidity and old age." Remembering the profitable discussions which those papers evoked, there are, I think, other subjects which might, at their present stage, be examined here in the same spirit of scientific enquiry. One of these is presented by the proposal for "widows' pensions," a question which, naturally, excites much interest and has been the subject of debate in Parliament. Recent answers to Parliamentary questions indicate that cost is one, perhaps the chief, stumbling block, and actuaries can judge better than most people how formidable that obstacle is. What is advocated, apparently, is something in the nature of a family annuity, continuing during the dependency of the children. It is doubtful whether in practice the problem could be so limited. The future of the home-keeping mother must, it seems to me, be safeguarded after her children have passed beyond her care and when through long disuse, enforced by the scheme itself, her industrial powers have become atrophied. Failing the acceptance by the State of a huge new burden, from which in present circumstances public men may well recoil, is it possible that the problem of the widow and orphans is capable of treatment by the methods of collective assurance? How assurance in this case should be arranged, whether it should be voluntary or compulsory, and what the contributions for particular benefits should be under various alternative conditions, are matters well worthy of the study of actuaries and I commend them to any who may be looking for an attractive subject for research. Even in its purely theoretical aspect the measurement of the risk here indicated and the determination of a premium is an interesting study, regard being had to the present limitations of the census material and the scientific devices to which the actuary must resort to overcome them.

It is desirable, I think, again to call your attention to a matter which was dealt with in the last Presidential address and has acquired additional importance in the last two years, a period of greatly rising prices and consequent adjustments, more or less permanent, in scales of salaries and wages. I

refer to the present situation in regard to Superannuation and Pension Funds. We are familiar, of course, with the Superannuation Funds of the great railway companies and public utility services, but the funds associated with many important commercial undertakings, which rarely come under public notice, are believed to be very numerous and to afford the hope of security in declining life to large numbers of our people. On the actuarial problems associated with these funds our *Journal* contains many notable papers, and so far as methods of valuation are concerned little remains to be done, though there is still scope for original work in one or two directions. In regard, for instance, to the rates of mortality among pensioners and the extent to which select functions should be used in the valuation of disablement pensions, we still lack authoritative pronouncements based upon full investigation. But points such as this cannot take rank in importance with recent developments. The system under which both contribution and pension are based upon the salary of the individual was never without its difficulties, as every actuary who has undertaken the responsible work of advising on the position of superannuation funds will testify. The system assumed that the scales of emolument would be stable and would also govern the pensions when these enured. The assumption was not always realized, but the basis was fairly stable, and the difficulties which arose were generally capable of adjustment. With the great increase of salaries and wages in recent years a new situation has arisen, and an altered standard of future pensions has to be recognized. It is true that future contributions will also follow the general rise in emoluments, but whence is the corresponding increment to the reserves to be obtained? So far from this being in sight it is at least probable that, in some cases, a serious depreciation of the securities representing existing funds has to be faced. Some little comfort is to be derived, no doubt, from the fact that substantially higher rates of interest may be looked for in the future while the position as to the contributions for new entrants may even have been improved. But the value of these factors should not be exaggerated. In the coming adjustments delicate and difficult questions must arise as to the apportionment of new burdens between employers and employed. The need for skilled actuarial assistance will be insistent, and our profession should, in my judgment, be preparing itself by frank discussion of the problem among ourselves. I have reason to understand

that during the coming session we shall have at least one paper which will give us the opportunity of surveying this thorny field. One other point impresses itself forcibly upon me. I apprehend that the disclosure of serious valuation results will be followed by an outburst of the criticisms of actuarial methods and conclusions which to a certain type of mind are so much better than facing the real issue. Such a situation may call for a demonstration of those principles of professional loyalty and support of which the development is undoubtedly one of the chief purposes of the Institute.

The actuarial profession has always taken a special interest, for obvious reasons, in the decennial censuses of the population of the United Kingdom, and many things combine to stimulate this interest in regard to the census to be taken in the spring of next year. We have been gratified by the opportunity given to us by the authorities to express our views as to the items which should be included in the Householder's Schedule. We hope that we may be permitted to make representations as to the form in which the facts shall be tabulated and presented. We remember, with satisfaction, that on the occasion of the last census certain parts of the work and notably the preparation of the national life tables were, for the first time, placed in the hands of a member of our profession, and we regard with both respect and admiration the results of Mr. George King's labours in this connection. It is at least unlikely that the preparation of the forthcoming and future life tables will be entrusted to other than actuarial hands, while as regards the subject in its wider aspect the profession will learn with pleasure that a Fellow of the Institute is acting as secretary to the official body charged with the preliminary work and the general oversight of the census.

Our *Journal* contains many references to the census and the subject has been frequently debated among us, the last occasion of formal discussion on the general question having been, I think, in 1901, when we had before us a valuable paper by Sir Gerald Ryan, entitled "The Case for Census Reform." In these circumstances it is unnecessary for me to apologize for bringing to your notice the Census Act, 1920, which is the authority for the census of 1921. Differing from previous measures of the kind, this Act is permanent in its operation, and it will no longer be necessary for a separate Act to be obtained in respect of each decennial census. The authority to take a



particular census will be derived, in future, from an Order in Council, made under the Act of 1920, and duly laid before Parliament. The limitation of the Act that an Order shall not be made at shorter intervals than five years indirectly marks a great advance upon the previous practice, since it gives authority for that quinquennial enumeration for which all who have to deal seriously with statistics have long pleaded. In view of the cost of a census and of the time absorbed by the tabulation and presentation of its results, it is reasonable to doubt whether the full enquiry to which we are accustomed will, or need be repeated at shorter intervals than ten years, as heretofore, but the power to make even a limited enumeration of the people at intervals of five years should be of great value. It is also worthy of note that a census may be authorized at any time in respect of the population of a local area. The expense of such a census will be borne by the local authority, at whose instance it must be taken.

The provision of permanent censal machinery raises wider questions, as to the proper scope of a census, than might have presented themselves in regard to a single enumeration. The matters on which enquiry may be made are contained in the Schedule to the Act, and on such familiar and essential points as name, age, sex, occupation, condition as to marriage, &c., the Act provides that the Order in Council, if not objected to by either of the Houses of Parliament, shall confer the necessary authority. But a different procedure is laid down with regard to certain powers on the proper use of which actuaries and statisticians will certainly be disposed to lay great stress. These provide that enquiries may be made as to "any other matters "with respect to which it is desirable to obtain statistical "information with a view to ascertaining the social or civil "condition of the population", and authority to use these powers must be conferred by specific resolution of both Houses of Parliament. It is interesting to observe, in this connection, that the draft Order which was recently laid in regard to the Census of 1921 proposes, with the assent of Parliament, to use the special powers for two purposes. The first is to obtain particulars as to the number and ages of all living children, and step-children, under 16 years of age in respect of married men, widowers and widows. The second is to ascertain whether each of the parents of an enumerated child is living or dead.

I have no doubt that actuaries will be unanimous in the



view that particulars such as these are essential to a complete study of the condition of the people. The tabulation of statistics of families in relation to locality, to the ages of parents and to the industries and occupations of the parents, would obviously supply a notable addition to our scientific equipment, and on this ground alone would command our assent. But this apart, we are prepared to take our share in all statistical investigations which can add to the public knowledge on matters affecting the welfare of the community, and the collection of any material tending to increase our capacity in this direction will receive our warm support.

It is proposed by the draft Order that, on the present occasion, age shall be stated in "years and months" instead of in "years" only. I do not take this to indicate a desire for meticulous detail, but rather to represent an effort to overcome, by suggestion, the tendency to loose and inaccurate statement of ages which has hitherto been an exasperating feature of censal work, and in actuarial opinion has reflected seriously upon the authority of successive National Life Tables. Assuming this to be its purpose the new proposal will be endorsed by actuaries, who will be keenly interested in its outcome.

Section 5 of the new Census Act is worth quotation in full. It is to the following effect :

"It shall be the duty of the Registrar-General from time  
"to time to collect and publish any available statistical  
"information with respect to the number and condition of  
"the population in the interval between one census and  
"another, and otherwise to further the supply and provide  
"for the better co-ordination of such information, and the  
"Registrar-General may make arrangements with any  
"Government Department or local authority for the purpose  
"of acquiring any materials or information necessary for  
"the purpose aforesaid."

The powers conferred upon the Registrar-General by this Section would appear to open up the possibility of a great extension of the utility of the statistical side of his Department. It seems reasonable to hope that by co-operation between the Registrar-General and other public authorities we may obtain hereafter in systematized and co-ordinated fashion a great deal of statistical information which has not been available in the past, or has made its appearance under conditions such as to detract materially from its value for research purposes.

From this brief summary of the provisions of the Census Act it will be seen that the Act marks a distinct step forward and goes a long way to meet the criticisms which have been levelled against the arrangements which have hitherto been in force. In this connection it is, I think, important to recognize that the question as to what a census should include is viewed from various angles by different people. It is fitting that actuaries should have definite ideas, well in advance of those of the general community, on this subject, and it is, perhaps, inevitable that when the scope of the census falls short of our ideals the criticisms in which we are prompted to indulge should be aimed at the departmental officials, who are supposed to be responsible generally for the whole scheme of the census and specially responsible for its shortcomings. To those who hold that view I would recommend perusal of the debates in the House of Commons on the Census Bill of 1920. I do not think, and, if I did, it would be improper for me to suggest, that the Act, as passed, imposes any limitations of an unreasonable kind upon those whom it is the fashion to label the bureaucracy. But while the Act provides means for obtaining a comprehensive statement of the condition of the people, such as a census ought to yield, the debates show that members of Parliament are closely interested in the subject, suspicious of ambiguous phrases and determined to oppose anything that savours of intrusion into the private affairs of the citizen. For one reason or another the State has invaded this domain so much in recent years that one seems to be carried back a generation or more in reading the debates on the Census Bill. It is, nevertheless, right that the representatives of the people should be sensitive as to the nature of the enquiries proposed to be made in the census. I only mention the matter here to lay stress on the fact, which I think has sometimes been forgotten, that when statisticians put up demands they should, in justice to ministers and officials, remember that their proposals, if adopted, will have to be justified to Parliament, and must be supported by evidence of real need.

It will not, perhaps, be out of place to remind you that between taking the census and preparing for the tabulation of its more refined products, some little time elapses which should be turned to advantage by those who are specially interested. The classification of the facts with reference to industry, occupation, conjugal status, dependency and the like, offers a wide

field for exploration and discussion. To obtain data for the solution of problems coming before us in our practice, many of us must have had occasion to resort to the numerous volumes in which the results of the census are published. Others of us are keenly interested in social questions for the advancement of which we have found it necessary to refer to censal reports. It would be affectation to pretend that we have always found what we wanted and we have often been inclined to think that, within the limits of the facts gathered, something more might have been given to the world with little additional trouble. May I suggest to this meeting the possibility of another discussion upon the subject of censal tabulation. A paper of modest scope, if promptly submitted, would afford an adequate basis for debate.

My remarks may fittingly close with a reference to the recent labours of the Institute on behalf of the Royal Patriotic Fund Corporation. We have for many years undertaken the actuarial work required by the Corporation, accepting the duty as a privilege and an opportunity to acknowledge our share of the debt which the country owes to those who fought and died in its service in the Victorian era. Under the influence of the Great War the public conscience has awakened more fully to its duty to those who have fought its battles, and with the substantial provision for the sufferers made by the State it is no longer necessary to appeal to private benevolence. But the Royal Patriotic Fund Corporation, under the Presidency of H.R.H. The Duke of Connaught, continues to administer the honourable trust handed down to it from the last generation, and the membership of the Corporation, conferred by statute upon the President of the Institute, is one of the most highly prized attributes of the occupancy of this Chair. It is a matter for deep satisfaction that during the last two years the Institute has been able to give its assistance in the heavy work of revising the pensions and allowances paid by the Corporation, a task imposed by economic conditions which must have severely affected the recipients of small pensions. It is only fitting that I should add that in my attendances at the Committees of the Corporation on which your President serves, I have been impressed by the reality of the duties and especially by the great influence which, in a difficult period of financial adjustment, was exercised by my predecessor.

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*On a Short Method of Constructing Select Mortality Tables.* By  
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[Submitted to the Institute, 17 January 1921.]

1. AN apology is scarcely required for explaining an easy method of constructing select mortality tables; and it is hoped that the method now described and illustrated will prove useful to others, and that they also will find that it gives results sufficiently accurate for all ordinary purposes.

2. Hitherto, select mortality tables have been constructed only by very elaborate processes; and the amount of labour involved is practically prohibitive in such cases as occur now and then in actuarial practice. Possibly many actuaries do not often require special tables of the kind, but occasions do arise where it is of real importance to construct them. This happened to me in respect of the experience of the female annuitants of the Royal National Pension Fund for Nurses, and hence the method which I now suggest was devised.

3. The first select tables constructed for practical use were those of the late Dr. Sprague, whose loss, so recent, we have to deplore; and his method was set forth in 1878 in great fulness in his paper, *J.I.A.*, vol. xxi, p. 229. They were based upon the Institute of Actuaries  $H^M$  Experience, but that experience was complicated by the fact that the ages taken were those next birthday, and the first year of the select period was really only half a year, while the succeeding years ran also from half-year to half-year. More recent experiences are not subject to this disability.

4. Dr. Sprague took the  $H^{M(5)}$  Table as the ultimate, and he made it a point to join on closely the tables for the five years of selection with the ultimate table; and this resulted in the necessity of applying a certain amount of what may almost be called violence to the probabilities. Frequently, therefore, the methods have the appearance of being empirical and unsatisfactory. Nevertheless, under all the then circumstances, these select tables were a great success, and the profession owes a deep debt of gratitude to Dr. Sprague as their author. Dr. Sprague made use of the whole of the  $H^M$  Experience; but, for the five years of selection, he grouped the ages at entry into periods of five years, taking ages 20, 25, &c., as central to these periods; and, by keeping the periods quite separate from each other, the results were made to depend exclusively on quinary groups of

ages ; with the consequence that the irregularities in the data are not fully eliminated. It is almost necessary in such cases to have a broader basis than independent quinary groups on which to build the edifice.

5. The second set published of select tables was that based upon the Government Annuity Experience, 1883. Only four years were included in the select period ; and, for the ultimate table, the  $q_x$  of the crude data was graduated by Woolhouse's formula ; but the graduation was really very rough, and Dr. Sprague's report on Graduation included in the volume will be remembered, as also his controversy on the subject with the late Mr. Woolhouse. For the select period, Gompertz's formula was used in some way that has not been fully explained : and sufficient particulars have not been given to enable an independent investigator to reproduce the results.

6. The British Offices' Experience, 1863-1893, was taken out in select form, and the tables based thereon were prepared by the late Sir G. F. Hardy. These tables have been used to illustrate the present paper, and further reference to them at this stage is unnecessary.

7. The Government Annuity Experience, 1912. is another example of select tables on a large scale. The ultimate table, which is five years after selection, was graduated by Makeham's formula ; and the select period tables by the special adaptation of Makeham devised by Hardy for the British Offices' Experience. In the explanatory preface we are not supplied with a comparison between the expected deaths and the actual, and the only test given of the accuracy of the tables consists in the values of expectations of life five years after selection, and of annuities at the moment of selection. These show that, for the purpose of the grant of whole life annuities, the tables could hardly be improved upon ; but there is nothing really to prove that, if only sections of the tables be taken, there is no distortion caused by the application of Makeham's law ; and, hence, we cannot be quite sure that it would be prudent to base upon these tables the values of temporary or deferred annuities.

#### NURSES' EXPERIENCE.

8. As already mentioned, the short method now submitted was devised for the experience of the Royal National Pension Fund for Nurses, and that experience is of a very peculiar character. The Fund was established in 1887, and its business consists almost entirely in the grant of deferred annuities to



female nurses, although a very few immediate annuities are also issued. The great majority of the deferred annuities emerge at the ages 50, 55 and 60 ; and, during the deferred term, the nurses pay monthly contributions, which are returnable at any time with compound interest in the event of the nurse wishing to withdraw, or in the event of her death. A nurse can withdraw up to the moment of the annuity being entered upon ; and, no doubt, if she at that time be in poor health, she does withdraw. Hence the annuities, when they emerge, depend on select lives ; and, of course, when immediate annuities are granted, the lives are also select.

9. When the Fund was started, the rates of contribution were based upon the Government Female Annuity Tables, 1883 ; and the Council were very severely criticized at the time for adopting them, on the ground that the nurses' occupation is a very arduous one ; that the rate of mortality among them must be heavy ; and that they were therefore being asked for contributions very much in excess of what would be really necessary. The reply was, that the Fund was mutual ; and that, if it should be found that the contributions were in excess, then the excess payments would be returned to the nurses by way of bonus. The actual experience of thirty years, however, has falsified the prophecies of the critics, because it has been found that, so far from the nurses suffering from a heavy rate of mortality, they really enjoy great longevity. In Table I appended, a brief summary is given of the experience. During the first ten years of the Fund the number of annuities in possession was very small, because the deferred annuities did not begin to emerge until more than ten years had passed. Therefore, in Table I, the first section relates to the first fifteen years of the Fund, and it will be seen that, from the very beginning, the rates of mortality were very low. In these fifteen years the actual deaths were only 63 per-cent of the expected. The succeeding three sections of the table relate to quinquennial periods ; and, for the five years ending 31 December 1907, the ratio of actual to expected deaths was again 63 per-cent ; while, for the quinquennium ending 31 December 1912, it was only 50 per-cent ; and, for the last quinquennium, that ending 31 December 1917, it was only 47 per-cent. The ratio for the whole of the thirty years of the Fund's existence was only 51 per-cent ; and the low rates of mortality were found to exist in each of the four years of the select period, as well as in the ultimate table.



10. This result was so extraordinary that very careful investigations were carried out, to make certain that no frauds were being perpetrated, and that annuities were not being drawn in respect of deceased nurses ; and there can be no question that the rates of mortality recorded are perfectly free from suspicion. A further proof of the light mortality prevailing among the nurses is obtained from the quinquennial valuations of the Fund. At each valuation, a calculation was made to ascertain the sources of profit or loss. On each occasion a profit was disclosed ; but that was found to arise entirely in the deferred period of the annuities ; and it was shown conclusively that the annuities in possession resulted in loss, notwithstanding that the annuities have always been valued as if they had only just been granted, and that the rate of interest assumed was only  $2\frac{1}{2}$  per-cent. Nevertheless, the extraordinary vitality of the nurses drawing annuities swallowed up all excess interest on their own proportion of the funds, and left a debit against the profits from other sources. There can be no question of this fact, and hence it was felt to be very important to get out select tables, and thus to learn exactly how matters stand.

11. The short method of constructing select tables consists in preparing first of all an ultimate table, and then in constructing the select columns by means of factors formed from the ratios between the actual deaths and those expected by the ultimate table, and by which factors the ultimate  $q_x$  is to be multiplied.

12. For the nurses, the ultimate table was taken as that from their own experience four or more years after selection, to correspond with the Government Tables on which the Fund's business was based. It is given in Table II appended, and it will be seen how, even in the ultimate table, the great weight of the observations centres at ages 54, 59 and 64, these being the ages four years older than those at which the majority of the annuities emerge and come into possession.

13. To have taken out the rate of mortality age by age for the ultimate table, and then to have graduated it, would not have given good results, as the material is very rough. I tried my own method of constructing mortality tables, that by graduated pivotal values, and osculatory interpolation, *J.I.A.*, vol. xliii, pp. 109 and 167, but that also was found not to be altogether suitable in the present case, because of the great concentration of the data at a few points of age. Recourse was therefore had to the method of graduation by reference to a standard table,

the standard adopted being the Government 1883. Female, Ultimate.

14. The illustrative ultimate table of the Australian Mutual Provident Society, to be discussed farther on, was also constructed by reference, the  $O^{M(5)}$  Table being taken as the standard; and this method of construction by reference has been found to be so easy, and so satisfactory, that I think it should be borne in mind, even when ordinary mortality tables have to be prepared.

15. Of course the construction of tables by reference is no new thing. Griffith Davies adopted it when he prepared the Davies' Equitable Table, working against the Northampton. Mr. Lidstone, in 1892, contributed a most interesting paper to the *Journal* of the Institute on the subject, *J.I.A.*, vol. xxx, p. 212. The principles followed here are precisely those of Mr. Lidstone, but his process, for reasons which he explained, was more elaborate than mine; and I think that for ordinary purposes the plan that I have adopted will be found to be more convenient.

16. The deaths expected by the Government Ultimate Table were taken out for each age, and then the ratios of the actual to the expected deaths were calculated for decennial periods 40-49, 50-59, down to 80-89. These ratios were:

40-49	...	...	·1754
50-59	...	...	·5084
60-69	...	...	·4893
70-79	...	...	·5164
80-89	...	...	·6949

17. The ratio for the period 40-49 is abnormally small. There were 464 years of risk, with only 1 death, and this period does not harmonize with the rest of the table. The ratio was therefore taken arbitrarily at ·5000. This change is of extremely trifling importance, making a difference of 2 in the expected deaths.

18. The ratios above given were then transformed by inspection into ratios for the individual ages, and these were the factors by which to multiply the  $q_x$  of the standard table in order to obtain the  $q_x$  of the Nurses' ultimate. A trial was made of these first adjustments, with the result that there were 297 expected deaths against 299 actual, a difference of four when the two deaths mentioned in paragraph 17 are allowed for; and

a slight regraduation was performed, and the graduated  $q_x$  was again taken out, and gave almost the exact actual deaths.

19. The table had to be completed at the old ages. In the experience there were 10 at risk aged over 89, with 5 deaths, and the expected deaths were 5.6; the ratio of actual to expected being .8929. It is thus indicated that at the old ages the nurses' mortality rapidly rises towards that shown by the Government table, and therefore the factor was taken as unity at age 100, and the intervening values were inserted by inspection.

20. The  $q_x$  thus formed was not very smooth, on account of the irregularities of the Government Ultimate Table, and it was graduated by formula C of my paper, *J.I.A.*, vol. xli, p. 559, which is an efficient and simple formula of graduation. In Table II appended, will be found the numbers at risk and the deaths of the original data, the Government expected deaths, the factors formed as above explained, with the preliminary graduation of  $q_x$ ; and, lastly, the final graduated  $q_x$ , with the expected deaths. It will be seen that the expected deaths were 303.7, and the actual deaths 304, so that the original facts are closely adhered to, and at no part of the table is there any important deviation.

21. Having now the ultimate table, we pass to the select tables, of which there are four, given in Table III, (A), (B), (C) and (D). Entry ages from 36 to 74 inclusive were brought into account. The tables give first of all the original data, then the expected deaths by the ultimate table; and, lastly, the graduated table of  $q_x$ , with the expected deaths produced thereby. For year 0-1 the experience was taken for the entry ages 36-74 inclusive, as just mentioned. There were only 5 at risk below age 36, without any deaths, and 8 at risk without any deaths at ages above 74, and these extremes were ignored. The actual deaths were 26, and the deaths expected by the ultimate table were 39.33, the ratio being .6611. This ratio was used as a uniform factor for the whole column, by which the  $q_x$  of the ultimate table was multiplied, and the result appears in the graduated  $q_x$  of the select table. The expected deaths are 25.99, being only .01 less than the actual.

22. The years of the select period 1-2, 2-3 and 3-4, were similarly dealt with, and the ratios came out as follows:

Year 1-2	...	...	.7580
„ 2-3	...	...	.6635
„ 3-4	...	...	.9167

Here we find that the ratio for the third year is smaller than that for the second, and this ought not to be ; and we are faced with the fact that a horizontal adjustment is required, in order to produce a sequence in the death rates of greater probability. This horizontal adjustment is quite justifiable, because the lives pass across the table from one select column to the next, and not down the table : and, where the experience is limited, irregularities occur. In a large experience the horizontal adjustment might not be needed. An investigation was made, and it was found that by transferring 2·5 deaths from the second to the third year, suitable factors were obtained, the factor for the second year becoming ·6930, and that for the third ·7327. The final factors for the select period were :

Year 0-1	...	...	·6611
„ 1-2	...	...	·6930
„ 2-3	...	...	·7327
„ 3-4	...	...	·9167

The factor for the fourth year being ·9167, and not far from unity, it is clear that selection practically wears off within the four years.

23. The following are the expected and the actual deaths for the select period :

Assurance Year			Actual Deaths	Expected Deaths
0-1			26	25·99
1-2			29	26·52
2-3			24	26·54
3-4			31	30·95
Total	...	...	110	110·00

Taking all the select columns together, the expected deaths and the actual agree. Of course, with such ragged data as we have for these tables, we must expect deviations age by age between the expected and the actual deaths ; but these deviations are never really large, and they change sign very frequently, and the accumulated deviations are never important, thus showing that the graduation is good in each of the columns.

24. It is of course a question whether a uniform multiplier for the whole of each column is legitimate ; and, to throw light

on that question, an investigation was made by means of the British Offices' Experience, which will be dealt with immediately. No matter what may be the entry age, all the lives passing through the select period are acted upon by very similar forces; and, presumably, the rates of mortality are thereby affected in similar ways, and in the same directions; but it must not be assumed without enquiry that they are affected in exactly the same proportions, as is done when a uniform factor is employed for the whole column. It may be that the factors should increase, or should decrease, or should vary otherwise, with advancing entry age; and that remains to be ascertained, but the experience of the nurses is not of a kind suited to the purpose.

25. The financial effect of the light mortality of the nurses is very great, and in the appended Table IV will be found specimen values, by the Nurses' Experience, of annuities at the date of purchase, and after four or more years from purchase; and also the corresponding values by the Government Table, on which the business of the Fund was based. The annuity values are calculated at  $2\frac{1}{2}$  per-cent interest and at 4 per-cent, and it will be seen that the Government  $2\frac{1}{2}$  per-cent values do not differ much on the average from the Nurses' 4 per-cent values. The following are the sums of the specimen values for the seven ages included in the table, select and ultimate, the Nurses' values at 4 per-cent, and the Government at  $2\frac{1}{2}$  per-cent:

—	Select	Ultimate
Nurses' 4 per-cent ...	99·512	99·125
Government $2\frac{1}{2}$ per-cent	98·881	86·319

Thus is explained the loss above mentioned, which arose at each valuation, on the current annuity portion of the business of the Royal National Pension Fund for Nurses.

#### BRITISH OFFICES' EXPERIENCE, 1863-1893.

26. The British Offices' Experience provides an ideal instrument by means of which to investigate from every point of view the forces and influences which produce the result known as "Selection"; and it was thought to be well worth the trouble involved, to test by it the short method of constructing select tables, which had seemed to be satisfactory for the small



and peculiar experience of the Nurses. Not only is the British Offices' Experience in itself eminently suitable for the purpose, but there are available the magnificent select tables of the late Sir George F. Hardy, with which to compare any others that may be constructed on different plans, Hardy's tables having been proved to adhere closely to the original data, while at the same time they possess qualities of the highest value.

27. It had been shown that the effects of selection in the case of the British Offices do not become exhausted until ten years have elapsed from the time of entry of the lives assured, and Hardy therefore constructed ten columns for the select period; and carried them on by a corresponding ultimate table, the  $O^{M(10)}$ , to which they fit accurately. I have no doubt but that the short method of construction could be used quite satisfactorily if applied to the ten years period; but to have done so would have involved much additional labour, and would have necessitated a great amount of tabular matter which is really superfluous for present purposes. It has therefore been deemed sufficient to limit the illustration to five years for the select period, and to use the  $O^{M(5)}$  Table for the ultimate.

28. Entry ages 15 to 69, inclusive, have been dealt with, although the appended graduated tables run only from entry ages 20 to 64. Similar graduated tables had been prepared for the whole range, 15 to 69, but they were too long to be convenient for the pages of the *Journal*, and therefore the trouble was undertaken of constructing others of shorter range; but, except for their difference in length, the two sets are hardly distinguishable, the values of  $q_x$  almost always agreeing to the fourth place of decimals, and usually even to the fifth place. The extremes, 15 to 19 and 65 to 69, taken together, comprised only 2.3 per-cent of the total numbers at risk, and 2.7 per-cent of the total deaths; and to ignore them does not affect the results.

29. The appended Table V gives factors for extended sets of age groups. It is divided into two sections, (A) and (B): Section (A) relating to the original data, or, what is the same thing, to the graduated tables as constructed by myself; and Section (B) to Hardy's tables. In Section (A), the factor is obtained by dividing the actual deaths in the age group by the  $O^{M(5)}$  expected deaths; while, in Section (B), the numerator is the deaths expected by Hardy's table, and, the denominator, the  $O^{M(5)}$  expected deaths. There are, first, the factors for quinary age groups, 15 to 19, 20 to 24, and so on down to 65 to



69; then there are factors for groups including fifteen entry ages each, 20 to 34, 35 to 49, and 50 to 64; and, lastly, there are the average factors for the whole of each column, given for the ranges of ages 20 to 64, and 15 to 69, respectively.

30. Examining first Section (A), and running the eye down the respective columns of factors for quinary age groups, it will be seen that these are very irregular, being sometimes greater, and sometimes less, than the average factors which are uniform for each column. There are deviations due to roughness in the original data, notwithstanding the considerable magnitude of the experience. No law is disclosed which governs the progression of the factors according to advancing entry age.

31. The case is, however, different when the table is examined in the larger groups of fifteen entry ages each, because for assurance years 0 to 1 and 1 to 2, the factors increase with entry age, as also for assurance year 2 to 3, but not so regularly, or to the same extent. For assurance years 3 to 4 and 4 to 5, this tendency is not apparent.

32. Passing now to Section (B) of the table, it will be seen that for assurance year 0 to 1 the factors for quinary age groups increase regularly with advancing age, as, of course, also the factors for the longer age groups, thus following the indications given by the figures of the original data; but that, for the other four assurance years, the reverse is the case; and Hardy's factors diminish regularly with advancing entry age, thus going contrary to such slight indications as there are here in the original data. This is not easy to understand, and is due probably to the exigencies of Makeham's law. The effect on the mortality table as a whole is, however, very small indeed; and the results of the two graduations, Hardy's and my own, are nearly identical, both also keeping very close to the original facts.

33. The following are the factors actually used in the construction of the appended graduated tables, Table VI, (A) to (E), for the respective assurance years:

Year 0-1	...	...	·44269
„ 1-2	...	...	·65178
„ 2-3	...	...	·74278
„ 3-4	...	...	·78741
„ 4-5	...	...	·81162

They run with suitable regularity, and, therefore, there is no need for adjustment of the columns horizontally. The smallness

of the factor for year 4 to 5 shows that in the British Offices' Experience the effects of selection are not exhausted by the end of the five years, there being a gap between the select and the ultimate portions of the tables; and that it was therefore a wise course to extend the select period to ten years in the official tables.

34. The Select Tables are given at full length in Table VI, (A) to (E), and they speak for themselves. They contain, first, the original data; and then the  $O^{M(5)}$  expected deaths, which are used in the computation of the factors; and, lastly, the new values of  $q_x$ , with the expected deaths, are placed alongside the values by Hardy's tables, so that comparison may be made age by age. It will be seen how closely the two sets run together. The following are the totals for each assurance year of the actual deaths by the data, and of the expected deaths by my own tables, and by Hardy's, respectively:

Assurance Year	Actual Deaths	EXPECTED DEATHS	
		King	Hardy
0-1	1909	1908·3	1907·4
1-2	2697	2697·1	2698·3
2-3	3005	3004·9	2972·7
3-4	3162	3162·1	3157·1
4-5	3261	3261·4	3323·4
Totals	14034	14033·8	14063·9

The total of the actual deaths for all the five years of the select period is 14034, while the total of the expected deaths by the new tables is 14033·8, and, by Hardy's, 14063·9, the deviation by the new tables being only -0·2, and, by Hardy's +29·9. From this point of view, therefore, the new tables keep, by the merest trifle, closer to the original data than do Hardy's; but, if the individual assurance years be looked at separately, the comparison comes out a little more favourably to the new tables.

35. Table VI, (A) to (E) is repeated in summary form in Table VII, (A) to (E), the same facts being recorded in quinary age groups, with columns added to show for each age group the deviation from the actual deaths, of the deaths expected by the new tables, and by Hardy's tables, respectively. These columns of deviations show with what remarkable fidelity the new tables run with Hardy's, the signs being the same, except in a few cases

where the deviations themselves are very small; and the magnitude of the deviations differing but little. For assurance years 2 to 3, and 4 to 5, Hardy departs somewhat more from the original data, but the extra deviations of these years nearly balance each other.

36. The grand summary given in Table VII (F) emphasizes the wonderful similarity in the aggregate of the new tables to Hardy's; and, I think furnishes an excellent testimonial to the short method of construction.

37. The probability of dying within the five years of the select period, by the original data, and by the new tables and Hardy's respectively, is given in the following short tabular statement, and is not without interest. The probability is the average for each quinary age group, 20 to 24, down to 60 to 64. The denominator of  $q$  for each assurance year is the sum of the numbers at risk in the age group, and the numerator is the sum of the actual deaths, and of the expected deaths by King and Hardy, respectively. We thus have, for the age group,  $q_{[x]}$ ,  $q_{[x]+1}$ , &c., and hence of  $p_{[x]}$ ,  $p_{[x]+1}$ , &c.; and from these we pass to  $q_{[x]\overline{5}|}$ . Once more there is clearly brought out the similarity of the new tables to the old, and how both sets interpret well the original data. It is worthy of note, however, that these probabilities, as also the grand summary of Table VII (F), show that the mortality at the older entry ages is a little under-estimated.

Entry Ages	Original Data $q_{[x]\overline{5} }$	KING		HARDY	
		$q_{[x]\overline{5} }$	Deviation $\times 10^5$	$q_{[x]\overline{5} }$	Deviation $\times 10^5$
20 to 24	·02400	·02343	— 57	·02357	— 43
25 „ 29	·02380	·02515	+ 135	·02530	+ 150
30 „ 34	·02766	·02794	+ 28	·02808	+ 42
35 „ 39	·03239	·03237	— 2	·03244	+ 5
40 „ 44	·03874	·03922	+ 48	·03925	+ 51
45 „ 49	·05117	·04983	— 134	·04979	— 138
50 „ 54	·07061	·06596	— 465	·06582	— 479
55 „ 59	·09245	·09088	— 157	·09069	— 176
60 „ 64	·13004	·12652	— 352	·12634	— 370
Totals	·49086	·48130	+ 211 — 1167	·48128	+ 248 — 1206

38. It is not impossible that the new tables might be brought into still closer harmony with the data, by making allowance

for the tendency in some of the assurance years towards increase in the factors with advancing entry age, discussed in paragraph 31 above. That could be done without difficulty by taking the three factors for the age groups 20 to 24, 35 to 49, and 50 to 64, and transforming them into factors for individual ages, in the way explained in paragraph 18, where the construction of the Nurses' Ultimate Table was in question. There would, however, be but little practical gain secured by this sacrifice of the simplicity of the uniform factor system.

#### AUSTRALASIAN EXPERIENCE, 1849-1903.

39. I have to thank our old friend Mr. Richard Teece, and his Co-Directors, for their kindness in allowing me to use for publication the experience of healthy male lives assured in the Australian Mutual Provident Society, for the fifty-five years from the commencement of the Society, down to 1903; and it is now taken for the purpose of this paper, as an illustration of the short method of constructing select tables. It is, however, unnecessary here to go into great detail, because it is proposed to submit to the Actuarial Society of Australasia a paper, in which this experience will be dealt with more fully. At present only sufficient will be given to make the illustration clear.

40. In the appended tables the experience itself is given, both ultimate and select. As above stated, it is the experience of healthy male lives; and it may be said that practically all the lives effected their policies in Australasia. The Society did not open its London establishment until after the experience had closed; but I understand that a few lives had been assured at home through local agencies. There cannot, however, have been many of them, and therefore the experience is really that of lives assured in Australasia.

41. The first process is, to prepare an ultimate table; and that might have been done by one or other of the more usual methods of construction. The construction of the Nurses' Ultimate Table by the method of reference had, however, been so interesting to me, that I adopted it for the Australian Mutual data, and the  $O^{M(5)}$  Table was taken as the standard.

42. An unexpected difficulty arose at the outset, through the very defective graduation after age 80 of the  $O^{M(5)}$  Table as given in the volume of British Offices Life Tables, published in 1902. The  $O^{M(5)}$  Table is supposed to follow strictly Makeham's

law, and it seems strange that at any portion of it the graduation should be erratic. It appears, however, that Makeham's  $p_x$ , and consequently  $q_x$ , was not retained in its integrity in the published table; but was recalculated, for some reason which I do not understand, by dividing  $l_{x+1}$  by  $l_x$ . That produces up to about age 80 Makeham's  $q_x$  to five places of decimals, although here and there the fifth place is not quite accurate. But above age 80, where the official  $l_x$  is cut down to four figures or less, the recalculated  $q_x$  does not follow Makeham's law in individual cases, although it does so on the average; and hence, from age 80 onwards the published table is not suited for a standard. In the volume of Principles and Methods, page 153, Makeham's real  $\text{colog } p_x$  as originally calculated from the constants is, however, given, and that was used for the AMP Ultimate.

43. The experience was first taken in quinary age groups from group 20-24, down to group 90-94, and the ratio of actual deaths in each group to the  $O^{M(5)}$  expected deaths was calculated. The following are the figures:

Age Group	Ratio	Age Group	Ratio
20-24	·592	60-64	·803
25-29	·614	65-69	·824
30-34	·645	70-74	·853
35-39	·703	75-79	·785
40-44	·731	80-84	·749
45-49	·730	85-89	·943
50-54	·759	90-94	·909
55-59	·726		

44. It will be seen that the foregoing ratios run irregularly, and that they must be adjusted in some way before the ratios for individual ages can be arrived at. After careful examination, it was decided to combine, in certain parts of the table, the age groups, and to use groups of varying age ranges for the purpose of graduation. The different lengths of groups present no difficulty when graduation is effected by the graphic method used by Milne for the Carlisle Table. The following are the groupings and ratios actually employed in the graduation:



Age Group	Ratio	Age Group	Ratio
20-29 (10 years)	·611	50-54 (5 years)	·759
30-34 (5 years)	·645	55-69 (15 years)	·779
35-39 (5 years)	·703	70-89 (20 years)	·828
40-49 (10 years)	·731	90-94 (5 years)	·909

45. Parallelograms were set up on cross-ruled paper, the base representing the number of years in the age group, and the altitude, the figures of the ratios; and, of course, the area of each parallelogram is, the number of years in the base, multiplied by the altitude. A curve was drawn through the tops of the parallelograms, and was read off to give the altitude at each individual age. In Table VIII appended, these ratios, as read on the curve, are given to three places of decimals. They run fairly well; but, in order to obtain a really smooth ultimate table, they were graduated by formula C mentioned above in paragraph 20; and the result of that graduation is given under the heading "Factors Graduated." The  $O^{M(5)}$  Table, as originally calculated from the constants, being in itself perfectly smooth, it was, for the AMP, the ratios which were graduated by the summation formula, and not the  $q$  after multiplication, as in the case of the Nurses' Table. See paragraph 20 above. The factors were used to carry the ultimate table down to age 78; and then, from the four values 75 to 78 inclusive, and assuming the factor for age 95 to be unity, the table was completed by means of a fourth difference.

46. The graduated table appears at the right of appended Table VIII, as also the expected deaths, and the deviations. It will be seen that the actual deaths are 6365, and the expected 6362·5, so that there is a deviation of only -2·5. Also, the deviations at individual ages are never large, and they change sign very frequently. It is submitted that this construction is successful. It may be called the  $(AMP)^{M(5)}$  Table. In Table IX appended, a summary of Table VIII is given in quinary age groups, from which may be seen at a glance the nature and weight of the deviations.

47. The  $(AMP)^{M(5)}$  Table was then employed in the construction of the select columns, there being five of these, the select period being assumed to run into the ultimate five years after the entry of the lives.

48. In Table X appended, the ratios are given, for quinary



age groups, between the actual deaths in the various select columns, and those expected by the (AMP)<sup>M(5)</sup> Table. It will be noticed that they run irregularly, just as in the case of the British Offices' Experience, and that in this form they do not throw much light upon the progression of the factors in the several columns with the advancing entry ages. The table was therefore divided into two portions of twenty-five ages each, for entry ages 15 to 39, and 40 to 64, respectively; and the ratios for these age groups follow in Table X. It will be observed that, contrary to the indications of the British Offices' Experience, the ratios almost uniformly decrease with advancing entry age; and, probably, a very accurate adjustment of the select period could be obtained by calculating, from these groups of twenty-five ages each, graduated factors diminishing slightly age by age. It seemed, however, scarcely necessary to undertake this additional work, as a uniform factor for all entry ages gives results sufficiently close to the data for all ordinary purposes. The average ratios for the whole of the entry ages from 15 to 64 are given at the foot of Table X. There is irregularity, however, in these ratios; because that for assurance year 2 to 3 is greater than that for assurance year 3 to 4; and that anomaly requires correction by horizontal adjustment. Fourteen deaths were transferred from year 2 to 3 to year 3 to 4, and the ratios, after this adjustment, are as follows:

Year	Ratio
0-1	·63370
1-2	·77656
2-3	·83315
3-4	·86312
4-5	·87485

These run with suitable regularity, and were adopted for the final graduation. The factor for the year 4-5 is not very large, and therefore it would appear that selection in Australasia does not fully wear off within five years. The materials, however, as tabulated do not make it possible to investigate this point farther.

49. In Table XI, (A) to (E), are given the tables for the five years of selection, and the right-hand columns show the deviations, age by age, of the expected from the actual deaths. It will be

seen that the deviations are, for the most part, quite small, and change sign very frequently ; and the changes of sign occur all down the columns, so that at no point is there any large accumulated deviation. It will also be observed in each table that, for each assurance year, the actual deaths are exactly reproduced, when allowance is made for the transfer of fourteen deaths mentioned in paragraph 49.

50. Table XI is repeated in summary form in Table XII, the facts being recorded in quinary age groups ; and I think this affords sufficient evidence that the construction of the select portion of the table, as well as of the ultimate, is quite satisfactory.

51. The characteristics of the Australasian mortality are well worthy of study. As regards the ultimate tables, when the  $(AMP)^{M(5)}$  is compared with the  $OM^{(5)}$ , it will be seen that the Australasian mortality is very light. From age 20 to about age 40 it is never more than 70 per-cent of the British, and up to age 70 it is less than 80 per-cent ; and it is only at the really old ages that the two tables approximate to each other. Also, as regards the select portion, we have the same relative position, except that, for the year 0-1 of assurance, the Australasian mortality is lighter than the British up to about age 36, but after that age it is heavier. In all the other assurance years, however, the Australasian mortality is the lighter throughout.

#### ADDENDUM.

When all the tables for this paper had been completed, and when the explanatory letterpress had been to a large extent sketched out, I was showing the results to a friend from the Far East, and he called my attention to the “Japanese Three Offices Life Tables”, which had been prepared by a Committee of Actuaries from the experience of the “Meiji”, the “Teikoku”, and the “Nippon”, the three oldest life assurance companies in Japan, under the superintendence of Mr. Kaitaro Ebihara, F.I.A., whom many of us in London know. It seems that the select tables in the Japanese collection were constructed by the short method set forth in this paper ; but I was entirely ignorant of the fact ; and, in so far as I am concerned, all that is in my paper is as original as anything on such a hackneyed subject as the construction of mortality tables can be. The Japanese

come first in the order of time. Nevertheless I venture to think that my paper is possibly not without value, seeing that I go farther than the Japanese, and that my paper contains an investigation into the principles of the short method, and as to its effects when applied to such an experience as that of the British Offices. Moreover, it includes two interesting new mortality tables, that for the Nurses, and that for Australasia, which are valuable in themselves.

The explanations of the Japanese are almost apologetic, and are contained in one brief paragraph; and there are only a few figures setting forth the results. The Japanese began with five years for the select period; but the fifth merges absolutely into the ultimate, so that practically there are only four. The factors used, by which to multiply the ultimate  $q_x$  in order to obtain the select, are:

Year 0-1	...	...	·62
„ 1-2	...	...	·87
„ 2-3	...	...	·95
„ 3-4	...	...	·97
„ 4-5	...	...	1·00

The following are the results in brief summary:

Assurance Year	At Risk	Actual Deaths	EXPECTED DEATHS	
			Ultimate	Actual Graduated
0-1	325,508	2312	3743·12	2320·32
1-2	235,771	2476	2838·95	2489·88
2-3	195,540	2341	2462·15	2339·04
3-4	163,009	2097	2152·88	2088·31
4-5	?	1885	1888·61	1888·61
Totals	...	11111	13085·71	11126·16

In conclusion, I take this opportunity of congratulating our Japanese brethren on their enterprise in striking out a new path, and on the success they have achieved.

TABLE I.

*Royal National Pension Fund for Nurses.  
Annuities in Possession, Females. Mortality Experience.*

Year of Currency of Annuity	Actual Deaths	Expected Deaths, Govt. Table, 1883	Ratio of Actual to Expected Deaths
Fifteen years ending 31 December 1902			
1	5	6	·83
2	2	7	·29
3	7	7	1·00
4	3	6	·50
Over 4	14	23	·61
All durations	31	49	·63
Five years ending 31 December 1907			
1	2	6	·33
2	6	8	·75
3	5	10	·50
4	5	11	·45
Over 4	44	64	·69
All durations	62	99	·63
Five years ending 31 December 1912			
1	10	13	·77
2	10	15	·67
3	6	17	·35
4	7	18	·39
Over 4	79	162	·49
All durations	112	225	·50
Five years ending 31 December 1917			
1	9	16	·56
2	11	22	·50
3	6	28	·21
4	16	28	·57
Over 4	167	349	·48
All durations	209	443	·47
Thirty years from the Foundation of Fund in 1887 to 31 December 1917			
1	26	41	·63
2	29	52	·56
3	24	62	·39
4	31	63	·49
Over 4	304	598	·51
All durations	414	816	·51

TABLE II.  
Nurses' Ultimate Table.

Age $x$	ORIGINAL DATA		Govt. Ultimate Expected Deaths	PRELIMINARY ADJUSTMENT		GRADUATED TABLE	
	At Risk	Actual Deaths		Factor	$q_x$	$q_x$	Expected Deaths
40	10	1	.1	.500	.00541	.00541	.1
1	14	...	.2	.500	.00552	.00559	.1
2	18	...	.2	.500	.00563	.00571	.1
3	26	...	.3	.500	.00575	.00581	.2
4	40	...	.5	.500	.00587	.00589	.2
45	54	...	.6	.500	.00598	.00597	.3
6	61	...	.7	.500	.00611	.00606	.4
7	64	...	.8	.500	.00623	.00618	.4
8	73	...	.9	.500	.00635	.00634	.5
9	104	...	1.4	.500	.00649	.00657	.7
50	128	1	1.7	.502	.00664	.00687	.9
1	139	...	2.0	.505	.00711	.00726	1.0
2	158	1	2.4	.508	.00767	.00771	1.2
3	156	4	2.5	.509	.00827	.00818	1.3
4	1157	3	20.2	.510	.00890	.00865	10.0
55	1091	10	19.8	.509	.00923	.00907	9.9
6	953	6	17.7	.508	.00942	.00944	9.0
7	806	9	15.6	.505	.00981	.00977	7.9
8	736	11	15.0	.502	.01025	.01008	7.4
9	1198	17	25.1	.497	.01040	.01043	12.5
60	1056	12	23.1	.492	.01077	.01087	11.5
1	951	13	22.2	.487	.01136	.01145	10.9
2	864	8	21.4	.485	.01202	.01221	10.5
3	747	13	19.9	.485	.01291	.01315	9.8
4	1058	12	30.9	.487	.01422	.01424	15.1
65	946	11	30.3	.490	.01567	.01546	14.6
6	829	9	28.8	.492	.01711	.01676	13.9
7	748	21	28.1	.492	.01845	.01815	13.6
8	703	13	28.3	.492	.01983	.01965	13.8
9	613	15	26.6	.492	.02135	.02134	13.1
70	520	15	24.3	.493	.02304	.02330	12.1
1	454	11	23.2	.496	.02535	.02562	11.6
2	382	7	21.6	.500	.02826	.02837	10.8
3	337	9	21.2	.506	.03183	.03160	10.6
4	293	8	20.4	.510	.03554	.03534	10.4
75	237	12	18.3	.515	.03970	.03962	9.4
6	163	8	13.7	.525	.04427	.04450	7.3
7	132	15	12.2	.545	.05019	.05003	6.6
8	99	3	9.9	.565	.05630	.05627	5.6
9	70	1	7.6	.585	.06346	.06330	4.4
80	41	12	4.8	.605	.07115	.07120	2.9
1	26	2	3.3	.625	.08033	.08007	2.1
2	20	3	2.8	.645	.08973	.09002	1.8
3	17	1	2.6	.665	.10117	.10116	1.7
4	18	...	3.0	.685	.11315	.11361	2.0
Carried forward	18310	297	576.2	...	...	...	290.2

TABLE II (*continued*).*Nurses' Ultimate Table.*

Age $x$	ORIGINAL DATA		Govt. Ultimate Expected Deaths	PRELIMINARY ADJUSTMENT		GRADUATED TABLE	
	At Risk	Actual Deaths		Factor	$qx$	$qx$	Expected Deaths
Brought forward	18310	297	576.2	...	...	...	290.2
85	16	1	2.9	.705	.12759	.12747	2.0
6	10	...	2.0	.725	.14145	.14271	1.4
7	11	...	2.4	.745	.15956	.15929	1.8
8	11	1	2.5	.765	.17737	.17705	1.9
9	10	...	2.5	.785	.19743	.19577	2.0
90	10	...	2.7	.805	.21505	.21493	2.1
1	10	5	2.9	.825	.24087	.23409	2.3
2	...	...	...	.845	.25976	.25313	...
3	...	...	...	.865	.28371	.27557	...
4	...	...	...	.885	.30400	.30493	...
95	...	...	...	.905	.34372	.34474	...
6	...	...	...	.925	.37060	.39850	...
7	...	...	...	.945	.42114	.46975	...
8	...	...	...	.965	.47304	.56200	...
9	...	...	...	.985	.64404	.67878	...
100	...	...	...	1.000	.77778	.82360	...
1	...	...	...	...	.99999	.99999	...
Totals	18388	304	594.1	...	...	...	303.7



TABLE III (A).

*Nurses' Select Table. Year 0 to 1.*

Entry Age [ $x$ ]	Attained Age [ $x$ ]	ORIGINAL DATA		Nurses' Ultimate Expected Deaths	GRADUATED TABLE	
		At Risk	Actual Deaths		$q(x)$	Expected Deaths
36	36	6	...	·03	·00330	·02
7	7	5	...	·03	·00334	·02
8	8	4	...	·02	·00340	·01
9	9	9	...	·05	·00348	·03
40	40	17	...	·09	·00358	·06
1	1	21	...	·12	·00370	·08
2	2	13	...	·07	·00377	·05
3	3	20	1	·12	·00384	·08
4	4	29	...	·17	·00389	·11
45	45	55	1	·33	·00395	·22
6	6	34	...	·21	·00401	·14
7	7	37	...	·23	·00409	·15
8	8	45	...	·29	·00419	·19
9	9	31	...	·20	·00434	·13
50	50	1610	4	11·06	·00454	7·31
1	1	79	...	·57	·00480	·38
2	2	61	...	·47	·00510	·31
3	3	55	1	·45	·00541	·30
4	4	80	1	·69	·00572	·46
55	55	1019	10	9·24	·00600	6·11
6	6	90	1	·85	·00624	·56
7	7	62	...	·61	·00646	·40
8	8	75	...	·76	·00666	·50
9	9	44	...	·46	·00690	·30
60	60	647	3	7·03	·00719	4·65
1	1	63	1	·72	·00757	·48
2	2	47	...	·57	·00807	·38
3	3	37	1	·49	·00869	·32
4	4	34	...	·48	·00941	·32
65	65	44	...	·68	·01022	·45
6	6	32	...	·54	·01108	·35
7	7	23	1	·42	·01200	·28
8	8	13	1	·26	·01299	·17
9	9	10	...	·21	·01411	·14
70	70	12	...	·28	·01540	·18
1	1	6	...	·15	·01694	·10
2	2	4	...	·11	·01876	·08
3	3	4	...	·13	·02089	·08
4	4	4	...	·14	·02336	·09
Totals	...	4481	26	39·33	...	25·99

TABLE III (B).

*Nurses' Select Table. Year 1 to 2.*

Entry Age [ $x$ ]	Attained Age [ $x$ ] + 1	ORIGINAL DATA		Nurses' Ultimate Expected Deaths	GRADUATED TABLE	
		At Risk	Actual Deaths		$q[x]+1$	Expected Deaths
36	37	6	...	·03	·00350	·02
7	8	5	...	·03	·00356	·02
8	9	4	...	·02	·00365	·01
9	40	9	...	·05	·00375	·03
40	41	17	...	·10	·00387	·07
1	2	18	...	·10	·00396	·07
2	3	13	...	·08	·00403	·05
3	4	19	...	·11	·00408	·08
4	45	29	...	·17	·00414	·12
45	46	53	...	·32	·00420	·22
6	7	34	...	·21	·00428	·15
7	8	37	...	·23	·00439	·16
8	9	44	...	·29	·00455	·20
9	50	31	...	·21	·00476	·15
50	51	1484	4	10·77	·00503	7·46
1	2	73	...	·56	·00534	·39
2	3	58	...	·47	·00567	·33
3	4	53	...	·46	·00599	·32
4	55	75	...	·68	·00629	·47
55	56	912	7	8·61	·00654	5·96
6	7	78	...	·76	·00677	·53
7	8	62	1	·62	·00699	·43
8	9	69	...	·72	·00723	·50
9	60	41	...	·45	·00753	·31
60	61	606	10	6·94	·00793	4·81
1	2	60	1	·73	·00846	·51
2	3	44	1	·58	·00911	·40
3	4	34	1	·48	·00987	·34
4	65	29	...	·45	·01071	·31
65	66	42	3	·70	·01161	·49
6	7	30	...	·54	·01258	·38
7	8	22	...	·43	·01362	·30
8	9	12	...	·26	·01479	·18
9	70	10	...	·23	·01615	·16
70	71	12	...	·31	·01775	·21
1	2	4	1	·11	·01966	·08
2	3	4	...	·13	·02190	·09
3	4	4	...	·14	·02449	·10
4	75	4	...	·16	·02746	·11
Totals	...	4141	29	38·24	...	26·52

TABLE III (c).

*Nurses' Select Table. Year 2 to 3.*

Entry Age [ <i>x</i> ]	Attained Age [ <i>x</i> ] + 2	ORIGINAL DATA		Nurses' Ultimate Expected Deaths	GRADUATED TABLE	
		At Risk	Actual Deaths		<i>q</i> [ <i>x</i> ]+2	Expected Deaths
36	38	6	...	·03	·00377	·02
7	9	5	...	·03	·00385	·02
8	40	4	...	·02	·00396	·02
9	1	8	...	·04	·00410	·03
40	42	17	...	·10	·00418	·07
1	3	18	1	·10	·00426	·08
2	4	13	...	·08	·00432	·06
3	45	18	...	·11	·00437	·08
4	6	27	...	·16	·00444	·12
45	47	51	...	·32	·00453	·23
6	8	33	...	·21	·00465	·15
7	9	34	...	·22	·00481	·16
8	50	44	...	·30	·00503	·22
9	1	30	...	·22	·00532	·16
50	52	1316	3	10·15	·00565	7·44
1	3	66	...	·54	·00599	·40
2	4	54	...	·47	·00634	·34
3	55	50	1	·45	·00665	·33
4	6	68	...	·64	·00692	·47
55	57	786	4	7·68	·00716	5·63
6	8	70	2	·71	·00739	·52
7	9	60	2	·63	·00764	·46
8	60	66	...	·72	·00796	·53
9	1	34	...	·39	·00839	·29
60	62	544	7	6·64	·00895	4·87
1	3	58	...	·76	·00964	·56
2	4	41	...	·58	·01043	·43
3	65	32	...	·49	·01133	·36
4	6	27	...	·45	·01228	·33
65	67	36	...	·65	·01330	·48
6	8	28	1	·55	·01440	·40
7	9	21	2	·45	·01564	·33
8	70	10	...	·23	·01707	·17
9	1	9	1	·23	·01877	·17
70	72	11	...	·31	·02079	·23
1	3	2	...	·06	·02315	·05
2	4	3	...	·11	·02589	·08
3	75	4	...	·16	·02903	·12
4	6	4	...	·18	·03261	·13
Totals	...	3708	24	36·17	...	26·54

TABLE III (D).

*Nurses' Select Table. Year 3 to 4.*

Entry Age [ $x$ ]	Attained Age [ $x$ ] + 3	ORIGINAL DATA		Nurses' Ultimate Expected Deaths	GRADUATED TABLE	
		At Risk	Actual Deaths		$q[x]+3$	Expected Deaths
36	39	6	...	·03	·00482	·03
7	40	5	...	·03	·00496	·02
8	1	4	...	·02	·00512	·02
9	2	8	...	·05	·00523	·04
40	43	14	...	·08	·00533	·07
1	4	16	...	·09	·00540	·09
2	45	11	...	·07	·00547	·06
3	6	16	...	·10	·00556	·09
4	7	27	...	·17	·00567	·15
45	48	47	...	·30	·00581	·27
6	9	31	...	·20	·00602	·19
7	50	29	...	·20	·00630	·18
8	1	42	2	·30	·00666	·28
9	2	27	1	·21	·00707	·19
50	53	1165	13	9·53	·00750	8·74
1	4	63	...	·54	·00793	·50
2	55	46	...	·42	·00831	·38
3	6	43	...	·41	·00865	·37
4	7	65	1	·64	·00896	·58
55	58	680	8	6·85	·00924	6·28
6	9	59	1	·62	·00956	·56
7	60	54	...	·59	·00996	·54
8	1	61	...	·70	·01050	·64
9	2	27	...	·33	·01119	·30
60	63	477	3	6·27	·01205	5·75
1	4	54	...	·77	·01305	·70
2	65	36	...	·56	·01417	·51
3	6	29	...	·49	·01536	·45
4	7	25	1	·45	·01664	·42
65	68	33	...	·65	·01801	·59
6	9	26	1	·55	·01956	·51
7	70	14	...	·33	·02136	·30
8	1	9	...	·23	·02349	·21
9	2	7	...	·20	·02601	·18
70	73	11	...	·35	·02897	·32
1	4	2	...	·07	·03240	·06
2	75	1	...	·04	·03632	·04
3	6	4	...	·18	·04079	·16
4	7	4	...	·20	·04586	·18
Totals	...	3278	31	33·82	...	30·95

TABLE IV.

*Values of Annuities at Date of Purchase and four or more years after Purchase, by the Table for Nurses, and by the Government Table 1883, Females.*

*Interest  $2\frac{1}{2}$  per-cent.*

AT DATE OF PURCHASE			4 OR MORE YEARS AFTER PURCHASE		
Entry Age	Nurses	Govt. Females	Attained Age	Nurses	Govt. Females
40	22.732	19.524	44	21.340	18.057
45	21.124	17.949	49	19.568	16.331
50	19.324	16.190	54	17.627	14.434
55	17.378	14.329	59	15.543	12.482
60	15.265	12.333	64	13.253	10.367
65	12.982	10.296	69	10.865	8.308
70	10.631	8.260	74	8.446	6.340

*Interest 4 per-cent.*

AT DATE OF PURCHASE			4 OR MORE YEARS AFTER PURCHASE		
Entry Age	Nurses	Govt. Females	Attained Age	Nurses	Govt. Females
40	18.015	15.904	44	17.159	14.917
45	17.049	14.879	49	16.037	13.739
50	15.899	13.669	54	14.738	12.374
55	14.588	12.328	59	13.269	10.908
60	13.084	10.818	64	11.558	9.236
65	11.369	9.204	69	9.681	7.540
70	9.508	7.520	74	7.683	5.854

TABLE V.

*British Offices' Experience.*

*Factors for various groups of Entry Ages, by which to multiply  $O^{M(5)} q_x$ , so as to obtain the  $O^{M}$  Select  $q[x]$ ,  $q[x]+1$ ,  $q[x]+2$ ,  $q[x]+3$ , and  $q[x]+4$ , which will reproduce (a) the Actual Deaths by the Original Data, and (b) the Expected Deaths by Hardy's graduation.*

*(a) Actual Deaths by Original Data reproduced.*

Entry Ages of Group	Year 0 to 1	Year 1 to 2	Year 2 to 3	Year 3 to 4	Year 4 to 5
15 to 19	·44234	·79310	·64338	·83012	1·00000
20 „ 24	·38631	·70067	·79816	·80301	·83463
25 „ 29	·39467	·57836	·70280	·76349	·80586
30 „ 34	·45659	·61243	·74286	·76082	·82809
35 „ 39	·40808	·66289	·74961	·79603	·81861
40 „ 44	·49054	·63230	·73257	·79211	·75281
45 „ 49	·43869	·67948	·70693	·86323	·83749
50 „ 54	·51884	·76255	·80456	·80909	·81025
55 „ 59	·51102	·72396	·76923	·76683	·75301
60 „ 64	·51036	·69020	·73416	·68731	·91971
65 „ 69	·44982	·77187	·65767	·96721	·95717
20 „ 34	·41797	·61691	·73813	·76999	·82072
35 „ 49	·44347	·65707	·73253	·81281	·80166
50 „ 64	·51471	·73624	·77953	·77179	·81262
20 „ 64	·44269	·65178	·74278	·78741	·81162
15 „ 69	·44278	·65535	·74027	·79060	·81608
<i>(b) Hardy's Expected Deaths reproduced.</i>					
15 to 19	·39969	·66552	·75551	·80502	·84200
20 „ 24	·40367	·66362	·75280	·80060	·83775
25 „ 29	·41188	·66103	·74869	·79740	·83639
30 „ 34	·42327	·65738	·74327	·79292	·83298
35 „ 39	·43754	·65325	·73646	·78751	·82950
40 „ 44	·45483	·64825	·72940	·78189	·82533
45 „ 49	·47217	·64363	·72191	·77644	·82161
50 „ 54	·48950	·63949	·71609	·77152	·81909
55 „ 59	·50493	·63568	·71119	·76772	·81670
60 „ 64	·51675	·63451	·70711	·76737	·81606
65 „ 69	·52941	·63122	·70826	·76393	·81441
20 „ 34	·41479	·66005	·74727	·79614	·83521
35 „ 49	·45230	·64905	·73022	·78264	·82595
50 „ 64	·49961	·63741	·71276	·76946	·81770
20 „ 64	·44231	·65210	·73479	·78618	·82841
15 „ 69	·44284	·65199	·73469	·78610	·82836



TABLE VI (A).  
O<sup>[M]</sup> Select Table. Year 0 to 1.

Age [ <i>x</i> ]	ORIGINAL DATA		O <sup>[M]</sup> Expected Deaths	KING		HARDY	
	At Risk	Actual Deaths		$q_{[x]}$	Expected Deaths	$q_{[x]}$	Expected Deaths
20	5768	17	37.6	.00289	16.7	.00261	15.1
1	10010	26	66.0	.00292	29.2	.00264	26.4
2	12392	27	82.4	.00294	36.4	.00268	33.2
3	15099	31	101.5	.00297	44.8	.00272	41.1
4	17861	57	121.5	.00301	53.8	.00276	49.3
25	20401	50	140.6	.00305	62.2	.00281	57.3
6	21501	51	150.1	.00309	66.4	.00286	61.5
7	22024	69	156.2	.00314	69.2	.00292	64.3
8	22249	53	160.4	.00319	71.0	.00298	66.3
9	22606	82	165.5	.00324	73.2	.00305	68.9
30	23113	72	172.7	.00331	76.5	.00312	72.1
1	21215	76	161.7	.00337	71.5	.00321	68.1
2	20535	75	159.6	.00344	70.6	.00329	67.6
3	19670	67	156.6	.00352	69.2	.00339	66.7
4	18505	76	151.0	.00361	66.8	.00350	64.8
35	17790	71	148.9	.00371	66.0	.00361	64.2
6	16510	57	142.0	.00381	62.9	.00374	61.7
7	14944	52	132.4	.00392	58.6	.00388	58.0
8	14157	53	129.5	.00405	57.3	.00403	57.1
9	13337	44	126.0	.00418	55.7	.00420	56.0
40	12821	60	125.4	.00433	55.5	.00438	56.2
1	11235	42	114.0	.00449	50.4	.00458	51.5
2	10434	58	110.2	.00467	48.7	.00480	50.1
3	9517	61	104.6	.00487	46.3	.00504	48.0
4	8749	51	100.3	.00507	44.4	.00530	46.4
45	8202	46	98.4	.00531	43.6	.00558	45.8
6	7152	43	89.8	.00556	39.8	.00590	42.2
7	6466	31	85.4	.00584	37.8	.00624	40.3
8	5860	32	81.3	.00614	36.0	.00661	38.7
9	5346	38	78.2	.00648	34.6	.00702	37.5
50	5193	49	80.2	.00684	35.5	.00746	38.7
1	4045	21	66.1	.00723	29.2	.00795	32.2
2	3670	37	63.5	.00766	28.1	.00849	31.2
3	3193	27	58.7	.00814	26.0	.00907	29.0
4	2825	34	55.3	.00866	24.5	.00971	27.4
55	2467	17	51.4	.00922	22.7	.01040	25.7
6	2110	22	46.9	.00984	20.8	.01117	23.6
7	1700	23	40.4	.01051	17.9	.01200	20.4
8	1477	23	37.5	.01125	16.6	.01291	19.1
9	1363	24	37.1	.01205	16.4	.01390	18.9
60	1348	25	39.4	.01293	17.4	.01499	20.2
1	806	12	25.3	.01389	11.2	.01618	13.0
2	712	12	24.0	.01493	10.6	.01748	12.4
3	541	10	19.6	.01608	8.7	.01889	10.2
4	438	5	17.1	.01732	7.6	.02044	9.0
Totals	467357	1909	4312.3	...	1908.3	...	1907.4

TABLE VI (B).  
 $O^{(M)}$  Select Table. Year 1 to 2.

Age [ $x$ ]+1	ORIGINAL DATA		$O^{(M)}$ Expected Deaths	KING		HARDY	
	At Risk	Actual Deaths		$q_{[x]+1}$	Expected Deaths	$q_{[x]+1}$	Expected Deaths
21	5128	30	33.8	.00430	22.1	.00437	22.4
2	8939	44	59.4	.00433	38.7	.00442	39.5
3	11182	49	75.1	.00438	49.0	.00446	49.9
4	13603	57	92.5	.00443	60.3	.00451	61.3
25	16215	81	111.7	.00449	72.8	.00457	74.1
26	18530	81	129.3	.00455	84.3	.00463	85.8
7	19684	71	139.6	.00462	90.9	.00469	92.3
8	20192	97	145.6	.00470	94.9	.00476	96.1
9	20540	70	150.4	.00477	98.0	.00484	99.4
30	20901	98	156.1	.00487	101.8	.00493	103.0
31	21401	93	163.1	.00497	106.4	.00502	107.4
2	19698	89	153.1	.00506	99.7	.00512	100.9
3	19072	89	151.8	.00519	99.0	.00523	99.7
4	18254	106	149.0	.00532	97.1	.00536	97.8
35	17195	89	143.9	.00546	93.9	.00549	94.4
36	16615	89	142.9	.00561	93.2	.00564	93.7
7	15391	89	136.4	.00577	88.8	.00580	89.3
8	13888	94	127.1	.00596	82.8	.00597	82.9
9	13183	82	124.6	.00616	81.2	.00616	81.2
40	12494	79	122.2	.00637	79.6	.00637	79.6
41	11928	83	121.1	.00662	79.0	.00660	78.7
2	10478	65	110.6	.00688	72.1	.00685	71.8
3	9734	65	107.0	.00716	69.7	.00713	69.4
4	8946	65	102.5	.00747	66.8	.00742	66.4
45	8174	63	98.1	.00782	63.9	.00775	63.3
46	7710	71	96.8	.00819	63.1	.00811	62.5
7	6690	57	88.3	.00860	57.5	.00850	56.9
8	6062	60	84.1	.00905	54.9	.00893	54.1
9	5479	53	80.2	.00954	52.3	.00940	51.5
50	5012	49	77.4	.01007	50.5	.00991	49.7
51	4908	62	80.2	.01065	52.3	.01047	51.4
2	3797	51	65.7	.01128	42.8	.01108	42.1
3	3440	51	63.3	.01199	41.2	.01175	40.4
4	2964	46	58.0	.01275	37.8	.01249	37.0
55	2660	36	55.4	.01358	36.1	.01329	35.4
56	2312	33	51.4	.01448	33.5	.01416	32.7
7	1980	41	47.0	.01548	30.7	.01511	29.9
8	1583	29	40.2	.01656	26.2	.01616	25.6
9	1395	28	38.0	.01774	24.7	.01730	24.1
60	1283	24	37.5	.01904	24.4	.01855	23.8
61	1262	25	39.6	.02045	25.8	.01991	25.1
2	762	19	25.7	.02198	16.7	.02140	16.3
3	673	23	24.4	.02367	15.9	.02302	15.5
4	512	10	20.0	.02550	13.1	.02480	12.7
65	421	11	17.8	.02751	11.6	.02674	11.3
Totals	432270	2697	4137.9	...	2697.1	...	2698.3

TABLE VI (c).  
O<sup>(M)</sup> Select Table. Year 2 to 3.

Age [ $x$ ]+2	ORIGINAL DATA		O <sup>(M)</sup> Expected Deaths	KING		HARDY	
	At Risk	Actual Deaths		$q_{[x]+2}$	Expected Deaths	$q_{[x]+2}$	Expected Deaths
22	4694	25	31.2	.00494	23.2	.00502	23.6
3	8194	45	55.1	.00499	40.9	.00507	41.5
4	10357	70	70.4	.00505	52.3	.00512	53.0
25	12571	69	86.6	.00512	64.4	.00518	65.1
6	15045	69	105.0	.00518	77.9	.00525	79.0
27	17368	79	123.1	.00527	91.5	.00532	92.4
8	18382	87	132.5	.00536	98.5	.00540	99.3
9	18993	117	139.0	.00544	103.3	.00549	104.3
30	19442	93	145.2	.00555	107.9	.00558	108.5
1	19726	109	150.3	.00566	111.6	.00569	112.2
32	20081	111	156.0	.00577	115.9	.00580	116.5
3	18608	110	148.1	.00591	110.0	.00592	110.2
4	18044	122	147.2	.00606	109.3	.00606	109.3
35	17239	104	144.3	.00622	107.2	.00621	107.1
6	16205	99	139.4	.00639	103.5	.00637	103.2
37	15728	115	139.4	.00658	103.5	.00655	103.0
8	14593	88	133.5	.00680	99.2	.00675	98.5
9	13134	88	124.1	.00702	92.2	.00696	91.4
40	12453	95	121.8	.00726	90.4	.00719	89.5
1	11838	93	120.2	.00754	89.3	.00745	88.2
42	11332	85	119.7	.00784	88.8	.00773	87.6
3	10013	72	110.0	.00816	81.7	.00803	80.4
4	9276	86	106.3	.00851	78.9	.00836	77.5
45	8463	71	101.6	.00891	75.4	.00873	73.9
6	7763	78	97.5	.00933	72.4	.00913	70.9
47	7310	68	96.5	.00980	71.6	.00956	69.9
8	6338	65	88.0	.01031	65.3	.01004	63.6
9	5784	62	84.6	.01087	62.9	.01056	61.1
50	5184	57	80.1	.01148	59.5	.01113	57.7
1	4773	50	78.0	.01214	57.9	.01175	56.1
52	4639	78	80.3	.01286	59.7	.01244	57.7
3	3556	41	65.4	.01366	48.6	.01318	46.9
4	3278	51	64.1	.01453	47.6	.01399	45.9
55	2816	63	58.7	.01547	43.6	.01488	41.9
6	2515	28	55.9	.01650	41.5	.01586	39.9
57	2161	49	51.3	.01764	38.1	.01692	36.6
8	1873	37	47.6	.01887	35.3	.01808	33.9
9	1511	33	41.1	.02022	30.6	.01935	29.2
60	1335	28	39.0	.02170	29.0	.02073	27.7
1	1215	29	38.1	.02331	28.3	.02225	27.0
62	1181	31	39.8	.02505	29.6	.02390	28.2
3	731	21	26.5	.02698	19.7	.02571	18.8
4	626	17	24.5	.02906	18.2	.02768	17.3
65	487	11	20.6	.03135	15.3	.02983	14.5
6	395	15	18.0	.03383	13.4	.03218	12.7
Totals	407250	3005	4045.6	...	3004.9	...	2972.7

TABLE VI (D).  
 $O^{[M]}$  Select Table. Year 3 to 4.

Age $[x] + 3$	ORIGINAL DATA		$O^{[M]}$ Expected Deaths	KING		HARDY	
	At Risk	Actual Deaths		$q[x]+3$	Expected Deaths	$q[x]+3$	Expected Deaths
23	4365	24	29.3	.00529	23.1	.00539	23.5
4	7686	52	52.3	.00535	41.1	.00545	41.9
25	9713	59	66.9	.00542	52.7	.00552	53.6
6	11890	57	83.0	.00550	65.4	.00559	66.5
7	14239	75	101.0	.00558	79.5	.00567	80.7
28	16507	90	119.0	.00568	93.8	.00576	95.1
9	17489	93	128.0	.00576	100.7	.00585	102.3
30	18004	96	134.5	.00588	105.9	.00595	107.1
1	18508	105	141.0	.00600	111.0	.00607	112.3
2	18892	127	146.8	.00612	115.6	.00619	116.9
33	19191	105	152.8	.00627	120.3	.00633	121.5
4	17836	112	145.5	.00643	114.7	.00647	115.4
35	17220	94	144.1	.00659	113.5	.00664	114.3
6	16553	117	142.4	.00677	112.1	.00681	112.7
7	15587	122	138.1	.00698	108.8	.00701	109.3
38	14985	112	137.1	.00720	107.9	.00722	108.2
9	13967	108	132.0	.00744	103.9	.00745	104.1
40	12670	98	123.9	.00770	97.6	.00770	97.6
1	11970	89	121.5	.00799	95.6	.00798	95.5
2	11358	98	119.9	.00832	94.5	.00829	94.2
43	10861	92	119.4	.00865	93.9	.00862	93.6
4	9681	103	110.9	.00902	87.3	.00898	86.9
45	8890	89	106.7	.00945	84.0	.00938	83.4
6	8128	82	102.1	.00989	80.4	.00981	79.7
7	7480	60	98.7	.01039	77.7	.01028	76.9
48	7024	75	97.5	.01093	76.8	.01080	75.9
9	6052	76	88.5	.01152	69.7	.01137	68.8
50	5540	82	85.6	.01217	67.4	.01199	66.4
1	4956	69	81.0	.01287	63.8	.01267	62.8
2	4595	71	79.5	.01363	62.6	.01341	61.6
53	4409	72	81.1	.01448	63.8	.01422	62.7
4	3426	52	67.0	.01540	52.8	.01510	51.7
55	3138	52	65.4	.01640	51.5	.01607	50.4
6	2665	46	59.2	.01750	46.6	.01713	45.7
7	2413	45	57.3	.01870	45.1	.01828	44.1
58	2073	44	52.7	.02001	41.5	.01955	40.5
9	1814	33	49.4	.02143	38.9	.02092	37.9
60	1468	22	42.9	.02300	33.8	.02243	32.9
1	1291	38	40.5	.02471	31.9	.02408	31.1
2	1151	35	38.8	.02656	30.6	.02587	29.8
63	1116	32	40.5	.02860	31.9	.02783	31.1
4	691	27	27.0	.03080	21.3	.02997	20.7
65	590	9	24.9	.03324	19.6	.03231	19.1
6	472	15	21.5	.03586	16.9	.03486	16.5
7	377	8	18.5	.03872	14.6	.03764	14.2
Totals	388931	3162	4015.7	...	3162.1	...	3157.1

TABLE VI (E).  
 $Q^{(M)}$  Select Table. Year 4 to 5.

Age [x] + 4	ORIGINAL DATA		$Q^{(M)}$ Expected Deaths	KING		HARDY	
	At Risk	Actual Deaths		$q[x]+4$	Expected Deaths	$q[x]+4$	Expected Deaths
24	4144	22	28.2	·00552	22.9	·00571	23.7
25	7238	61	49.9	·00559	40.5	·00578	41.8
6	9218	56	64.3	·00567	52.3	·00586	54.0
7	11351	59	80.5	·00575	65.3	·00594	67.4
8	13618	70	98.2	·00585	79.7	·00603	82.1
29	15840	83	115.9	·00594	94.1	·00614	97.3
30	16774	107	125.3	·00606	101.7	·00625	104.8
1	17248	115	131.4	·00618	106.6	·00637	109.9
2	17796	110	138.3	·00631	112.3	·00650	115.7
3	18124	113	144.3	·00646	117.1	·00664	120.3
34	18496	121	150.9	·00662	122.4	·00680	125.8
35	17172	121	143.7	·00679	116.6	·00698	119.9
6	16591	112	142.7	·00698	115.8	·00717	119.0
7	15897	121	140.8	·00719	114.3	·00737	117.2
8	14953	117	136.8	·00743	111.1	·00760	113.6
39	14427	109	136.3	·00767	110.7	·00785	113.3
40	13438	113	131.4	·00794	106.7	·00812	109.1
1	12248	103	124.3	·00824	100.9	·00842	103.1
2	11528	88	121.7	·00857	98.8	·00874	100.8
3	10949	106	120.3	·00892	97.7	·00910	99.6
44	10511	97	120.5	·00930	97.8	·00948	99.6
45	9362	81	112.3	·00974	91.2	·00991	92.8
6	8573	81	107.7	·01019	87.4	·01037	88.9
7	7801	84	103.0	·01071	83.5	·01088	84.9
8	7193	66	99.8	·01127	81.1	·01143	82.2
49	6789	84	99.3	·01187	80.6	·01204	81.7
50	5814	76	89.8	·01254	72.9	·01270	73.8
1	5342	70	87.3	·01326	70.8	·01343	71.7
2	4787	69	82.9	·01405	67.3	·01422	68.1
3	4419	70	81.3	·01493	66.0	·01509	66.7
54	4242	69	83.0	·01588	67.4	·01603	68.0
55	3288	41	68.5	·01691	55.6	·01707	56.1
6	3048	74	67.7	·01803	55.0	·01820	55.5
7	2572	45	61.1	·01928	49.6	·01943	50.0
8	2327	46	59.1	·02062	48.0	·02078	48.4
59	2002	42	54.5	·02209	44.2	·02226	44.6
60	1743	42	50.9	·02371	41.3	·02387	41.6
1	1446	35	45.4	·02547	36.8	·02562	37.0
2	1233	32	41.6	·02738	33.8	·02754	34.0
3	1100	24	40.0	·02948	32.4	·02964	32.6
64	1062	30	41.5	·03175	33.7	·03193	33.9
65	660	22	27.9	·03426	22.6	·03442	22.7
6	575	27	26.2	·03696	21.3	·03715	21.4
7	448	23	22.0	·03992	17.9	·04012	18.0
8	365	24	19.4	·04315	15.7	·04336	15.8
Totals	373752	3261	4017.9	...	3261.4	...	3328.4



TABLE VII (A).  
*Q<sup>(M)</sup> Select Table. Summary in Age Groups. Year 0 to 1.*

Entry Ages	Attained Ages	ORIGINAL DATA		OM(5) Expected Deaths	KING		HARDY	
		At Risk	Actual Deaths		Expected Deaths	Deviation	Expected Deaths	Deviation
20 to 24	20 to 24	61130	158	409.0	180.9	+ 22.9	165.1	+ 7.1
25 " 29	25 " 29	108781	305	772.8	342.0	+ 37.0	318.3	+ 13.3
30 " 34	30 " 34	103038	366	801.6	354.6	- 11.4	339.3	- 26.7
35 " 39	35 " 39	76738	277	678.8	300.5	+ 23.5	297.0	+ 20.0
40 " 44	40 " 44	52756	272	554.5	245.3	+ 26.7	252.2	- 19.8
45 " 49	45 " 49	33026	190	433.1	191.8	+ 1.8	204.5	+ 14.5
50 " 54	50 " 54	18926	168	323.8	143.3	- 21.7	158.5	- 9.5
55 " 59	55 " 59	9117	109	213.3	94.4	- 14.6	107.7	- 1.3
60 " 64	60 " 64	3845	64	125.4	55.5	- 8.5	64.8	+ 0.8
20 to 64	Totals	467357	1909	4312.3	1908.3	+ 85.2 - 85.9 ± 171.1	1907.4	+ 55.7 - 57.3 ± 113.0

TABLE VII (B). *Year 1 to 2.*

20 to 24	21 to 25	55067	261	372.5	242.9	- 18.1	247.2	- 13.8
25 " 29	26 " 30	99847	417	721.0	469.9	+ 52.9	476.6	+ 59.6
30 " 34	31 " 35	95620	466	760.9	496.1	+ 30.1	500.2	+ 34.2
35 " 39	36 " 40	71571	433	653.2	425.6	- 7.4	426.7	- 6.3
40 " 44	41 " 45	49260	341	539.3	351.5	+ 10.5	349.6	- 8.6
45 " 49	46 " 50	30953	290	426.8	278.3	- 11.7	274.7	- 15.3
50 " 54	51 " 55	17769	246	322.6	210.2	- 35.8	206.3	- 39.7
55 " 59	56 " 60	8553	155	214.1	139.5	- 15.5	136.1	- 18.9
60 " 64	61 " 65	3630	88	127.5	83.1	- 4.9	80.9	- 7.1
20 to 64	Totals	432270	2697	4137.9	2697.1	+ 93.5 - 93.4 ± 186.9	2698.3	+ 102.4 - 101.1 ± 203.5



TABLE VII (c).  
[30] Select Table. Summary in Age Groups. Year 2 to 3.

Entry Ages	Attained Ages	ORIGINAL DATA		OM(3) Expected Deaths	KING		HARDY	
		At Risk	Actual Deaths		Expected Deaths	Deviation	Expected Deaths	Deviation
20 to 24	22 to 26	50861	278	348.3	258.7	- 19.3	262.2	- 15.8
25 " 29	27 " 31	93911	485	690.1	512.8	+ 27.8	516.7	+ 31.7
30 " 34	32 " 36	90177	546	735.0	545.9	- 0.1	546.3	+ 0.3
35 " 39	37 " 41	67746	479	639.0	474.6	- 4.4	470.6	- 8.4
40 " 44	42 " 46	46847	392	535.1	397.2	+ 5.2	390.3	- 1.7
45 " 49	47 " 51	29389	302	427.2	317.2	+ 15.2	308.4	+ 6.4
50 " 54	52 " 56	16804	261	324.4	241.0	- 20.0	232.3	- 28.7
55 " 59	57 " 61	8095	167	217.1	161.3	- 5.7	154.4	- 12.6
60 " 64	62 " 66	3420	95	129.4	96.2	+ 1.2	91.5	- 3.5
20 to 64	Totals	407250	3005	4045.6	3004.9	+ 49.4 - 49.5 ± 98.9	2972.7	+ 38.4 - 70.7 ± 109.1

TABLE VII (d). Year 3 to 4.

20 to 24	23 to 27	47893	267	332.5	261.8	- 5.2	266.2	- 0.8
25 " 29	28 " 32	89400	511	669.3	527.0	+ 16.0	533.7	+ 22.7
30 " 34	33 " 37	86387	550	722.9	569.4	+ 19.4	573.2	+ 23.2
35 " 39	38 " 42	64950	505	634.4	499.5	- 5.5	499.6	- 5.4
40 " 44	43 " 47	45040	426	537.8	423.3	- 2.7	420.5	- 5.5
45 " 49	48 " 52	28167	373	432.1	340.3	- 32.7	335.5	- 37.5
50 " 54	53 " 57	16051	267	330.0	259.8	- 7.2	254.6	- 12.4
55 " 59	58 " 62	7797	172	224.3	176.7	+ 4.7	172.2	+ 0.2
60 " 64	63 " 67	3246	91	132.4	104.3	+ 13.3	101.6	+ 10.6
20 to 64	Totals	388931	3162	4015.7	3162.1	+ 53.4 - 53.3 ± 106.7	3157.1	+ 56.7 - 61.6 ± 118.3

TABLE VII (E).  
O<sup>M</sup> Select Table, Summary in Age Groups., Year 4 to 5.

Entry Ages	Attained Ages	ORIGINAL DATA		O <sup>M</sup> (5) Expected Deaths	KING		HARDY	
		At Risk	Actual Deaths		Expected Deaths	Deviation	Expected Deaths	Deviation
20 to 24	24 to 28	45569	268	321.1	260.7	- 7.3	269.0	+ 1.0
25 " 29	29 " 33	85782	528	655.2	531.8	+ 3.8	548.0	+ 20.0
30 " 34	34 " 38	83109	592	714.9	580.2	- 11.8	595.5	+ 3.5
35 " 39	39 " 43	62590	519	634.0	514.8	- 4.2	525.9	+ 6.9
40 " 44	44 " 48	43440	409	543.3	441.0	+ 32.0	448.4	+ 39.4
45 " 49	49 " 53	27151	369	440.6	357.6	- 11.4	362.0	- 7.0
50 " 54	54 " 58	15477	275	339.4	275.6	+ 0.6	278.0	+ 3.0
55 " 59	59 " 63	7524	175	232.4	188.5	+ 13.5	189.8	+ 14.8
60 " 64	64 " 68	3110	126	137.0	111.2	- 14.8	111.8	- 14.2
20 to 64	Totals	373752	3261	4017.9	3261.4	+ 49.9 - 49.5 ± 99.4	3328.4	+ 88.6 - 21.2 ± 109.8

TABLE VII (F). All Years 0 to 5.

Entry Ages	Attained Ages	ORIGINAL DATA		O <sup>M</sup> (5) Expected Deaths	KING		HARDY	
		At Risk	Actual Deaths		Expected Deaths	Deviation	Expected Deaths	Deviation
20 to 24	24 to 28	260520	1232	1783.4	1205.0	- 27.0	1209.7	- 22.3
25 " 29	29 " 33	477721	2246	3508.4	2383.5	+ 137.5	2393.3	+ 147.3
30 " 34	34 " 38	458331	2520	3735.3	2546.2	+ 26.2	2554.5	+ 34.5
35 " 39	39 " 43	343595	2213	3239.4	2215.0	+ 2.0	2219.8	+ 6.8
40 " 44	44 " 48	237343	1840	2710.0	1858.3	+ 18.3	1861.0	+ 21.0
45 " 49	49 " 53	148686	1524	2159.8	1485.2	- 38.8	1485.1	- 38.9
50 " 54	54 " 58	85027	1217	1640.2	1129.9	- 87.1	1129.7	- 87.3
55 " 59	59 " 63	41086	778	1101.2	760.4	- 17.6	760.2	- 17.8
60 " 64	64 " 68	17251	464	651.7	450.3	- 13.7	450.6	- 13.4
20 to 64	Totals	2069560	14034	20529.4	14033.8	+ 181.0 - 184.2 ± 368.2	14063.9	+ 209.6 - 179.7 ± 389.3

TABLE VIII.  
*Australian Ultimate Table (AMP)<sup>M(5)</sup>.*

Age <i>x</i>	ORIGINAL DATA		QM <sup>(5)</sup> Expected Deaths	FACTORS		GRADUATED ULTIMATE TABLE		
	At Risk	Actual Deaths		Milne's Method	Graduated	<i>q<sub>x</sub></i>	Expected Deaths	Deviation
20	194	2	1.3	.607	.60500	.00394	0.8	- 1.2
1	397	1	2.6	.607	.60560	.00399	1.6	+ 0.6
2	732	1	4.9	.607	.60670	.00403	2.9	+ 1.9
3	1180	7	7.9	.608	.60780	.00408	4.8	- 2.2
4	1764	6	12.0	.609	.60887	.00414	7.3	+ 1.3
25	2613	8	18.0	.611	.60990	.00420	11.0	+ 3.0
6	3726	20	26.0	.612	.61118	.00427	15.9	- 4.1
7	4853	19	34.4	.614	.61293	.00435	21.1	+ 2.1
8	6102	27	44.0	.616	.61558	.00444	27.1	+ 0.1
9	7535	35	55.2	.619	.61972	.00454	34.2	- 0.8
30	9036	25	67.5	.624	.62597	.00468	42.3	+ 17.3
1	10499	58	80.0	.631	.63470	.00484	50.8	- 7.2
2	11942	50	92.8	.640	.64577	.00502	59.9	+ 9.9
3	13315	72	106.0	.654	.65846	.00524	69.8	- 2.2
4	14574	95	118.9	.676	.67162	.00548	79.9	- 15.1
35	15902	111	133.1	.692	.68399	.00572	91.0	- 20.0
6	16863	91	145.0	.699	.69455	.00597	100.7	+ 9.7
7	17719	87	157.0	.703	.70286	.00623	110.4	+ 23.4
8	18391	125	168.3	.708	.70902	.00649	119.4	- 5.6
9	18689	134	176.6	.713	.71356	.00674	126.0	- 8.0
40	18940	150	185.2	.716	.71713	.00701	132.8	- 17.2
1	18938	141	192.2	.720	.72028	.00731	138.4	- 2.6
2	18754	136	198.0	.723	.72331	.00764	143.3	+ 7.3
3	18593	155	204.3	.726	.72636	.00798	148.4	- 6.6
4	18102	140	207.4	.729	.72944	.00836	151.3	+ 11.3
45	17836	134	214.0	.733	.73252	.00879	156.8	+ 22.8
6	17299	158	217.3	.736	.73562	.00924	159.8	+ 1.8
7	16599	174	219.1	.739	.73885	.00975	161.8	- 12.2
8	15807	161	219.4	.742	.74231	.01030	162.8	+ 1.8
9	15190	170	222.2	.745	.74608	.01092	165.9	- 4.1
50	14557	160	224.9	.749	.75019	.01159	168.7	+ 8.7
1	13926	172	227.6	.754	.75454	.01233	171.7	- 0.3
2	13282	183	229.9	.759	.75889	.01314	174.5	- 8.5
3	12619	162	232.1	.764	.76298	.01403	177.0	+ 15.0
4	11929	194	233.3	.768	.76655	.01499	178.8	- 15.2
55	11285	163	235.1	.770	.76944	.01603	180.9	+ 17.9
6	10602	170	235.6	.772	.77165	.01715	181.8	+ 11.8
7	9944	171	236.2	.773	.77326	.01836	182.6	+ 11.6
8	9265	170	235.4	.774	.77445	.01968	182.3	+ 12.3
9	8553	179	232.8	.775	.77536	.02111	180.6	+ 1.6
60	7803	170	227.9	.776	.77604	.02267	176.9	+ 6.9
1	7029	172	220.6	.777	.77653	.02437	171.3	- 0.7
2	6294	185	212.3	.778	.77689	.02620	164.9	- 20.1
3	5558	153	201.9	.778	.77723	.02823	156.9	+ 3.9
4	4884	166	191.1	.778	.77784	.03043	148.6	- 17.4
Carried forward	499614	5063	6907.3	...	...	...	5095.7	+ 32.7

TABLE VIII (continued).

Australian Ultimate Table (AMP)<sup>M(5)</sup>.

Age <i>x</i>	ORIGINAL DATA		QM(5) Expected Deaths	FACTORS		GRADUATED ULTIMATE TABLE		
	At Risk	Actual Deaths		Milne's Method	Graduated	<i>q<sub>x</sub></i>	Expected Deaths	Deviation
Brought forward	499614	5063	6907·3	...	...	...	5095·7	+ 32·7
65	4336	137	183·0	·779	·77912	·03289	142·6	+ 5·6
6	3774	148	171·9	·780	·78145	·03559	134·3	- 13·7
7	3258	126	160·2	·782	·78514	·03861	125·8	- 0·2
8	2816	138	149·7	·788	·79022	·04202	118·3	- 19·7
9	2350	110	135·1	·796	·79643	·04578	107·6	- 2·4
70	1984	116	123·4	·804	·80325	·04995	99·1	- 16·9
1	1623	83	109·2	·812	·81010	·05453	88·5	+ 5·5
2	1324	83	96·5	·818	·81638	·05951	78·8	- 4·2
3	1038	65	82·0	·822	·82169	·06488	67·3	+ 2·3
4	810	63	69·3	·826	·82584	·07063	57·2	- 5·8
75	623	38	57·7	·830	·82881	·07681	47·9	+ 9·9
6	491	36	49·3	·831	·83074	·08344	41·0	+ 5·0
7	373	23	40·6	·832	·83184	·09055	33·8	+ 10·8
8	298	37	35·2	·832	·83234	·09818	29·3	- 7·7
9	214	31	27·4	·832	·83248	·10640	22·8	- 8·2
80	155	16	21·5	·832	·83249	·11527	17·9	+ 1·9
1	112	14	16·8	·832	·83263	·12489	14·0	+ 0·0
2	78	4	12·7	·832	·83314	·13531	10·6	+ 6·6
3	59	11	10·4	·832	·83430	·14666	8·7	- 2·3
4	35	6	6·7	·832	·83635	·15906	5·6	- 0·4
85	28	7	5·8	·832	·83958	·17263	5·0	- 2·0
6	17	3	3·8	·832	·84425	·18759	3·2	+ 0·2
7	13	3	3·1	·832	·85066	·20409	2·7	- 0·3
8	8	2	2·1	...	·85909	·22237	1·8	- 0·2
9	4	0	1·1	...	·86983	·24271	1·0	+ 1·0
90	3	1	0·9	...	·88318	·26535	0·8	- 0·2
1	2	0	0·6	...	·89945	·29068	0·6	+ 0·6
2	2	1	0·7	...	·91894	·31905	0·6	- 0·4
3	...	...	...	...	·94199	·35087	...	...
4	...	...	...	...	·96890	·38662	...	...
95	...	...	...	...	·99999	·42679	...	...
6	...	...	...	...	...	·47338	...	...
7	...	...	...	...	...	·52840	...	...
8	...	...	...	...	...	·59385	...	...
9	...	...	...	...	...	·67173	...	...
100	...	...	...	...	...	·76405	...	...
1	...	...	...	...	...	·87280	...	...
2	...	...	...	...	...	·99999	...	...
Totals	525442	6365	8484·0	...	...	...	6362·5	- 2·5

TABLE IX.

(AMP)<sup>M(5)</sup> *Ultimate Table. Summary in Age Groups.*

Attained Ages	ORIGINAL DATA		OM(5) Expected Deaths	GRADUATED TABLE	
	At Risk	Actual Deaths		Expected Deaths	Deviation
20 to 24	4267	17	28.7	17.4	+ 0.4
25 „ 29	24829	109	177.6	109.3	+ 0.3
30 „ 34	59366	300	465.2	302.7	+ 2.7
35 „ 39	87564	548	780.0	547.5	- 0.5
40 „ 44	93327	722	987.1	714.2	- 7.8
45 „ 49	82731	797	1092.0	807.1	+ 10.1
50 „ 54	66313	871	1147.8	870.7	- 0.3
55 „ 59	49649	853	1175.1	908.2	+ 55.2
60 „ 64	31568	846	1053.8	818.6	- 27.4
65 „ 69	16534	659	799.9	628.6	- 30.4
70 „ 74	6779	410	480.4	390.9	- 19.1
75 „ 79	1999	165	210.2	174.8	+ 9.8
80 „ 84	439	51	68.1	56.8	+ 5.8
85 „ 89	70	15	15.9	13.7	- 1.3
90 „ 94	7	2	2.2	2.0	0.0
Totals	525442	6365	8484.0	6362.5	+ 84.3 - 86.8

TABLE X.

*Factors by which to multiply (AMP)<sup>M(5)</sup> Ultimate  $q_x$ , so as to obtain the (AMP)<sup>[M]</sup> Select  $q_{[x]}$ ,  $q_{[x]+1}$ ,  $q_{[x]+2}$ ,  $q_{[x]+3}$ ,  $q_{[x]+4}$ , which will reproduce by Age Groups the Actual Deaths of the Original Data.*

Entry Ages of Groups	Year 0 to 1	Year 1 to 2	Year 2 to 3	Year 3 to 4	Year 4 to 5
15 to 19	.54348	.92199	.78740	.35398	.80000
20 „ 24	.73482	.70968	.75294	1.17048	.93407
25 „ 29	.67868	.91399	.97983	.76220	1.13965
30 „ 34	.69990	.81609	.76923	.84472	.84306
35 „ 39	.59313	.64951	.89286	.83777	.81831
40 „ 44	.71808	.87156	.79239	.99836	.82770
45 „ 49	.44610	.61350	.95436	.52521	.70513
50 „ 54	.54830	.73239	.82153	.64789	.81461
55 „ 59	.45249	.72464	.72816	1.07317	.90909
60 „ 64	.53763	1.02273	1.72414	.96386	.97561
15 to 39	.66542	.78393	.85255	.84958	.91896
40 „ 64	.57373	.76408	.88118	.80347	.80778
15 to 64	.63370	.77656	.86345	.83165	.87485
<i>Factors adopted for the final Graduation.</i>					
15 to 64	.63370	.77656	.83315	.86312	.87485

TABLE XI (A).  
(AMP)<sup>[M]</sup> Select Table. Year 0 to 1.

Age [x]	ORIGINAL DATA		(AMP) <sup>M(5)</sup> Ultimate Expected Deaths	SELECT GRADUATED TABLE		
	At Risk	Actual Deaths		$q[x]$	Expected Deaths	Deviation
15	397	...	1.5	·00243	1.0	+ 1.0
6	612	...	2.4	·00243	1.5	+ 1.5
7	922	4	3.5	·00244	2.2	- 1.8
8	1214	1	4.7	·00245	3.0	+ 2.0
9	1628	5	6.3	·00247	4.0	- 1.0
20	2170	8	8.5	·00250	5.4	- 2.6
1	2872	11	11.5	·00253	7.3	- 3.7
2	3186	7	12.8	·00255	8.1	+ 1.1
3	3454	8	14.1	·00258	8.9	+ 0.9
4	3794	12	15.7	·00262	9.9	- 2.1
25	4109	10	17.3	·00266	10.9	+ 0.9
6	4234	7	18.1	·00271	11.5	+ 4.5
7	4383	18	19.1	·00276	12.1	- 5.9
8	4483	20	19.9	·00281	12.6	- 7.4
9	4394	9	19.9	·00288	12.7	+ 3.7
30	4682	13	21.9	·00297	13.9	+ 0.9
1	4243	16	20.5	·00307	13.0	- 3.0
2	4108	17	20.6	·00318	13.1	- 3.9
3	3985	15	20.9	·00332	13.2	- 1.8
4	3728	12	20.4	·00347	12.9	+ 0.9
35	3623	11	20.7	·00362	13.1	+ 2.1
6	3281	10	19.6	·00378	12.4	+ 2.4
7	3062	11	19.1	·00395	12.1	+ 1.1
8	2954	17	19.2	·00411	12.1	- 4.9
9	2601	8	17.5	·00427	11.1	+ 3.1
40	2703	11	18.9	·00444	12.0	+ 1.0
1	2101	14	15.4	·00463	9.7	- 4.3
2	1854	10	14.2	·00484	9.0	- 1.0
3	1715	6	13.7	·00506	8.7	+ 2.7
4	1558	13	13.0	·00530	8.3	- 4.7
45	1566	8	13.8	·00557	8.7	+ 0.7
6	1182	6	10.9	·00586	6.9	+ 0.9
7	1019	2	9.9	·00618	6.3	+ 4.3
8	952	4	9.8	·00653	6.2	+ 2.2
9	862	4	9.4	·00692	6.0	+ 2.0
50	887	11	10.3	·00734	6.5	- 4.5
1	647	5	8.0	·00781	5.1	+ 0.1
2	547	2	7.2	·00833	4.6	+ 2.6
3	464	3	6.5	·00889	4.1	+ 1.1
4	418	...	6.3	·00950	4.0	+ 4.0
55	397	4	6.4	·01016	4.1	+ 0.1
6	259	2	4.4	·01087	2.8	+ 0.8
7	210	3	3.9	·01163	2.5	- 0.5
8	177	...	3.5	·01247	2.2	+ 2.2
9	185	1	3.9	·01338	2.5	+ 1.5
60	175	1	4.0	·01437	2.5	+ 1.5
1	92	1	2.2	·01544	1.4	+ 0.4
2	49	1	1.3	·01660	0.8	- 0.2
3	36	1	1.0	·01789	0.6	- 0.4
4	26	1	0.8	·01928	0.5	- 0.5
Totals	98200	364	574.4	...	364.0	Nil.



TABLE XI (B).  
(AMP)<sup>[M]</sup> Select Table. Year 1 to 2.

Age [x] + 1	ORIGINAL DATA		(AMP) <sup>M(5)</sup> Ultimate Expected Deaths	SELECT GRADUATED TABLE		
	At Risk	Actual Deaths		$q[x]+1$	Expected Deaths	Deviation
16	315	2	1.2	.00298	0.9	- 1.1
7	459	1	1.8	.00299	1.4	+ 0.4
8	709	...	2.7	.00301	2.1	+ 2.1
9	941	8	3.7	.00303	2.9	- 5.1
20	1182	2	4.7	.00306	3.6	+ 1.6
21	1602	4	6.4	.00310	5.0	+ 1.0
2	2123	7	8.6	.00313	6.6	- 0.4
3	2277	7	9.3	.00317	7.2	+ 0.2
4	2502	9	10.4	.00321	8.0	- 1.0
25	2812	6	11.8	.00326	9.2	+ 3.2
26	3119	12	13.3	.00332	10.4	- 1.6
7	3230	10	14.1	.00338	10.9	+ 0.9
8	3391	17	15.1	.00345	11.7	- 5.3
9	3458	14	15.7	.00353	12.2	- 1.8
30	3461	15	16.2	.00363	12.6	- 2.4
31	3693	12	17.9	.00376	13.9	+ 1.9
2	3370	9	16.9	.00390	13.1	+ 4.1
3	3318	8	17.4	.00407	13.5	+ 5.5
4	3220	21	17.6	.00426	13.7	- 7.3
35	3014	21	17.2	.00444	13.4	- 7.6
36	2917	12	17.4	.00464	13.5	+ 1.5
7	2691	10	16.8	.00484	13.0	+ 3.0
8	2497	12	16.2	.00504	12.6	+ 0.6
9	2416	9	16.3	.00523	12.6	+ 3.6
40	2129	10	14.9	.00544	11.6	+ 1.6
41	2226	17	16.3	.00568	12.6	- 4.4
2	1773	12	13.5	.00593	10.5	- 1.5
3	1519	10	12.1	.00620	9.4	- 0.6
4	1447	9	12.1	.00649	9.4	+ 0.4
45	1297	9	11.4	.00683	8.9	- 0.1
46	1301	7	12.0	.00718	9.4	+ 2.4
7	1018	2	9.9	.00757	7.7	+ 5.7
8	900	8	9.3	.00800	7.2	- 0.8
9	810	5	8.8	.00848	6.9	+ 1.9
50	764	8	8.9	.00900	6.9	- 1.1
51	755	6	9.3	.00957	7.2	+ 1.2
2	574	6	7.5	.01020	5.9	- 0.1
3	468	5	6.6	.01090	5.1	+ 0.1
4	409	6	6.1	.01164	4.8	- 1.2
55	373	3	6.0	.01245	4.6	+ 1.6
56	338	5	5.8	.01332	4.5	- 0.5
7	232	1	4.3	.01426	3.3	+ 2.3
8	181	4	3.6	.01528	2.8	- 1.2
9	156	...	3.3	.01639	2.6	+ 2.6
60	164	5	3.7	.01760	2.9	- 2.1
61	155	3	3.8	.01892	2.9	- 0.1
2	76	3	2.0	.02035	1.5	- 1.5
3	44	...	1.2	.02192	1.0	+ 1.0
4	33	3	1.0	.02363	0.8	- 2.2
65	24	...	0.8	.02554	0.6	+ 0.6
Totals	77883	375	482.9	...	375.0	Nil.

TABLE XI (c).  
(AMP)<sup>[M]</sup> *Select Table.* Year 2 to 3.

Age [ <i>x</i> ] + 2	ORIGINAL DATA		(AMP) <sup>M(5)</sup> Ultimate Expected Deaths	SELECT GRADUATED TABLE		
	At Risk	Actual Deaths		<i>q</i> [ <i>x</i> ] + 2	Expected Deaths	Deviation
17	269	...	1.0	.00321	0.9	+ 0.9
8	403	2	1.6	.00322	1.3	- 0.7
9	640	...	2.5	.00325	2.1	+ 2.1
20	863	4	3.4	.00328	2.8	- 1.2
1	1061	4	4.2	.00332	3.5	- 0.5
22	1431	3	5.8	.00336	4.8	+ 1.8
3	1904	3	7.8	.00340	6.5	+ 3.5
4	2059	5	8.5	.00345	7.1	+ 2.1
25	2271	11	9.5	.00350	7.9	- 3.1
6	2555	10	10.9	.00356	9.1	- 0.9
27	2820	13	12.3	.00362	10.2	- 2.8
8	2934	12	13.0	.00370	10.9	- 1.1
9	3061	14	13.9	.00378	11.6	- 2.4
30	3157	14	14.8	.00390	12.3	- 1.7
1	3175	15	15.4	.00403	12.8	- 2.2
32	3378	12	17.0	.00418	14.1	+ 2.1
3	3107	10	16.3	.00437	13.6	+ 3.6
4	3052	8	16.7	.00457	13.9	+ 5.9
35	2932	17	16.8	.00477	14.0	- 3.0
6	2755	17	16.4	.00497	13.7	- 3.3
37	2689	11	16.8	.00519	14.0	+ 3.0
8	2474	15	16.1	.00541	13.4	- 1.6
9	2298	14	15.5	.00562	12.9	- 1.1
40	2225	17	15.6	.00584	13.0	- 4.0
1	1967	13	14.4	.00609	12.0	- 1.0
42	2038	12	15.6	.00637	13.0	+ 1.0
3	1644	7	13.1	.00665	10.9	+ 3.9
4	1400	9	11.7	.00697	9.8	+ 0.8
45	1320	10	11.6	.00732	9.7	- 0.3
6	1198	12	11.1	.00770	9.2	- 2.8
47	1203	11	11.7	.00812	9.7	- 1.3
8	952	8	9.8	.00858	8.2	+ 0.2
9	831	10	9.1	.00910	7.6	- 2.4
50	761	8	8.8	.00966	7.4	- 0.6
1	711	9	8.8	.01027	7.3	- 1.7
52	703	12	9.2	.01095	7.7	- 4.3
3	548	6	7.7	.01169	6.4	+ 0.4
4	435	3	6.5	.01249	5.4	+ 2.4
55	375	4	6.0	.01336	5.0	+ 1.0
6	346	4	5.9	.01429	4.9	+ 0.9
57	308	6	5.7	.01530	4.7	- 1.3
8	219	4	4.3	.01640	3.6	- 0.4
9	169	2	3.6	.01759	3.0	+ 1.0
60	148	2	3.4	.01889	2.8	+ 0.8
1	149	1	3.6	.02030	3.0	+ 2.0
62	142	7	3.7	.02183	3.1	- 3.9
3	68	5	1.9	.02352	1.6	- 3.4
4	43	1	1.3	.02535	1.1	+ 0.1
65	30	1	1.0	.02740	0.8	- 0.2
6	23	1	0.8	.02965	0.7	- 0.3
Totals	71244	399	462.1	...	385.0	- 14.0

TABLE XI (D).  
(AMP)<sup>[M]</sup> Select Table. Year 3 to 4.

Age [x] + 3	ORIGINAL DATA		(AMP) <sup>M(5)</sup> Ultimate Expected Deaths	SELECT GRADUATED TABLE		
	At Risk	Actual Deaths		q[x]+3	Expected Deaths	Deviation
18	226	...	0.9	.00334	0.8	+ 0.8
9	353	1	1.4	.00337	1.2	+ 0.2
20	566	1	2.2	.00340	1.9	+ 0.9
1	746	...	3.0	.00344	2.6	+ 2.6
2	942	2	3.8	.00348	3.3	+ 1.3
23	1306	12	5.3	.00352	4.6	- 7.4
4	1725	7	7.1	.00357	6.1	- 0.9
25	1875	4	7.9	.00363	6.8	+ 2.8
6	2060	13	8.8	.00369	7.6	- 5.4
7	2354	10	10.2	.00375	8.8	- 1.2
28	2588	9	11.5	.00383	9.9	+ 0.9
9	2690	12	12.2	.00392	10.5	- 1.5
30	2834	7	13.3	.00404	11.4	+ 4.4
1	2899	10	14.0	.00418	12.1	+ 2.1
2	2907	12	14.6	.00433	12.6	+ 0.6
33	3129	16	16.4	.00452	14.1	- 1.9
4	2907	8	15.9	.00473	13.8	+ 5.8
35	2826	18	16.2	.00494	14.0	- 4.0
6	2710	18	16.2	.00515	14.0	- 4.0
7	2543	8	15.8	.00538	13.7	+ 5.7
38	2485	14	16.1	.00560	13.9	- 9.1
9	2299	13	15.5	.00582	13.3	+ 0.3
40	2101	9	14.7	.00605	12.7	+ 3.7
1	2047	13	15.0	.00631	12.9	- 0.1
2	1820	14	13.9	.00659	12.0	- 2.0
43	1873	13	14.9	.00689	12.9	- 0.1
4	1535	15	12.8	.00722	11.1	- 3.9
45	1283	7	11.3	.00759	9.7	+ 2.7
6	1213	13	11.2	.00798	9.7	- 3.3
7	1117	13	10.9	.00842	9.4	- 3.6
48	1106	6	11.4	.00889	9.8	+ 3.8
9	888	10	9.7	.00943	8.3	- 1.7
50	767	1	8.9	.01000	7.7	+ 6.7
1	709	2	8.7	.01064	7.5	+ 5.5
2	675	6	8.9	.01134	7.7	+ 1.7
53	651	4	9.1	.01211	7.9	+ 3.9
4	516	4	7.7	.01294	6.7	+ 2.7
55	412	5	6.6	.01384	5.7	+ 0.7
6	353	7	6.1	.01480	5.2	- 1.8
7	329	3	6.0	.01585	5.2	+ 2.2
58	286	7	5.6	.01699	4.8	- 2.2
9	205	6	4.3	.01822	3.7	- 2.3
60	158	...	3.6	.01957	3.1	+ 3.1
1	141	5	3.4	.02103	3.0	- 2.0
2	136	4	3.6	.02261	3.1	- 0.9
63	123	3	3.5	.02437	3.0	- 0.0
4	58	3	1.8	.02626	1.5	- 1.5
65	38	...	1.2	.02839	1.1	+ 1.1
6	29	1	1.0	.03072	0.9	- 0.1
7	22	1	0.8	.03333	0.7	- 0.3
Totals	65561	370	444.9	...	384.0	+ 14.0

TABLE XI (E).

(AMP)<sup>[M]</sup> *Select Table.* Year 4 to 5.

Age [x] + 4	ORIGINAL DATA		(AMP) <sup>M(5)</sup> Ultimate Expected Deaths	SELECT GRADUATED TABLE		
	At Risk	Actual Deaths		q[x]+4	Expected Deaths	Deviation
19	185	1	0.7	·00341	0.6	— 0.4
20	317	...	1.2	·00345	1.1	+ 1.1
1	498	2	2.0	·00349	1.7	— 0.3
2	651	1	2.6	·00353	2.3	+ 1.3
3	864	4	3.5	·00357	3.1	— 0.9
24	1194	5	4.9	·00362	4.3	— 0.7
25	1577	9	6.6	·00367	5.8	— 3.2
6	1683	6	7.2	·00374	6.3	+ 0.3
7	1884	7	8.2	·00381	7.2	+ 0.2
8	2142	7	9.5	·00388	8.3	+ 1.3
29	2380	11	10.8	·00397	9.4	— 1.6
30	2449	15	11.5	·00409	10.0	— 5.0
1	2586	14	12.5	·00423	10.9	— 3.1
2	2661	14	13.4	·00439	11.7	— 2.3
3	2688	17	14.1	·00458	12.3	— 4.7
34	2846	17	15.6	·00479	13.6	— 3.4
35	2650	13	15.2	·00500	13.3	+ 0.3
6	2619	9	15.6	·00522	13.7	+ 4.7
7	2494	13	15.5	·00545	13.6	+ 0.6
8	2339	13	15.2	·00568	13.3	+ 0.3
39	2279	10	15.4	·00590	13.4	+ 3.4
40	2104	20	14.7	·00613	12.9	— 7.1
1	1944	9	14.2	·00640	12.4	+ 3.4
2	1896	10	14.5	·00668	12.7	+ 2.7
3	1668	10	13.3	·00698	11.6	+ 1.6
44	1731	16	14.5	·00731	12.7	— 3.3
45	1410	10	12.4	·00769	10.8	+ 0.8
6	1191	8	11.0	·00808	9.6	+ 1.6
7	1110	8	10.8	·00853	9.5	+ 1.5
8	1020	7	10.5	·00901	9.2	+ 2.2
49	1011	9	11.0	·00955	9.7	+ 0.7
50	831	7	9.6	·01014	8.4	+ 1.4
1	714	7	8.8	·01079	7.7	+ 0.7
2	659	3	8.7	·01150	7.6	+ 4.6
3	621	7	8.7	·01227	7.6	+ 0.6
54	601	6	9.0	·01311	7.9	+ 1.9
55	484	3	7.8	·01402	6.8	+ 3.8
6	391	6	6.7	·01500	5.9	— 0.1
7	327	5	6.0	·01606	5.3	+ 0.3
8	309	9	6.1	·01722	5.3	— 3.7
59	256	3	5.4	·01847	4.7	+ 1.7
60	181	4	4.1	·01983	3.6	— 0.4
1	144	6	3.5	·02132	3.1	— 2.9
2	123	1	3.2	·02292	2.8	+ 1.8
3	126	4	3.6	·02470	3.1	— 0.9
64	111	3	3.4	·02662	3.0	+ 0.0
65	49	2	1.6	·02877	1.4	— 0.6
6	38	2	1.4	·03114	1.2	— 0.8
7	24	1	0.9	·03378	0.8	— 0.2
8	21	...	0.9	·03676	0.8	+ 0.8
Totals	60081	374	427.5	...	374.0	Nil.

TABLE XII (A).

(AMP)<sup>[M]</sup> Select Table. Summary in Age Groups. Year 0 to 1.

Entrv Ages	Attained Ages	ORIGINAL DATA		(AMP) <sup>M(5)</sup> Ultimate Expected Deaths	SELECT GRADUATED TABLE	
		At Ri k	Actual Deaths		Expected Deaths	Deviation
15 to 19	15 to 19	4773	10	18.4	11.7	+ 1.7
20 „ 24	20 „ 24	15476	46	62.6	39.6	- 6.4
25 „ 29	25 „ 29	21603	64	94.3	59.8	- 4.2
30 „ 34	30 „ 34	20746	73	104.3	66.1	- 6.9
35 „ 39	35 „ 39	15521	57	96.1	60.8	+ 3.8
40 „ 44	40 „ 44	9931	54	75.2	47.7	- 6.3
45 „ 49	45 „ 49	5581	24	53.8	34.1	+ 10.1
50 „ 54	50 „ 54	2963	21	38.3	24.3	+ 3.3
55 „ 59	55 „ 59	1228	10	22.1	14.1	+ 4.1
60 „ 64	60 „ 64	378	5	9.3	5.8	+ 0.8
Totals	...	98200	364	574.4	364.0	+ 23.8 - 23.8

TABLE XII (B). Year 1 to 2.

15 to 19	16 to 20	3606	13	14.1	10.9	- 2.1
20 „ 24	21 „ 25	11316	33	46.5	36.0	+ 3.0
25 „ 29	26 „ 30	16659	68	74.4	57.8	- 10.2
30 „ 34	31 „ 35	16615	71	87.0	67.6	- 3.4
35 „ 39	36 „ 40	12650	53	81.6	63.3	+ 10.3
40 „ 44	41 „ 45	8262	57	65.4	50.8	- 6.2
45 „ 49	46 „ 50	4793	30	48.9	38.1	+ 8.1
50 „ 54	51 „ 55	2579	26	35.5	27.6	+ 1.6
55 „ 59	56 „ 60	1071	15	20.7	16.1	+ 1.1
60 „ 64	61 „ 65	332	9	8.8	6.8	- 2.2
Totals	...	77883	375	482.9	375.0	+ 24.1 - 24.1

TABLE XII (C). Year 2 to 3.

15 to 19	17 to 21	3236	10	12.7	10.6	+ 0.6
20 „ 24	22 „ 26	10220	32	42.5	35.4	+ 3.4
25 „ 29	27 „ 31	15147	68	69.4	57.8	- 10.2
30 „ 34	32 „ 36	15224	64	83.2	69.3	+ 5.3
35 „ 39	37 „ 41	11653	70	78.4	65.3	- 4.7
40 „ 44	42 „ 46	7600	50	63.1	52.6	+ 2.6
45 „ 49	47 „ 51	4458	46	48.2	40.2	- 5.8
50 „ 54	52 „ 56	2407	29	35.3	29.4	+ 0.4
55 „ 59	57 „ 61	993	15	20.6	17.1	+ 2.1
60 „ 64	62 „ 66	306	15	8.7	7.3	- 7.7
Totals	...	71244	399	462.1	385.0	+ 14.4 - 28.4

TABLE XII (D).

(AMP)<sup>[M]</sup> *Select Table. Summary in Age Groups. Year 3 to 4.*

Entry Ages	Attained Ages	ORIGINAL DATA		(AMP) <sup>M(5)</sup> Ultimate Expected Deaths	SELECT GRADUATED TABLE	
		At Risk	Actual Deaths		Expected Deaths	Deviation
15 to 19	18 to 22	2833	4	11.3	9.8	+ 5.8
20 „ 24	23 „ 27	9320	46	39.3	33.9	-12.1
25 „ 29	28 „ 32	13918	50	65.6	56.5	+ 6.5
30 „ 34	33 „ 37	14115	68	80.5	69.6	+ 1.6
35 „ 39	38 „ 42	10752	63	75.2	64.8	+ 1.8
40 „ 44	43 „ 47	7021	61	61.1	52.8	- 8.2
45 „ 49	48 „ 52	4145	25	47.6	41.0	+16.0
50 „ 54	53 „ 57	2261	23	35.5	30.7	+ 7.7
55 „ 59	58 „ 62	926	22	20.5	17.7	- 4.3
60 „ 64	63 „ 67	270	8	8.3	7.2	- 0.8
Totals	...	65561	370	444.9	384.0	+39.4 -25.4

TABLE XII (E). *Year 4 to 5.*

15 to 19	19 to 23	2515	8	10.0	8.8	+ 0.8
20 „ 24	24 „ 28	8480	34	36.4	31.9	- 2.1
25 „ 29	29 „ 33	12764	71	62.3	54.3	-16.7
30 „ 34	34 „ 38	12948	65	77.1	67.5	+ 2.5
35 „ 39	39 „ 43	9891	59	72.1	63.0	+ 4.0
40 „ 44	44 „ 48	6462	49	59.2	51.8	+ 2.8
45 „ 49	49 „ 53	3836	33	46.8	41.0	+ 8.0
50 „ 54	54 „ 58	2112	29	35.6	31.2	+ 2.2
55 „ 59	59 „ 63	830	18	19.8	17.3	- 0.7
60 „ 64	64 „ 68	243	8	8.2	7.2	- 0.8
Totals	...	60081	374	427.5	374.0	+20.3 -20.3

TABLE XII (F). *All Years 0 to 5.*

15 to 19	15 to 23	16963	45	66.5	51.8	+ 6.8
20 „ 24	20 „ 28	54812	191	227.3	176.8	-14.2
25 „ 29	25 „ 33	80091	321	366.0	286.2	-34.8
30 „ 34	30 „ 38	79648	341	432.1	340.1	- 0.9
35 „ 39	35 „ 43	60467	302	403.4	317.2	+15.2
40 „ 44	40 „ 48	39276	271	324.0	255.7	-15.3
45 „ 49	45 „ 53	22813	158	245.3	194.4	+36.4
50 „ 54	50 „ 58	12322	128	180.2	143.2	+15.2
55 „ 59	55 „ 63	5048	80	103.7	82.3	+ 2.3
60 „ 64	60 „ 68	1529	45	43.3	34.3	-10.7
Totals	...	372969	1882	2391.8	1882.0	+75.9 -75.9



## ABSTRACT OF THE DISCUSSION.

MR. C. W. KENCHINGTON said that the paper exemplified the scientific spirit of enquiry which had on so many former occasions led the author to valuable discoveries. Mr. King had found that the Nurses' experience produced peculiar results, and he had spared no trouble to ascertain the real cause. His method of attacking the problem was eminently practical. In Mr. King's view simplicity was of primary importance. Elaboration of detail was quite secondary. The public spirit of the author in communicating his investigations to the Institute deserved their acknowledgments.

He thought that Mr. King must have felt some disappointment when he found that he had been forestalled in point of time by the authors of the Japanese Tables. But it was probable that if he had been aware of the use of the method in the construction of the Japanese Tables, he would have adopted it for the Nurses' experience and would not have troubled to go any further in submitting the results of his investigation to the Institute. It was, therefore, not altogether to be regretted that Mr. King had overlooked the fact of the method having been used before.

Mr. King had referred to the complicated methods which formerly had been used for the construction of Select Tables. So far from being complicated, the new method could be described upon the proverbial half-sheet of notepaper. The method was not advocated by Mr. King as invariably applicable. He expressed the hope that it would prove useful to others, and that it would be found to give results sufficiently accurate for all ordinary purposes; and he stated that the applicability of the method had to be ascertained. In that connection he (the speaker) might quote the following passage from the review in the *Journal of the Japanese Offices' Tables*: "It necessarily follows from the way in which the ratios were obtained that the total expected and actual deaths in each year of duration are in substantial agreement. The deviations in each year of duration are also fairly evenly spread as positive and negative quantities, but an extended comparison over the first four years shows that the rate of mortality is too heavily weighted at the early entry ages, 20 to 34, whilst for the group of entry ages, 35 to 39, the expected deaths (1,194.5) are in defect of the actual (1,319) by no less than 124.5. While, therefore, the method was extremely simple to apply, and doubtless gave sufficiently good results for practical use, it cannot be said to have satisfactorily met all the requirements of a good graduation."

Mr. King, in investigating the applicability of the method, made use of the  $O^{M(5)}$  Table as the ultimate table for the purpose of examining the  $O^{[M]}$  data for the first five years of assurance. He thought that the  $O^{M(5)}$  Table was not quite suitable for the purpose. It was clear, from the ratios Mr. King gave, and as Mr. King himself pointed out, that selection had not worn off at the end of five years, and the necessary balancing of the first five years with the second five years of assurance was therefore lost sight of in the investigation which Mr. King had made. At the same time he (the speaker)

thought that the  $O^{[M]}$  Tables, as graduated by Sir George Hardy, provided a useful means of showing that Mr. King's method really gave a very close approximation to the official graduation. He had prepared the following table, showing in columnar form, according to duration of assurance, the ratios of  $q_{[x]+n}$  to  $q_{x+n}$ :

$O^{[M]}$  Official Graduation, Ratio of  $q_{[x]+n}$  to  $q_{x+n}$ .

$x$	$n$									
	0	1	2	3	4	5	6	7	8	9
20	·393	·652	·743	·788	·826	·860	·892	·924	·956	·985
30	·412	·650	·736	·786	·824	·861	·894	·926	·957	·986
40	·444	·646	·728	·780	·822	·861	·898	·930	·960	·988
50	·483	·640	·719	·774	·821	·863	·900	·934	·962	·989
60	·516	·638	·712	·771	·821	·865	·903	·936	·965	·989
70	·538	·640	·712	·772	·824	·868	·906	·939	·967	·990

*Maximum variation in Ratios.*

Character of progression according to age indicated as follows: Increasing, I; Decreasing, D.

·145 I	·014 D	·031 D	·017 D	·005 D	·008 I	·014 I	·015 I	·011 I	·005 I
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From this table it appeared that, with the exception of year of assurance 0, there was a very close approximation to constancy in the ratios treated as a function of age, thereby showing that the official graduation did not differ appreciably from that now suggested by Mr. King.

He had tested other tables and had found that the ratios showed very similar progression. For instance, in the Government Annuity Experience, 1910, which Mr. King himself quoted, the ratios approached constancy, as a function of the age, almost for the whole of the first five years of assurance. But while that might be sufficient testimony to the value of the method, it did not follow that one would be successful in obtaining the most appropriate ratios in the absence of detailed work upon the graduation. Therefore, it seemed to him that Mr. King's method was one which furnished a very valuable means of making a first investigation.

One of the first problems that confronted the actuary in the construction of Select Tables was to determine the number of years to be excluded in the construction of the Ultimate Table. In existing Select Tables the number of years excluded had been arrived at by somewhat arbitrary means. The five years' exclusion of the  $H^{[M]}$  Table, the five years' of the  $O^{[SM]}$  Table and the ten years' of the  $O^{[M]}$  Table, all appeared to be somewhat arbitrary. The method suggested by Mr. King would help in obtaining the

best approximation to the number of years to be excluded. Suppose they had an entirely new experience; it might be determined in the first instance that they would exclude ten years as being the utmost limit of practical tabulation to which they would be prepared to go. Then, having obtained a comparatively satisfactory Ultimate Table of mortality, they would, with that, calculate the expected deaths for each year of assurance, and, after having obtained the ratios of the actual to the expected, they would be able to ascertain at what shorter interval than the ten years it would be feasible to assume that the effect of selection had worn off. It would not, however, for that purpose be necessary to go to quite the amount of trouble that Mr. King had done in working out his expected deaths. A year ago Mr. Henry had submitted a method of approximate valuation, depending for its application on the calculation of constants and the multiplying of a first and second summation by those constants. With certain limitations as to range the method was equally applicable to functions not involving the rate of interest and could quite readily be used in the calculation of expected deaths. In order to illustrate the shortening of the work rendered possible by the method, he had had the expected deaths for the  $O^{[M]}$  data for the first ten years of assurance calculated by constants derived from the  $O^{[M]}$  Ultimate Table. The whole of the work involved had been done with the aid of listing and tabulating machines in something less than four hours. Having used the method extensively, he should himself have been quite content to employ it for the purpose in question without making any test whatsoever as to the accuracy of the approximation, but as he intended to mention the matter before the Institute that evening he had had a test made by calculating the exact expected deaths age by age, using the  $q_x$ 's from the Ultimate Table. The resulting figures were as follows:

*Expected Deaths  $O^{[M]}$  Data. Year of Assurance 5.*

Ages attained	Approximate	Actual
20-34	1016.0	1016.0
35-49	1909.6	1910.0
50-64	1030.8	1035.2

He suggested, therefore, that the use of the approximate method would help very materially in the application of Mr. King's method. He would not enter into a description of the approximate method at the moment, as there might be an opportunity of developing it at greater length before the close of the present session.

The next point to which he would refer was that of the progression of the ratios according to the duration of assurance. The ratios first obtained would not necessarily be the most suitable for adoption. Mr. King had found it necessary himself, in both the Nurses' experience and the Australian Mutual Provident experience, to vary the ratios. If Mr. King had taken out ratios for the whole of the ten years of assurance of the British Offices

Select Tables, he would probably have agreed that in that experience also it was necessary to vary the ratios in order to ensure the degree of smoothness desirable in a standard table. This smoothness, he thought, could best be obtained by graduating the actual deaths and not the ratios, because if the ratios were graduated they would not obtain equality between the expected and the actual deaths, whereas if the actual deaths were graduated on the assumption, for instance, that they could be represented by such a function as  $a + bx + cx^2 + dx^3$ , they would obtain, over the whole duration of assurance in which they were analysing selection, exact equality between actual and expected deaths. He had graduated the O<sup>M</sup> Table in this way, and the ratios were found to represent the original facts satisfactorily except for year of assurance 0.\* For year of assurance 0, in order to give the rates of mortality with any approach to representation of the original data, it was necessary to assume that  $q_{[x]}$  was of the form  $m + nq_x$ , the constant  $m$  being a negative quantity. It might be mentioned that the ratios increased with age for years of assurance 0 to 4, but decreased with age for years of assurance 5 to 9. It seemed rather peculiar that the progression of the ratios was, as Mr. King had stated, exactly the reverse of that shown by Sir George Hardy's graduation of the experience. He had calculated for decennial ages at entry the values of  $q_{[x]+n}$  for the whole of the ten years in which selection was analysed, and had computed the 3 per-cent annuity-values. The differences between the values so obtained and the official values were never great, being +·002 at age 20 at entry, +·001 at age 30, -·001 at age 40, -·006 at age 50, and -·011 at age 60. Those differences in annuity-values, he ventured to think, were so small that one was led to the conclusion that, at any rate for most practical purposes, so far as the British Offices experience was concerned, a graduation by Mr. King's method would have led to almost identical results with those brought out by the much more elaborate process of Sir George Hardy.

MR. G. J. LIDSTONE wrote as follows :

It is gratifying to me that Mr. King, in his present valuable communication, endorses the method of graduating a small experience by reference to a standard table, to which method I drew attention in one of the earliest of my own papers. Further experience has confirmed my view that this is the proper method to adopt when (as Mr. Todhunter remarked—*J.I.A.*, vol. xxxvii, p. 520) "the "most (the data) did was to indicate the general nature of the "mortality under that particular class. . . . That being so, the "obvious course was to go to some other source of information in "regard to the progression of the rate of mortality, to find some "suitable standard by which to test the mortality under that class." In his application of the method to the AMP<sup>(5)</sup> experience, Mr. King reverts to Griffith Davies' method of ratios ( $q'_x : q_x$ ) and rejects the new process which I suggested for reasons given in my paper. Had he tried that process he would have found it particularly applicable

\* The detailed results of the graduation have been left with the Assistant-Secretary of the Institute, in order that they may be inspected by anyone interested.—EDS.



to the case in hand, for the ungraduated curve of  $r_x = \log p'_x - \log p_x$  would have indicated that it could suitably be graduated in the form  $r_x = \alpha + \beta c^x$  and hence\* (as shown in my paper) that the graduated values of  $\text{colog } p_x$  or  $\mu_x$  could be represented by a change in the Makeham constants A and B—or, in other words, that the experience could be graduated by a Makeham curve having the same value of  $c$  as in the standard table. A change in A and B is equivalent to expressing the graduated values of  $\text{colog } p$  or  $\mu$  in the form  $K \times (\text{standard value}) + L$ . This method of variation by a constant and a percentage can be extended to the case in which the standard table does not follow Makeham's law, and it can be applied equally well to  $q_x$  [*cf.* Hardy's Lectures, p. 35, footnote, where a similar method is suggested for the adjustment of a graduation otherwise determined]. In this way we have two constants at our disposal, and these can be found in any convenient way (*e.g.*, moments, aggregate method or least squares). In the case of the AMP<sup>(5)</sup> experience, I find by a short and rough method—which could certainly be improved—that the AMP<sup>(5)</sup> experience can be adequately represented in the form

$$q_x^{\text{AMP}^{(5)}} = q_x^{\text{O}^{m(5)}} / 1.2 - .00135$$

This, of course, gives perfectly smooth graduated values, and the following comparison of actual and expected deaths seems to show that the graduation is satisfactory. The deviation in the abnormal group, 55–59, namely, +59, is remarkably close to Mr. King's figure, +55; and the two graduations differ principally in the way in which the obvious shortage of deaths in that group is spread over the remaining groups.

Attained Ages	At Risk	Actual Deaths	Expected Deaths	Deviation	Expected Deviation†
20–24	4,267	17	18	+ 1	± 3
25–29	24,829	109	115	+ 6	9
30–34	59,366	300	308	+ 8	14
35–39	87,564	548	532	– 16	18
40–44	93,327	722	697	– 25	21
45–49	82,731	797	798	+ 1	23
50–54	66,313	871	867	– 4	24
55–59	49,649	853	912	+ 59	24
60–64	31,568	846	835	– 11	23
65–69	16,534	659	644	– 15	20
70–74	6,779	410	391	– 19	16
75–79	1,999	165	172	+ 7	10
80–84	439	51	56	+ 5	6
85–	77	17	15	– 2	3
	525,442	6,365	6,360	± 179	...

† By Hardy's formula,  $.8 \sqrt{\text{Expected Deaths}}$ .

\* Since  $\beta c^x = c^x - t$  (when  $t$  is a constant  $= \log \beta / \log c$ ), a curve of  $c^x$  drawn on a transparent scale (say on tracing-paper) can easily be shifted over the diagram of  $r_x$ , and it can be seen by inspection whether any part of the curve raised by a constant above our base-line will sufficiently represent and graduate the values of  $r_x$ .—*Cf.* Calderon, *J.I.A.*, vol. xxxv, pp. 173–5.

It may be suggested that the system of representing deviations from the standard table by means of a constant and a percentage instead of a percentage only might very probably be usefully applied in exhibiting the results of specialized mortality investigations. If, for example, in any class of lives the extra risk is found steadily to increase or to decrease as the age advances we should evidently get a better representation of the special mortality by the use of the two constants than by using the percentage only; and the constants are very easily found approximately as the exposed to risk and the expected deaths according to the standard table will be ascertained.

As regards Mr. King's method of graduating the rates of mortality during the period of selection, I may perhaps be allowed to say that (unknown, of course, to him) I happened to use what is substantially the same method twelve years ago, when I had occasion to form select tables based on the very special experience of the office with which I was then connected. The ultimate table was graduated by Mr. King's method of osculatory interpolation with satisfactory results. The experience being too slender to enable each year of assurance to determine its own ratio, the ratios  $q_{[x]+t} : q_{x+t}$  were got out according to the  $O^{[M]}$  Table and found to be practically independent of the age, except for year 0, which was specially dealt with. The ratios for age at entry 40 were used in conjunction with the graduated ultimate table to find the expected deaths during the period of selection. The intention was to use the results to obtain adjusted values by means of variation by a constant and percentage, in the way just described; but the experience being very slender the use of the unmodified  $O^{[M]}$  ratios was found to give sufficiently good results. The principle of variation by a constant and a percentage was, however, applied successfully to year 0. It was found that while, in the  $O^{[M]}$  Table, the ratio  $q_{[x]} : q_x^{ult}$  steadily increased with the age, the ratio of  $(q_{[x]} + \cdot 001)$  to  $q_x^{ult}$  was practically constant, namely, about  $\cdot 545$ ; and that the same relation sufficiently represented the connection between  $q_{[x]}$  and  $q_x^{ult}$  in the special experience. The following table, giving the ratios deduced from the  $O^{[M]}$  Table, may be of interest in connection with the foregoing remarks:

*Table showing the value, according to the  $O^{[M]}$  Table, of the Ratio  $[\cdot 001 + q_{[x]}] : q_x$  given in the First Column and  $q_{[x]+t} : q_{x+t}$  (given in subsequent columns).*

Age at Entry	YEARS ELAPSED SINCE DATE OF ENTRY									
	0	1	2	3	4	5	6	7	8	9
20	·544	·653	·742	·789	·826	·859	·892	·923	·954	·985
25	·544	·652	·740	·788	·825	·860	·892	·924	·955	·985
30	·544	·651	·737	·786	·825	·860	·893	·925	·956	·985
35	·545	·649	·734	·784	·824	·860	·894	·926	·957	·986
40	·546	·646	·729	·781	·823	·861	·895	·928	·958	·986
45	·546	·643	·724	·778	·822	·862	·897	·929	·959	·987
50	·547	·641	·720	·775	·822	·862	·898	·931	·961	·988
55	·548	·639	·716	·773	·821	·863	·900	·933	·962	·988
60	·550	·638	·713	·771	·821	·864	·901	·934	·963	·989



In conclusion, I should like to congratulate my friend, Mr. King, on the addition of another highly valuable and important paper to the long tale of those for which the Institute is so much indebted to him.

MR. A. HENRY doubted whether sufficient attention had been given hitherto to the question of select mortality in relation to pension fund problems. Mr. King's method was a very admirable one for the purposes of life assurance experience, but it might not be entirely suitable for pension funds, although that was a case in which probably the ordinary actuary would want to use it to some considerable extent. He thought that the method might break down since the selection that occurred in the earlier years of retirement in pension funds—when retirement was usually due to ill-health—might very probably be substantially different from that occurring at ages 60 or 65—when retirement took place at the option of the individual and the question of ill-health only entered to a relatively small extent. Possibly Mr. King, if he were dealing with a case of that kind, would advocate making a separate investigation in regard to the ages below 60, or whatever the retiring age would be. He believed that would be an essential point to bear in mind in the application of the method.

From the theoretical point of view, he thought that the conversion of ratios for quinquennial ages into ratios for individual ages by Milne's method introduced an error when the exposed-to-risk was changing rapidly. The weight of the data was lost sight of, and though fairly correct results would be obtained where the exposed-to-risk was constant, an error was introduced where the exposed-to-risk was increasing, which was counterbalanced later on when the exposed-to-risk decreased. But he did not think that a theoretical consideration of this kind, when one came to the practical application of the method, was of much importance.

There was a point a little outside the scope of the paper on which he thought an opinion had never been expressed in their discussions. It was the point to which Mr. King drew attention when he referred to the trouble that he had had in dealing with the  $q$ 's at the old ages of the  $OM^{(5)}$  experience. Mr. King had found there that the theoretical  $q$ 's had been arbitrarily adjusted in order to make them agree with the  $l$ 's which had been cut down in the life table. That had been done in the new experience and it had been done in other life tables. If he might say so without offence to Mr. King, he thought Mr. King had done it himself in the Text Book table. His personal opinion was that the  $q$  was the original basis of all the tables, and he did not think it should be tampered with in any way. In his view the  $l$  was purely an auxiliary function and it seemed quite inconsistent that one should get different results for the  $q$  according as one used 100,000 or 10,000 as the radix. In connection with the commutation columns the same question arose; the values began to decrease, and one had to consider the point at which one should begin to cut down. He thought the cutting down should be done with reference to the degree of accuracy

required in the final figures. If the figures were not cut down at the 10,000's, then in passing from the 10,000's to the 9,000's an extra decimal was sometimes put in to get greater accuracy. It ought to be done not at that stage but halfway down, at the 5,000's. He mentioned these points in the hope that at some time or other the members would give a considered view on them. Perhaps the next editor of the Text Book might incorporate something about them.

He had been very gratified to hear from Mr. Kenchington that the few suggestions which he (the speaker) had thrown out in regard to approximate valuations had borne some sort of fruit. But he might add, if it did not sound egotistical to say so, that he was not surprised that Mr. Kenchington had found them of value in other ways, because, although he had made the suggestion originally for the purpose of an ordinary valuation, the principle of the method could equally well be applied to the calculation of expected deaths, expected death strain or any other function of that description.

Mr. W. P. ELDERTON said that before proceeding to the main subject of the paper he would refer to a few incidental points. The first was with regard to the calculation of the  $l$  column of the  $O^M$  Table. He was convinced that, whatever Mr. King's view had been when the tables were prepared for Part II of the Text Book, the opinion he had now come to was right, and that it was a mistake to modify the rates of mortality after they had been fixed. The second arose in connection with the table in which Mr. King compared the probability of dying within five years by the  $O^M$  unadjusted data with the probabilities according to his own graduation and Sir George Hardy's. It was rather disconcerting to find that in both graduations the probability of dying within five years was appreciably under-stated at the older ages, so that in basing term premiums for five years on that probability, something like 4 per-cent loading had to be added before one got back to the original data. The only other point was the appalling vitality, from the annuity point of view, of the nurses. Their mortality had apparently been only about one-half that expected when the annuities were granted.

The method of constructing select tables which Mr. King gave assumed in its simplest form that the rates of mortality at the date of selection, or at any subsequent time during the period of selection, were proportionate to the ultimate mortality, and, as he understood Mr. King's application of the method, Mr. King cross-graduated as little as possible. For simplicity, there was a great deal to be said for that. Mr. King suggested, however, that it might be possible to graduate the ratios age by age. He (the speaker) thought that the difficulty would be to make the graduated ratios for the select period run smoothly into the ultimate. Clearly, if one used the same ratios all the way through it was easier to make them run smoothly than if they varied from age to age. Having graduated the ratios for age 0, they might work on proportions instead of actual differences in the ratios, and thus run them into the ultimate table satisfactorily. But no method of application that occurred to him would deal with one type of selection, which he believed arose

fairly frequently. It had been referred to by Mr. Henry in the case of pension mortality, where there was in a sense negative selection, and its effect was that it ran off at different periods at different ages. He was not quite sure that if the O<sup>SM</sup> Table was examined it would not be found that there, too, selection lasted for different periods at different ages; and it was obviously difficult to effect a smooth junction between the select and ultimate parts of a table when that peculiarity in the selection occurred. Except in the case of pension funds, he was not quite sure that that peculiarity was real selection, where selection was defined as a factor introduced at the date when a contract was granted. He thought that it might be due in part to the withdrawal of healthy lives or some other extraneous cause, such as he had suggested some years ago.

His general impressions in regard to the method suggested in the paper were that it was as accurate a method as one could hope to obtain at present in many cases, and that it could clearly be extended, as Mr. King had indicated, to follow the more minute variations of selection within certain limits in those cases where it was necessary to refine. He thought, too, that the method indicated the first step that must be taken in examining selection. It assumed, as it were, a simple form, and by examining deviations from that form the student would, he thought, learn far more about selection than by attempting brilliant graduations by empirical methods.

The CHAIRMAN (Sir Joseph Burn) considered that a paper like Mr. King's was of great interest because it gave students an actual experience to consider. Mr. King had stated what had happened in a certain pension fund, and it seemed to him very useful for those unacquainted with such an experience to examine the problem for themselves. The problem was as follows: There were a large number of nurses who were going to pay a certain contribution each year, and who would be entitled, on attaining age 50, 55 or 60, to receive one of two things—an annuity or a lump sum, the latter being equivalent to all their money back, with compound interest. What was the first question which in practice would present itself to actuaries? The first point which should be considered was, he thought, what amount of selection would there be against the fund? He was not sure whether he had correctly understood Mr. Elderton's remarks on that point, but it seemed to him that there was such a selection, and it was likely to be more powerful than anything they had experienced. He was not surprised to see it was even as great as shown in the paper, because he would expect that any nurse as young as 50 or 55, who took an annuity, would be very careful to see that she had a good chance of receiving it for a good many years. What would actually happen in that pension fund was this—a large number of members would receive all their contributions back, with compound interest, and these lump sum claims at a particular age would really take the place of nearly all the deaths expected for several years after that age, had the persons in question taken pensions instead of lump sums. That seemed to be borne out in the figures which were given in the paper.

MR. GEORGE KING, in reply, said that the paper was not meant to be final. He offered it to the members simply as an instrument that would be very effective in making further investigations. The remarks of Mr. Kenchington showed that there was great promise in that direction. He might say that he himself was trying to graduate the ratios age by age, but it was very difficult. It involved a double graduation, perpendicular and horizontal; and the introduction of a varying factor or any departure from a perpendicular graduation affected the horizontal graduation. That was a difficulty, but he did not know that it could not be overcome. That remained to be seen. There were two kinds of selection as regards assured lives. There was first of all the medical selection in favour of the office, and then there was reason to think there was an adverse selection through lapses, strongest in the earlier policy-years. Therefore, although it was almost an axiom that the factors by which one multiplied the ultimate  $q$  to get the select  $q$  must increase with the duration, it was not certain that this would be the case if a long period of selection were taken, because a reduction in the lapse rate might have the reverse effect.

With regard to the Chairman's remarks as to the selection exercised by the nurses, he personally did not see why the nurses, if they took a lump sum, should exercise more selection against the Fund than a person paying a lump sum. It seemed to him that the two processes were almost identical.

With regard to Mr. Lidstone's very interesting and very useful note, he should like to give a word of explanation, because he thought in a sense Mr. Lidstone had misunderstood him, and there might be an appearance of discourtesy to Mr. Lidstone. Mr. Lidstone said that he (Mr. King) had *rejected* his new process, but he could assure Mr. Lidstone that he did not reject it at all. It was a very admirable and very useful process, but the history of the matter was as follows. When he (Mr. King) had been trying to graduate the nurses' mortality he had had the impression that something had been written about graduation by reference. He had not, however, looked up the subject, but had devised the method which he adopted; and he had found this method so easy and so accurate, that it had been unnecessary for him to make any further enquiries. As a matter of fact, the paragraph in which he referred to Griffith Davies and Mr. Lidstone, had been inserted after the paper had been written, and it had been inserted solely in order that no one might imagine that he claimed any originality for graduation by reference to a standard table. Mr. Lidstone's article had been written nearly thirty years ago, and he might, perhaps, be forgiven if he had not remembered the details of it. He ventured to think, however, that for the purpose he had in view, even in graduating the AMP Ultimate Table, his (Mr. King's) method had very distinct advantages. His paper was on the construction of Select Tables, and not on the best possible construction of Ultimate Tables. He had become very much interested in graduation by reference, and therefore he had tried it on the Australian Table, although that was not



a very small experience; and he did not think one could have a better proof of its success than was given by Mr. Lidstone—that his (Mr. Lidstone's) graduation by his own method was almost identical with his (Mr. King's). He submitted that his method was a very great deal shorter than Mr. Lidstone's. He merely got the expected deaths by the Ultimate Table and the actual deaths; and he then got in age groups the ratios between them, and distributed the ratios into factors; he found that for this purpose Milne's graphic method of graduation was a very good one, because one did not need to trouble about having equal age ranges in the groups. Then, having got the factors, he could check the accuracy of the table at once, without forming  $q_x$  and getting the final table, by simply multiplying the factors into the expected deaths by the Standard Table; and if he found that the actual deaths were accurately reproduced then the table was all right. If he found, on the other hand, that they were not properly reproduced in the different portions of the table, he could simply slightly alter the factors, without any further trouble, instead of having to deal with the final  $q_x$ 's, and in the end he got a perfectly graduated  $q_x$ , fitting closely to the original facts. Then there was a further advantage. If Tables 2 and 8 of the paper were looked at, it would be seen at once, by glancing at the factors, what proportion, at any age, the mortality, for instance, of the Australian data bore to the home mortality. He did not think that could be done by Mr. Lidstone's method without a much more elaborate process. One could read off, without any trouble at all, that, up to a given age, the Australian mortality was not more than so-much per-cent of the home mortality. It was worth noticing that really the Australian mortality was probably better than it appeared to be in the paper, because there was every reason to think that the rate of mortality throughout the world had been improving, and the Australian experience was older than the British. It ran from 1849 to 1903, against 1863 to 1893 for the  $O^{M(5)}$  Table. Probably, if one had more recent Australian experience, it would be found even better than came out in his paper.

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*The Census of 1921. Some Remarks on Tabulation. By F. A. A. MENZLER, B.Sc., F.I.A., F.S.S., of the Government Actuary's Department.*

[Submitted to the Institute, 14 February 1921.]

Scope of  
Paper.

1. IT will be recalled that the President remarked in his recent Address that a paper of modest scope on the tabulation of the coming Census would be welcome as a means of promoting a discussion at this Institute. The present paper has accordingly been written in the hope that it may

serve to elicit suggestions, for possible submission to the authorities, as to points in censal tabulation in which actuaries are specially interested. The fact that important data as to ages of children in relation to the ages of parents will be available for the first time as a result of the forthcoming Census seems to make it specially advisable that the actuarial profession should consider the subject of tabulation while there is yet time to give effect to practicable proposals. For this reason the present paper is confined to questions of tabulation. Presumably there will be opportunity later for consideration of the many interesting questions that arise upon the subjects of graduation and adjustment and with regard to the National Life Tables.

It is proposed, therefore, (i) to refer briefly to the Householder's Schedule, its limitations, and the changes that will be made in it at the 1921 Census (pars. 2-7); (ii) to give an account with some criticism (chiefly of Vol. X) of the 1911 tabulation (pars. 8-50); and (iii) to deal with the question of the new data as to orphanhood and dependency (pars. 51-54). The actuary who is not particularly interested in the descriptive detail of the subject under (ii) will find criticism and suggestions in pars. 27-40 and 45-50. A general view of the tabulation is given in the Appendix.

#### (i) THE HOUSEHOLDER'S SCHEDULE.

2. There is of course a very considerable literature dealing with alleged defects in the Census and indeed in almost every branch of official statistics, but, putting aside the questions of imperfect co-ordination and of defects that are stated to exist in the law relating to Registration, it will be found that previous criticism of the Census has been chiefly directed to what are considered to be shortcomings in the questions asked on the Householder's Schedule rather than to the mode of presentation of the censal data. In any criticism that draws attention to lack of information on any given point in the published Census volumes it must clearly be remembered that the Registrar-General can only tabulate information that is given on the Householder's Schedule, and the inability of the social reformer, or even of the actuary, to find an answer to every statistical conundrum within the massive volumes which enshrine the results for England and Wales must largely be ascribed to causes other than the "official inertia" to which Sir Gerald Ryan refers, somewhat cautiously, in his paper on "The Case for Census Reform" (*J.I.A.*, vol.

Previous  
criticism of the  
Census.



xxxvi). It will be useful, therefore, to indicate a few of the more or less inevitable limitations on the Householder's Schedule before proceeding to discuss the question of tabulation.

3. There is scarcely any limit to the number of questions for each of which a good case could doubtless be made out by the enthusiast, social, economic or actuarial. Consanguineous marriages, wages and sickness have all been suggested as suitable additional points for enquiry, but it is obvious there are certain practical limits to the questions that can reasonably be asked if any degree of accuracy in the answers is to be obtained.

4. The first limitation is imposed by public opinion which can be relied upon, mainly perhaps through its self-elected interpreters, to resent any question that savours of the inquisitorial. It is only by the gradual education of public men and others to the necessity of adequate statistics that we have got to the present position as regards questions on the Householder's Schedule. Again, it is of little use to ask questions that will invite prejudiced answers. If a man is asked what his wages are he will at once scent the tax-gatherer. If the answer to a question is felt to be in any way a reflection on the individual, important anti-social facts will be concealed and the resulting data will be unreliable. It is well-known that for reasons of this kind the statistics as to Infirmary have been quite unreliable. Presumably this explains why the Infirmary question is to be omitted in the coming Census.

There is further the limitation imposed by our system of making the Householder responsible for completion of the Schedule, and it seems clear, as the Registrar-General says in his General Report on the last Census, that we have practically reached the limit of the questions that can usefully be asked unless, as in the United States, the duty of completing the Schedule is thrown upon a more or less skilled enumerator. Considerations of space, however, prevent further discussion of this aspect of the subject, beyond the remark that, in my opinion, it is very doubtful whether an elaborate Census, with enumerators upon the American model, would produce results at all in keeping with the cost of the undertaking.

5. The answer, therefore, to those critics who complain of the failure to obtain information at the Census as to wages, for example, is that the Census is not the best means of obtaining the information required, and that other means, such as special

Limitations  
on the  
Householder's  
Schedule.

official enquiries, must be adopted if reliable data are to be obtained and if the Census is not to be overweighted. It will then be necessary to show that the value of the information when collected will be commensurate with the expense of the enquiry.

Within the legitimate limits of census-taking, however, it would be idle to deny that much has been lacking in the past, especially from the actuarial point of view. As long ago as 1893 Mr. King drew attention to certain defects in the Census statistics in regard to dependency and while it may seem a long time between then and 1921, when these defects will be remedied for the first time, it must be recognized, on the one hand, that the information desired was not of such obvious public interest in former years as it has now become, and, on the other hand, that the means of representing the actuarial point of view to the authorities were less direct than is now the case.

6. It is, moreover, evident that each additional question on the Schedule involves for its appropriate tabulation a further expenditure of public money which is by no means a matter of simple proportion. At the time when Mr. King wrote his paper on "Family Annuities", Treasury control, reflecting, of course, the prevalent feeling of Parliament, was of a very stringent character. It is interesting in this connection to refer to an account given in the *Journal of the Royal Statistical Society* (vol. lvii, pp. 379-383), of the reception of a deputation of that body in 1894 by the President of the Local Government Board, himself a past President of the Society. The deputation put forward demands for a permanent Census Office and for a quinquennial Census, and, in his reply, the President remarked that if he could make out a sufficient case to the Treasury he would press it with all the means at his command. He added later that the cost of the proposal (for a permanent staff) was estimated at £1,500 per annum. No action was taken in regard to the proposals, and it is not unreasonable to assume that a potent influence in bringing about this result was a generally-accepted tradition of public economy.

7. The following table compares the questions asked in the Householder's Schedule of 1911 with those that will be asked in 1921. The latter are briefly set out in Part I of the Second Schedule of the Order in Council which has been laid before each House of Parliament in accordance with Section 1(1) of the Census Act,

Comparison of  
questions in  
Householder's  
Schedules of  
1911 and 1921.

1920. The new procedure for directing the taking of a Census was recently described in the Presidential Address and it must suffice to say here that both Houses of Parliament have passed the necessary resolutions required by the Census Act approving the special enquiries as to orphanhood and ages and size of family.

### ENGLAND AND WALES.

#### *Comparison of Questions on Householder's Schedules of 1911 and 1921.\**

1911.	1921.
Name.	Name.
Relation to head of family.	Relation to head of family.
Sex.	Sex.
Age last birthday: age in months for children under one year of age.	Age in years and months.
Whether unmarried, married or widowed.	Whether unmarried, married, widowed or divorced.
For each married woman:— (a) Completed number of years marriage has lasted. (b) Of children born alive to present marriage:— (1) Total children born alive. (2) No. of Children still living. (3) No. of Children who have died.	For each married man, widower or widow, the number and ages of all living children and step-children under 16 years of age. In respect of persons under 15 years of age, whether both parents alive, mother dead, father dead, or both parents dead.
Particulars as to profession or occupation.	Particulars as to profession or occupation.
Whether employer, employee or working on own account.	Whether employer, employee or working on own account.
—	Whether employed or unemployed; present or last employer, and employer's business.
Whether working at home.	Place of work, giving address.
—	Whether occupied in full-time or part-time attendance at an educational institution.
Birthplace.	Birthplace.
Nationality of persons not born in United Kingdom.	Nationality of persons not born in United Kingdom.
Infirmity.	—
Number of Rooms.	Number of Rooms.
Language spoken (Wales and Monmouth only).	Language spoken (Wales and Monmouth only).

\* Subsequently to the completion of the paper, Provisional Census Regulations have been issued giving the exact form of the new Schedule.

Infirmity and  
Fertility.

It will be observed that the 1911 questions as to Infirmity and Fertility will not be asked in 1921. As to Infirmity there is a general consensus of opinion that the data obtained in the past were incomplete and unreliable. With regard to Fertility, the data obtained in 1911 and published in Vol. xiii, Pt. 1, no doubt afford interesting material to the student of social phenomena, but it must be remembered that the effects of the war on statistics of this character\* would undoubtedly tend to vitiate any conclusions that might be drawn from similar data collected at the coming Census. Actuaries will be chiefly interested in the new method of asking for age in years and months, and in the questions as to orphanhood and dependency.

## (ii) GENERAL ACCOUNT AND CRITICISM OF 1911 TABULATION FOR ENGLAND AND WALES.

General  
Description of  
1911 Tabulation.

8. The tabulated results of the 1911 Census for England and Wales were issued in 13 main volumes. In addition there were a volume for each County, a summary volume, and a volume containing the General Report of the Registrar-General. It should be explained at the outset that the general scheme appears to be to print practically the whole of the tabulation in the 13 main volumes, and to extract the appropriate tables for the purpose of the County volumes. An examination of typical County volumes (Lancashire and Kent) did not reveal any information of importance that could not be found in the main volumes. This arrangement may be contrasted with that adopted in Scotland where the practice appears to be to issue County volumes first which contain a considerable amount of detail, though not as much as in the English County volumes, and which are described collectively as Volume I. A summary of the broad features of the tabulation in the County volumes together with particulars for the whole of Scotland is given in Volumes II and III. It may seem reasonable and, as regards printing, economical to relegate details to County volumes and to give a condensed view of the facts in one or two main or summary volumes. It is, however, submitted that there is something to be said for the procedure adopted for England and Wales, for it is conceivable that for many purposes all the information of one particular kind that is tabulated for various areas should be found in one place. The Scottish system, on the other hand, certainly produces an elegant handbook enabling

\* *i.e.*, statistics derived from a question of the type of the 1911 Fertility question.

a rapid survey of social conditions easily to be made. This may have certain advantages for journalistic and similar purposes, but for many practical purposes the English system seems to be preferable. The argument that a large amount of printing and duplication is saved is not of great importance when it is realized that once the type has been set extra copies for amalgamation into County volumes can be run off very cheaply.

9. An attempt has been made in the Appendix to give a general view of the tabulation and to indicate the more important tables. There is, of course, in addition a considerable number of Summary and Comparative Tables in the various volumes. It should be noted that the *de facto* population is the basis of the tabulation and that the administrative units of the country are the basis of the classification by areas. It may be of interest to those who have not ready access to the Census volumes, and of help to Census critics who have not yet made a close examination of the volumes, if an account of the tabulation based on an actual scrutiny is attempted. Description of tabulation is necessarily a somewhat dull subject, and it is feared that the appropriate brevity has not been attained.

Vols. I-V.

10. *Volumes I-IV* contain a bare enumeration of the population according to various areas, males and females being given separately except in the case of Vol. IV which deals with Ecclesiastical Areas. Vol. V is an Index to Vols. I-IV.

Vol. I.

11. *Volume I.—Population in Administrative Areas.*—Tables 1 to 10 relate to the various Administrative Areas and show :

- (1) Area in acres ;
- (2) Number of Families or Separate Occupiers (=Number of Householders' Schedules) ;
- (3) Population, Males and Females.

The largest and, as regards areas dealt with, the most detailed table is Table 10 (368 pages) which gives the above particulars down to the Civil Parish and Municipal Ward. Certain corresponding figures for 1901 are also shown for comparative purposes. Additional columns show the numbers of, and population enumerated in, Institutions, Large Establishments, Vessels, &c., included under (2) and (3) above, the Registrar-General stating that this information has been given to enable a better estimate to be made of the average size of the family in the various areas.

Table 12 gives information as to those administrative



curiosities, the "Detached Parts of Civil Parishes." The table gives the name of the Civil Parish to which the detached part belongs, and the names of the Parishes surrounding the detached part. The tabulation of these particulars extends over 29 pages, and it is indeed surprising to find that at the 1911 Census there were still in existence no fewer than 921 detached parts of Civil Parishes.

As illustrating some of the difficulties of carrying out the tabulation, reference may be made to Table 13 (18 pages) which shows the changes of boundaries of Civil Parishes during the inter-censal period 1901-1911.

Seamen on vessels were included in the enumeration only if such vessels were in port on the night of the Census or arrived there on the following day. Such persons are dealt with in Table 21, and are "distributed among the general population of the parishes contiguous to the waters in which the vessels were lying with as much accuracy as the information obtained allows."

Table 25 shows the constitution of the so-called "Outer Ring" of London, *i.e.*, that part of the Metropolitan Police District which is outside the Administrative County of London.

The volume concludes with Table 27—"Number and Home Ports of Persons absent at Sea on Fishing Vessels at the date of the Census." It may be remarked that these fishermen, who were at sea on the night of the Census and who did not arrive in port on the following day, are not included in the returns of population. The numbers in the table are approximate and are derived from a Return prepared by the Registrar-General of Shipping and Seamen.

Vol. II.                    12. *Volume II.—Population in Registration Areas.*—Tabulation follows the general form of Table 10, Vol. I.

Table 7 shows the differences between Registration and Administrative Counties.

Table 8 gives the aggregate numbers of Marriages, Births and Deaths in each Registration County and District in the inter-censal period.

The volume concludes with a repetition, no doubt for the purpose of completeness, of the table relating to fishermen, Table 27 of Vol. I, "Number and Home Ports of Persons absent at Sea on Fishing Vessels at the date of the Census."

Vol. III.                    13. *Volume III.—Population in Parliamentary Areas.*—Tabulation follows the general form of



Table 10, Vol. I, but the details as to persons enumerated in Institutions, &c., mentioned in connection with that table are omitted, and the numbers of "Electors on Register" in 1901 and 1911 are given instead.

Table 4 gives the constitution of Parliamentary Boroughs by Civil Parishes and the differences between Parliamentary and Municipal Boroughs.

The volume concludes with the usual absent fishermen table found at end of Volumes I and II.

Vol. IV. 14. *Volume IV.—Population in Ecclesiastical Areas.*—This volume gives for each Diocese, Ecclesiastical Parish or District information similar to that recorded in Vols. I-III, but the population is not separated into males and females, nor are the 1901 figures given for comparison. The Civil Parish, or Parishes, and the Administrative County, or County Borough, in which the Ecclesiastical Area is situated, are also given.

The tabulation extends over 531 pages, and in view of the non-coincidence of the Civil and Ecclesiastical Parishes must be a troublesome piece of work. It would be interesting to know what administrative purposes are served by this volume.

The volume concludes inevitably, if somewhat irrelevantly, with Table 6—"Number and Home Ports of Persons absent at Sea on Fishing Vessels at the date of the Census."

Vol. V. 15. *Volume V.—Index to Volumes I-IV.*—This is as far as can be discovered the only formal index to the Census tabulation. It is an index to "every defined area separately mentioned in the tables of the Census Volumes," i.e., Vols. I-IV.

The Index shows for each area:—

Description.

Administrative County.

Urban or Rural District.

Number of Registration District.

Population, 1911.

Page of Population Volume, I, II, III or IV.

Overlapping  
arising out of  
complexity  
of Local  
Government.

16. It is appropriate at this stage to draw attention to the overlapping that arises from the necessity for tabulating the population in the above four different ways. The non-coincidence of the Administrative, Registration, Parliamentary and Ecclesiastical areas arises out of historical considerations and a full account of the

origin of the various areas is given in the General Report on the Census and in the Introductions to Vols. I-IV. It may be of interest, however, to remark that Parliamentary areas follow more or less the ancient or geographical county. Thus we find that the Redistribution Commissioners in 1884 decided to adopt as a basis the Petty Sessional District which was always contained within the County boundaries. Registration counties are, however, aggregations of Poor Law Unions, and Poor Law Unions centreing as they did on market towns frequently extended over two or more counties. The most important areas at the present time, *i.e.*, the Administrative, which will in all probability absorb the others in course of time, were largely the creation of the Local Government Acts of 1888 and 1894, and are based partly on the Public Health areas that had grown up since 1848. Local Government reform is not a subject that need detain us, but, disregarding the opportunity for a few pungent remarks that the subject affords, it may be stated that there is apparently one area common to the three areas that have just been described, namely, the Civil Parish, and that in the case of one County at any rate, *i.e.*, Cumberland, the Administrative, Registration and Parliamentary Counties coincide. In the later volumes tabulation is according to administrative areas throughout, though certain tabulation in Vol. VII (Ages and Conditions as to Marriage) is repeated for Registration Counties and Districts.

Examples of  
confusion  
of boundaries.

It may be of interest to quote some of the examples given on page 28 of the General Report to illustrate the existing confusion of boundaries.

The Urban Districts of Brownhills, Oakengates and Spenny-moor each extend into three Poor Law Unions.

The Poor Law Union of Stamford extends into five Administrative Counties, and comprises one Municipal Borough and four Rural Districts.

The Parliamentary Borough of the Hartlepoons comprises the County Borough of West Hartlepool, part of the Municipal Borough of Hartlepool and part of the Rural District of Hartlepool.

The Civil Parishes of Crosby Ravensworth, Hutton-le-Hole, Foolow and Bigge's Quarter have 14, 12, 8 and 7 detached parts respectively.

information collected by the enumerators, and, while a high degree of accuracy is not claimed, it is stated that the figures given may be regarded as reasonably close approximations. The classification is according to Buildings used as Dwellings, and Buildings not used as Dwellings, each of which categories is sub-divided into certain obvious classes of buildings for which further information is given as to Number Inhabited, Separate Occupiers, Population, Uninhabited and Being Built.

The above tabulation is given for :—

Administrative Counties (inclusive and exclusive of County Boroughs),

Aggregate of Urban Districts (exclusive of County Boroughs),

Urban Districts,

Rural Districts.

Tables follow showing for the above areas the average number of persons per dwelling and per family.

Vol. VII. 18. *Volume VII.—Ages and Conditions as to*

*Marriage.*—It is perhaps only when this volume is reached that actuarial interest begins to stir. In Tables 1 to 7 marital condition is given age by age for each sex for England and Wales, London and the aggregates of administrative areas to which Mr. King's Sectional Life Tables relate. For the individual Administrative Counties and their sub-divisions we must turn to Table 8 (Ages only) and Table 9 (Ages and Conditions). Considerations of space naturally arise and quinquennial groups only are given after age 20. Table 8 tabulates the smaller areas down to the borough, and urban or rural district, whereas Table 9 stops short at urban districts with more than 50,000 inhabitants, and omits rural districts altogether. Table 9 evidently repeats for certain areas the age detail, for persons aged 15 and upwards, given in Table 8. The critic will find it a possibly interesting problem to try to avoid this duplication without grouping the ages under 20 and without destroying the homogeneity of Table 8. After some consideration it will probably appear that in this case, as in other parts of the tabulation, a reasonable compromise has been made.

As an illustration of the complexity of Local Government reference may be made to Table 8a which gives age particulars, as in Table 8, for Rural Districts extending into two Administrative Counties.

A considerable number of comparative tables (Tables 18–28)

is a feature of this volume as indeed of other volumes. As they are mostly exercises in simple proportion (*e.g.*, per 100,000 persons in various areas and so on) they do not call for much comment. It may be noted, in passing, that the heading to Table 18 is inadequate. It should be clearly explained that the tabulation is on the basis of 100,000 persons in each area.

Vol. VIII.

19. *Volume VIII.—Tenements in Occupation of Private Families.*—Tables 1 and 2 in this volume are a cross-tabulation, chiefly for certain obvious aggregates of Administrative Areas, of the number of private families according to number of rooms per tenement and number of persons in private families.

Table 3 brings in a further variable, the number of children under 10 years of age, and relates to London, each Metropolitan Borough, County Borough and Urban District with over 50,000 inhabitants.

Comparative tables designed to show the degree of overcrowding follow.

Vol. IX.

20. *Volume IX.—Birthplaces of Persons enumerated.*—Two subjects arise in this volume, namely, Aliens and Internal Migration.

Aliens are faithfully dealt with in Tables 3–5 as regards Country of Birth, Ages, Marital Condition, Occupation and, for England and Wales only, according to whether they are Residents or Visitors.

21. On the subject of Internal Migration, the tables simply show the resultant movement since birth, and it seems open to doubt whether much is to be learnt from the detailed classification, in respect of certain areas, of birthplaces in relation to ages. It would seem to be of greater interest to know the movement since, say, the previous Census, but this of course could only be ascertained by a special question on the Schedule. It must, however, be said that a Committee of the Royal Statistical Society has asked for “a greater amount of age-detail with reference to the inhabitants of certain selected towns and rural districts”, so it is to be assumed that the tabulation by ages is of value to the student of social phenomena. The tabulation by ages is described by the Registrar-General as “experimental”, but no reasons are given for the choice of areas.

Vol. X.  
Parts I and 2.

22. *Volume X.—Occupations and Industries.*—As might perhaps be expected, this important

volume requires a considerable amount of study and invites a certain amount of criticism. It is, moreover, a volume of distinct actuarial interest.

Grouping of  
Occupations.

23. It must first be explained that in the portion of Vol. X, Pt. 1, relating to Occupations, one system of occupational headings is adopted throughout, occupations being grouped under

23 Orders,  
78 Sub-Orders,  
474 Headings.

Effectively, however, there are only 472 headings in view of the grouping together in one total in Order XXIII of the figures relating to the headings "Students", "Scholars" and "Others." It would seem preferable to have the figures relating to "Others" separately if not for "Students" and "Scholars", but it is stated in the General Report (p. 146) that the three headings were amalgamated on account of the incompleteness of the returns. "Others", it may be remarked, are principally wives wholly engaged in domestic duties at home. In Part 2, which consists of one table only (Table 13), and which relates to the individual County Boroughs, and the sub-divisions of the Administrative County, some compression of headings had to be made, so that we find statistics relating to a minimum condensed list of occupations followed by particulars in italics for relatively important occupations. An Appendix to Vol. X contains an index to the occupations returned on the Schedules together with the order, sub-order and heading under which each occupation was grouped.

It must be added that, in view of the fact that the classification adopted, in the portion of Vol. X which is stated to relate to "Occupations", is a compromise between a personal and an industrial classification, considerable caution is necessary in using the statistics tabulated. This point is dealt with more fully in pars. 37-40 in conjunction with Tables 28 and 29 in which the classification is on a purely industrial basis.

England and  
Wales, London,  
Lancashire and  
Yorkshire.

24. The detailed tabulation for England and Wales is, with one exception (Table 7 relating to married men), repeated for London, Lancashire and Yorkshire separately, presumably because of the relative importance of these areas, including as they did some 13 millions of the population.



**Administrative  
Areas.**

25. The statistics as to Administrative Counties (inclusive of County Boroughs) are tabulated in Table 12 which follows the form of Table 1 (England and Wales), while Table 13 gives information under condensed headings, as explained above, for

England and Wales,

Aggregate of Urban Districts,

Aggregate of Rural Districts,

and each

Administrative County,

County Borough,

Metropolitan Borough,

Urban District with more than 50,000 inhabitants,

and the

Aggregate of Rural Districts in each Administrative County.

**Information  
Tabulated.**

26. Having dealt with the areas for which tabulation is made, we may now proceed to consider the actual tabulation, which for any occupational heading deals with ages, conditions as to marriage (with or without age-distribution) and economic status, *i.e.*, whether employer, employee, or working on own account, for which totals only are given for each heading. In connection with status it may be remarked that data as to unemployed will also be available as a result of the coming Census.

The following table gives, it is hoped, a clear view of the tabulation in the more important occupation tables, while it is thought that the specimens given of the actual forms of the tabulation may assist in the understanding of what follows.



## VOLUME X.

*Table showing the contents of the more important Tables relating to Occupations.*

The particulars shown are tabulated for each occupational "heading."

A cross (×) denotes the tabulation of the information indicated in the column heading.

Area	SEX		Age Groups	CONDITIONS AS TO MARRIAGE				Economic Status	Remarks
	M.	F.		SEX		AGE			
				M.	F.	M.	F.		
England & Wales	×	×	10- ...	—	×	—	—	—	Occupational headings in alphabetical order.
England & Wales	×	×	10- ...	—	—	—	—	—	
England & Wales London (Ad. Co.) Lancs. (Ad. Co. + C.B.) ... Yorkshire (Ad. Co. + C.B.)	×	×	10-, 15- 20-, 25-, 35-, ... 75 and over.	—	—	—	—	×	—
England & Wales	×	—	-20, 20-, 21-, 25-, 35-, ... 75 and over.	Mar- ried only	—	Mar- ried only	—	—	—
England & Wales London ... Lancashire ... Yorkshire ...	—	×	10-, 15-, 20-, 25-, 35-, ... 75 and over.	—	×	—	×	×	{ Marital condition is indicated by use of different kinds of type.
Each Ad. Co. + C.B.	×	×	10- ...	—	×	—	—	—	
England & Wales Subdivisions of Ad. Co.; Co. Boroughs ...	×	×	10-, 13-, 14-, ... 19, 20-, 25-, 35-, ... 65 and over.	—	×	—	—	—	This table forms Vol. X, Pt. 2.
England & Wales London ... Lancashire ... Yorkshire ...	×	×	Individual ages 10 to 21.	—	—	—	—	—	—

NOTE.—Ad. Co. + C.B. = Administrative County inclusive of County Boroughs.

*Specimens of Tabulation.—Volume X.*

TABLE 1.

*England and Wales*

Occupations at Ages 10 years and upwards	Persons	Males	FEMALES			
			Total	Unmarried	Married	Widowed
XII.—BUILDING, AND WORKS OF CONSTRUCTION.	946,707	946,127	580	48	45	487
1. <i>House-Building, &amp;c.</i>						
1. Builders ... ..	39,998	39,858	140	9	10	121
2. Builders' Labourers ... ..	66,026	66,026	...	...	...	...
3. Carpenters, Joiners ... ..	209,051	208,995	56	2	...	54
... ..	...	...	...	...	...	...
5. Bricklayers ... ..	102,756	102,752	4	...	...	4
... ..	...	...	...	...	...	...

Table 12 omits the column "Persons", but is otherwise identical in form with Table 1.

TABLE 3.  
*England and Wales. (Males and Females on opposite pages.)*

AGES OF MALES, 10 YEARS AND UPWARDS										WORKING AT HOME (included in Columns 12-15)				
10—	15—	20—	25—	35—	45—	55—	65—	75— and up- wards	Total	Occupations of Males aged 10 years and upwards	Employers	Working for Employers	Working on Own Account	Others, or No Statement
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
5,769	65,064	92,254	247,541	204,261	172,195	110,332	42,966	5,745	946,127	XII BUILDING, AND WORKS OF CONSTRUCTION.				
135	1,290	1,633	7,599	9,442	9,808	6,858	2,677	416	39,858	1. <i>House Building, &amp;c.</i>	21,963	12,630	4,349	916
190	3,621	5,562	16,413	17,622	13,341	6,905	2,164	208	66,026	1. Builders ... ..	...	66,026	...	...
1,742	16,862	22,921	53,822	36,754	36,479	26,688	11,839	1,888	208,995	2. Builders' Labourers	9,591	176,978	16,073	6,353
...	...	...	...	...	...	...	...	...	...	3. Carpenters, Joiners	...	...	...	...
249	3,601	8,034	31,028	21,993	19,350	13,117	4,799	581	102,752	5. Bricklayers ... ..	2,615	92,312	3,957	3,868
											...	...	...	...

Tables 4, 5 and 6 (London, Lancashire and Yorkshire, respectively).

Table 7 (Married Males).

Tables 8-11 (Females only. Marital condition indicated by use of different kinds of type).

} follow the same general form as Table 3.

Need for  
uniform system  
of age-groups.

27. A casual glance at the tabulation reveals a considerable amount of duplication and the use of several different systems of age-groupings. With regard to the latter point, there are obvious advantages, both from the point of view of avoiding duplication and of comparability, in the adoption of a uniform system of age-groups throughout. It is not suggested, of course, that the information in Table 14 (Occupations of persons under 21 at individual ages) is valueless, but it seems better to aim at a compromise and get all similar information together in one table. On the other hand, while persons under 21 were treated in such detail, persons aged 25 and over were tabulated in decennial groups which unfortunately ended in the digit "5." The important ages from the point of view of retirement and old-age pensions were thus concealed. A reasonable standard system of age-groups throughout would be one starting at 14 (in view of the new Education Act) with biennial groups to 20, quinquennial groups to 75, and one group for 75 and over. This suggestion could be adopted if the tables were printed as in Vol. X, Part 2, *i.e.*, the tabulation running from left to right instead of down the page in the normal way. Under this proposal Table 14 would disappear.

Duplication.

28. With regard to duplication it will probably be found after close examination that much of it is inevitable in view of the difficulty of including in one intelligible table statistics for each occupational heading showing the number of persons in each age-group according to marital condition, with further information as to totals in the various sub-divisions of economic status. It is theoretically possible to combine Tables 1, 3, 7 and 8, using differential type for marital condition as in Table 8, but it is to be assumed that totals in each age-group would be required irrespective of marital condition. There are in addition the totals in age-groups for each "Order." In the result there would be four or perhaps five kinds of type. Clearly such a table would be unsatisfactory, the conclusion being that the relegation of marital condition to a separate table is justifiable. Nevertheless it is suggested that Tables 1 and 3 and, if necessary, the corresponding tables for London, Lancashire and Yorkshire could usefully be amalgamated by inserting three extra columns for "Unmarried", "Married" and "Widowed" between columns (11) and (12) of Table 3. (*See specimen of tabulation*). Provision will also have to be made

in the coming tabulation for columns for "Unemployed", but there would be room for all these extra columns as well as for those required in connection with the age-groups proposed in par. 27, if, as there suggested, the printing were carried out as in Vol. X, Part 2. Males and females should be on opposite pages, and there would then be room for the complete tabulation of males in relation to marital condition, whereas at present only Married Men are separately dealt with (Table 7).

29. It is a matter for consideration whether the tabulation of males should not follow that of females in regard to marital condition. If Table 7 is compared with Table 3 (all males) it will be found that the proportion of married men varies greatly in different occupations, and no doubt there are similar variations in the proportion of widowers. Moreover, there are certain tendencies in industry which may render it of importance that there should be more precise information as to marital condition of males in different occupations. Such problems as possible pension schemes in connection with particular occupational groups, *e.g.*, trade unions, and differential wage schemes such as certain of the Dominions are experimenting upon, at once suggest themselves. On all these grounds, therefore, it is suggested that a table for men on the lines of Table 8 for women should be furnished in lieu of Table 7 for married men only.

The following table may be of interest in this connection :—

Marital  
Conditions of  
Males.

## Census of England and Wales, 1911.

## Proportion of Married Men to total Men in various groups of occupations at different ages.

ORDER	Sub- Order	Head- ing	OCCUPATIONS	AGES								
				Under 20	20-21	21-25	25-35	35-45	45-55	55-65	65-75	75 and over
			ALL	.002	.03	.17	.61	.81	.82	.78	.69	.51
IX	I	I	Coal and Shale Miners—Workers at the Face ... ..	.006	.06	.34	.71	.83	.83	.77	.68	.47
IX	—	—	IN AND ABOUT, AND WORKING AND DEALING IN THE PRODUCTS OF, MINES AND QUARRIES ... ..									
VI	—	—	CONVEYANCE OF MEN, GOODS, AND MESSAGES ... ..	.003	.06	.28	.68	.81	.82	.77	.68	.48
X	—	—	METALS, MACHINES, IMPLEMENTS AND CONVEYANCES ... ..	.002	.04	.22	.65	.83	.83	.78	.69	.51
XIX	I	16	*Boot and Shoe Makers ... ..	.002	.03	.19	.65	.83	.84	.79	.68	.48
XII	—	—	BUILDING, &c....	.002	.02	.19	.63	.81	.83	.79	.70	.51
XIX	I	8	*Tailors ... ..	.001	.02	.16	.64	.83	.83	.77	.66	.49
XVIII	—	—	TEXTILE FABRICS ... ..	.001	.03	.17	.62	.83	.84	.79	.67	.46
I	2	3	Local Government Officers ... ..	—	.002	.08	.57	.86	.89	.83	.71	.63
V	2	1	Commercial Clerks ... ..	.001	.01	.07	.51	.79	.81	.78	.69	.53
VII	I	5, 6, 7	Agricultural Labourers ... ..	.001	.03	.14	.52	.71	.73	.71	.65	.51
XXII	5	1	General Labourers ... ..	.002	.03	.16	.47	.62	.64	.63	.59	.46

NOTE.—The statistics relating to complete "orders" (titles shown in capital letters) and to the "headings" indicated by a \* include Employers, but their numbers are not such as to disturb the above proportionate figures.



Below age 25 the variation in the proportion of married men is very marked, those occupations where the maximum earning power is reached at an early age naturally showing a higher proportion of married men. In the age-groups 35-45 and 45-55 a more or less constant proportion is shown in almost all classes, only General and Agricultural Labourers falling markedly below the general average for the country as a whole. A comparison of the figures given at the older ages for Coal Miners with those for Agricultural Labourers and Local Government Officers would appear to afford some support for the view that the proportion of widowers may also vary appreciably with the occupation and, if this be so, added force is given to the above suggestion for the detailed tabulation of males in relation to marital condition.

**Makers and Dealers.** 30. In connection with many industrial problems it is important to preserve a clear distinction between "makers" and "dealers." In the 1911 tabulation there was a lack of uniformity in the treatment of the tabulation of "dealers." In some Orders "dealers" are treated as a separate Sub-order (*e.g.*, Order XVIII), in others they are included under different headings (*e.g.*, Order XIX). It seems desirable that "dealers" should be brought together as a separate Sub-order to each Order, an arrangement which will be found to have an important advantage when the following remarks as to the tabulation of economic status have been considered.

**Economic Status in relation to ages.** 31. As remarked previously in par. 26, totals only are given in respect of each occupational "heading" for Employers, Working for Employers and Working on Own Account. To these categories we must now add that of "Unemployed" in view of the new Schedule. There can be no doubt that the age-distributions of these classes are quite dissimilar, and, if reference is made to par. 8 of the Hardy and Wyatt Report in connection with the National Health Insurance Bill of 1911 (Cd. 5681), it will be found that this particular information was required and that an estimate had to be made roughly by comparing the age-distribution in certain special occupations, where the majority were returned either as employers or working on own account, with the age-distribution in analogous occupations where the great bulk of those enumerated were workers. The same distribution had to be assumed for those who were employers and for those

working on their own account.\* It is to be hoped, therefore, that this important defect in the tabulation will be remedied in the forthcoming Census.

32. The question, however, arises as to what should be the occupations or groups of occupations for which this tabulation of status by ages should be made. "Orders" are clearly too broad a grouping, and on the other hand it would lead to a bulky table if the tabulation were made for "headings" as in the case of marital condition. A reasonable and practicable compromise would be to tabulate for each of the existing 78 "sub-orders", the "sub-orders" being modified as previously suggested in connection with dealers. It would evidently be of great use to have the age-distribution of dealers separately according to economic status. The information suggested should be given in a separate table rather than incorporated in the main tables by means of differential type.

33. It may be noted in connection with this part of the subject that in certain Orders, *e.g.*, Order III (Professional Occupations and their Subordinate Services) and Order V (Commercial Occupations), no particulars as to status are given at present, but this will no doubt receive attention.

The suggested tabulation of status in relation to age will

\* The following is an illustration of the method understood to have been used by Sir George Hardy and Mr. Wyatt. Vol. X, Table 3, p. 14, of the 1911 Census gives the following data, under Order VI, for

(1) Livery Stable Keepers; Coach and Cab Proprietors.

(2) Carmen, Carriers, Carters, Wagoners (not Farm).

	AGES					Employers and Working on Own Account (E)	Working for Employers (W)	Others or No Statement
	...	20-	25-	35-	...			
(1)	...	553	2,068	2,773	...	10,989	8,278	1,992
(2)	...	37,720	89,695	67,185	...	278,443	19,803	249,477
								719
								9,163

Reducing the age-distribution figures proportionately to eliminate "Others," we may write

$$(1) \dots + 517 + 1,933 + 2,592 + \dots = 10,270 = 8,278E + 1,922W$$

$$(2) \dots + 36,479 + 86,743 + 64,974 + \dots = 269,280 = 19,803E + 249,477W$$

$$(2)^1 = (2) \times \frac{1,992}{249,477} = \dots + 291 + 693 + 519 + \dots = 2,150 = 158E + 1,992W$$

$$(3) = (1) - (2)^1$$

$$\dots + 226 + 1,240 + 2,073 + \dots = 8,120 = 8,120E$$

(3) gives the age-distribution of 8,120 "Employers" and "Working on Own Account."

throw some light, if only of a momentary character, on the debateable question as to the relationship (if any) of unemployment to age.

34. In serious examination of the occupational tabulation, attention should be directed to the important notes before Table 1, in which it is stated that persons returned as "Pensioners" or "Retired" from any specified profession or business (except Officers of the Army or Navy) are included under the headings "Pensioners" or "Retired" (Order XXIII). Inmates of Lunatic Asylums or of Workhouses where returned as Insane and yet described by their former occupations have, irrespective of age, been classed as "Retired." Other Inmates of Workhouses aged 60 years and upwards whose former occupations were returned have also been classed as "Retired." The note proceeds to state, somewhat humorously perhaps, that other Inmates of Institutions and Prisoners of all ages have been classed under their stated occupations on the assumption that their ordinary work has been relinquished for a time only. These notes are obviously of importance in relation to the question of occupational mortality. The usual Supplement dealing with this subject in connection with the 1911 Census has not yet been published.

35. The former occupations of "Pensioners" and "Retired" are separately tabulated for England and Wales in Table 17 with a note at the head of the table explaining its relationship to Table 3. It seems very desirable that a similar note should be before Table 3, instead of, or at any rate as well as, before Table 1. The age-distributions and former occupations of Paupers and Prisoners are given separately in Tables 18 and 19 and here again a note before Table 3 would be useful.

36. Despite the rather discouraging references to statistics in the Report of the recent Departmental Committee on the Employment Exchanges (Cmd. 1054, p. 16), it may be hoped that as a result of the operation of the new Unemployment Insurance Act a considerable volume of new statistics relating to unemployment will presently become available, and it is earnestly to be desired that, in the collection and presentation of those statistics, regard will be had to the treatment of occupations in the Census tabulation, even if it should be necessary to recast the latter. Those of us with Army experience will recall the very complete handbook of occupations and

Classification  
of "Retired,"  
&c.

Unemployment  
statistics in  
relation to  
Census  
tabulation.

their code numbers, which was issued in connection with a certain Army Order, and in which occupations were grouped, the group and code number being endorsed on the now almost forgotten Pay Book (A.B. 64) for Men or Record of Services Book (A.B. 439) for Officers. Some of us indeed will recall a severe mental struggle as to whether demobilisation would be more rapid under Group 43 (which included Students) or under Group 41 (which included Actuaries). It does seem that a similar system could be adopted in connection with the Unemployment Book, especially as co-ordination of the returns of the Labour Exchanges with those of the 1911 Census appears to have been achieved. (*See* Vol. X, p. ix). No doubt there are difficulties in the way, as there are in the way of every piece of co-ordination, but it should be the aim of enlightened administration to overcome them rather than to add to the already long list of criticisms of official statistics from the point of view of lack of co-ordination.

37. Before concluding this lengthy review of Industries and Services. Volume X, it is important to refer to Tables 28 and 29, which give full particulars as to the allocation to various Industries and Services of the persons who are classified in the preceding Occupation tables. It is probable that these tables are the least easy to appreciate of all the tables in the Census volumes. Table 28 in particular is far from lucid and the explanatory notes that precede it are in a size of type more appropriate to a work of a devotional character. Reference is made later, however, to the general question of type. This industrial classification is, of course, of economic rather than actuarial interest, but it is of the utmost importance that the distinction between it and the classification in the occupation tables should be appreciated, especially as the notes preceding the latter do not afford sufficient information as to the basis of classification nor do they refer to the industrial classification in Tables 28 and 29. This is a point that should certainly receive attention.

38. Theoretically, the classification of occupations Basis of Classification of Occupations. may be of a personal or industrial character and for its complete treatment would require a very extensive piece of cross-tabulation. At the last Census an experiment was made "to test the feasibility of introducing a consistently personal classification of occupations, which should tabulate by the nature of the process performed by the individual worker

those sections of the list which are now classified only according to the nature of the product or the material worked on " (Vol. X, p. viii). This was found to be impracticable owing to the incomplete character of the information on the Schedules, and it is stated that under present circumstances "any logically consistent tabulation in our census of workers by personal occupation is unattainable and that the present system of classification partly by occupation and partly by product worked on must be adhered to" (pp. viii and ix). It is this system which is adopted in the Occupation tables of Vol. X. Thus it is stated in a note preceding the Occupation tables that "clerks, messengers, porters, carmen and engine drivers or stokers, who are engaged in occupations which are of very similar character in the many industries with which they may be connected, are classified according to their personal occupations; in most other cases persons are classified according to the manufacture, trade or service with which they are connected." The Registrar-General goes on to say (p. ix): "While complete tabulation by personal occupation has thus proved, for the present at least, impossible, we have been able to secure consistently industrial tabulation by classifying the persons grouped under headings relating to personal occupations by the industry or service with which they are connected, thus ascertaining the total numbers employed in various industries or groups of industries and services." This industrial classification is shown in Tables 28 and 29, of which a brief account is given in the following paragraph.

Tables 28 and 29.

39. Table 28 shows the building up of the number of persons in an Industry or Service by adding to

(I) Persons classified to the Industry or Service in the Occupation Tables

a group of

(II) Additional Workers.

A deduction is then made of those persons under (I) who are included as "Additional Workers" in other Industries or Services. Males and Females are dealt with separately. The "Additional Workers" are dealt with in three broad groups:—

Group A—relating to persons engaged in production, repair and maintenance, &c.

Group C—relating to persons engaged in domestic offices and services, catering, &c.



Group B—relating to persons engaged in occupations other than A and C, *e.g.*, commercial, conveyance, professional, &c.

At the foot of each page of the table is an analysis under occupational headings of what are presumably important items included in the totals in the upper part of the page. No clear explanation is given of this subsidiary table.

In Table 29 the classes (I) and (II) above are separately redistributed in age-groups under the appropriate occupational headings, and in addition the totals for each heading are distributed into certain broad areas which, it is stated, were adopted after consultation with the Board of Trade. Clearly the age-distribution of (I) is given in the preceding Occupation tables, and to that extent there is a considerable amount of duplication. But if the full details for each heading were not given, it is probable that the economist would complain of having to look in two places to find the constitution of an Industry, a complaint which would be just as well founded as the one implied in connection with the suggested combination of Tables 1 and 3 (par. 28). The conclusion once again is that the duplication in these particular tables is more or less unavoidable, but it appears to be a more legitimate criticism to question the necessity for printing such a degree of detail. The form of tabulation is necessarily elaborate and the printing must be a very costly matter. It may be considered sufficient to print a summary table only with a brief account of the methods employed, giving the earnest seeker after truth the right to inspect the full tabulation.

Illustration  
from Building  
Industry.

40. The importance of the industrial classification may be illustrated by certain statistics relating to the Building Industry, in which the problem of the admission of dilutees is one of current interest.

In the Occupation tables we find Order XII, BUILDING and WORKS of CONSTRUCTION, with *Sub-Orders* (1) *House Building, &c.*; (2) *Other Works of Construction and Roads*. Under these sub-orders there are 19 and 5 headings respectively, the 19 headings under sub-order 1 including such typical headings as :

1. Builders.
2. Builders' Labourers.
3. Carpenters, Joiners.
4. Carpenters', Joiners'—Labourers.



5. Bricklayers.

6. Bricklayers' Labourers.

and so on.

From the titles of ORDER XII and *Sub-Order* (1), it might at first sight be inferred that all the persons classified to the heading "Bricklayers", for example, were engaged in House Building, but we read on p. lxxxi of the Introduction to Vol. X that "The 872,963 males classified to the occupational headings of the sub-order "House Building, &c.", included 87,274 who were returned as following these occupations in connection with some other industry or service, while there were, on the other hand, 29,299 males classified to various other occupational headings but returned as being employed in connection with building trades." These 29,299 additional males included the workers given in the following table :—

ORDER	<i>Sub-Order</i>	Heading	Description	Number of Additional Male Workers
VI	2	6, 7	Carmen, Motor Van Drivers, &c.	11,544
V	2	1	Commercial Clerks ... ..	9,002
VI	5	3	Messengers, Porters, Watchmen	2,218
XIII	2	1	Sawyers, Wood Cutting Machinists	1,787
XXII	5	2	Engine Drivers, Stokers, Firemen (not railway) ... ..	1,002
X	3	5	Blacksmiths, Strikers ... ..	737

In the following table a comparison is made, for the more important trades, of the number of men working at building trades *in connection with other industries or services* with the number classified to those trades in the Occupation Tables. The figures in cols. (1) and (2) of the table were extracted from Table 29 (Vol. X, Pt. 1, p. 722). The actual numbers of persons in the various trades employed in the House Building, &c., Industry are, of course, the differences between the numbers shown in cols. (1) and (2).

*Census of England and Wales, 1911.  
House Building, &c., Industry. Males only.*

ORDER XII, Sub- Order 1. Number of Heading	Description of Occupational Heading	(1) Number classified to Industry in Occupation Tables. (ORDER XII)	Number of (1) included as Additional Workers in other Industries or Services	(2) as a percentage of (1)
		(1)	(2)	(3)
1-19	ALL	872,963	87,274	10.0
3	Carpenters, Joiners ...	208,995	30,736	14.7
4	Carpenters', Joiners'— Labourers ...	5,371	1,683	31.3
5	Bricklayers ...	102,752	12,396	12.1
6	Bricklayers' Labourers ...	69,809	5,309	7.6
7	Masons ...	47,063	4,581	9.7
8	Masons' Labourers ...	15,950	1,405	8.8
13	Painters, Decorators ...	181,613	9,617	5.3
17	Plumbers ...	64,968	8,466	13.0
18	Gasfitters ...	17,116	9,803	57.3

Taking the 17,705 bricklayers and their labourers shown in col. (2) as an example, this figure included

3,016 in connection with iron and steel manufacture,

2,996 in connection with coal mining,

2,544 in railway companies' service,

1,830 in the various services of Local Authorities,

and

788 in iron founding, boiler making and general engineering and machine making works.

It is at once seen that the statistics given in the Occupation tables are inevitably a very unsafe guide as to the number of persons in a given occupation engaged in a particular industry, however essential to, or characteristic of, that industry the occupation in question may be. If such information is required, reference must be made to Tables 28 and 29.

Vol. XI.

41. *Vol. XI.—Infirmities.*—The statistics as to the

Totally Blind, Totally Deaf, Deaf and Dumb, Lunatic, Imbecile and Feeble-minded have long been considered to be unreliable and the question upon which they are based is being dropped in the forthcoming Census. It need only be said that Ages, Conditions as to Marriage and Occupations were tabulated for England and Wales, and totals only of males and females in the case of certain administrative areas.

Vol. XII.

42. *Vol. XII.—Language spoken in Wales and Monmouthshire.*—The numbers of males and females

aged 3 and upwards returned as speaking English only, Welsh only, both English and Welsh, or other Languages are tabulated for each Administrative County, Urban District and Rural District for the following age-groups :—

3-5, 5-10, 10-15, 15-25, 25-45, 45-65, 65 and over.

Vol. XIII.  
Part 1.

43. *Vol. XIII, Pt. 1.—Fertility of Marriage.*—

This Volume is chiefly occupied by Table 1 (333 pages) which gives for each duration of marriage, and each age of husband and wife at the Census, the number of couples, the number of children born and the number of children dead. The table, of course, relates to England and Wales only. Table 9 is of definite actuarial interest as it gives relative ages of husbands and wives irrespective of duration of marriage, together with the age-distributions of widows and widowers. Table 6 affords similar statistics for marriages of “completed fertility.” There is a considerable amount of more elaborate tabulation.

44. The promised Part 2 in which the data are to be examined in relation to occupation, birthplace, number of rooms occupied and locality has not yet appeared, but Dr. Stevenson has delivered a paper before the Royal Statistical Society entitled “The Fertility of Various Social Classes in England and Wales from the middle of the Nineteenth Century to 1911” (J.R.S.S., Vol. LXXXIII. p. 401) in which the data are closely examined. It would appear, however, that the possibilities of the data from an actuarial standpoint have not yet been fully explored. Reference may also be made to a treatment of the corresponding Scottish data by the methods of multiple correlation, a description of which is given in Volume III of the Report on the Scottish Census.

Lack of  
General Index.

45. *General Remarks on the 1911 Tabulation.*—

The most obvious defect in this tremendous piece of tabulation is the lack of a general index. It is conceivable for example that it might be desired to know quickly what is tabulated about the inmates of workhouses, or, in the more precise phraseology of the Census, “Pauper Inmates of Workhouse Establishments.” Unless the enquirer is really familiar with the volumes, it would take some little time to find out that particulars are given in Volume I, Tables 16/17, Volume VII, Table 14 and Volume X, Part I, Table 18. This scattering of information is, of course, inevitable. The only indexes are Vol. V, to which reference has been made in par. 10, relating to the areas for which tabulation is made in Vols. I-IV, and Table 2,

Vol. X which is an alphabetical list of the occupational headings adopted in the tabulation. There is thus no means of finding out without a close scrutiny what is tabulated on any given subject. There is at the beginning of each volume a "Table of Contents" which sets out in numerical order the headings of the various tables without the use of differential print, but this is clearly inadequate. A further general index to the valuable remarks of the Registrar-General in the General Report and in the Introductions to the various volumes is also required, and in view of the fact that publication of the volumes is spread over several years an index in each volume to the introduction and tables does not seem a very unreasonable demand. The index to any volume might be a consolidated index for that volume and previously issued volumes.

Headings,  
printing, &c.

46. A close examination of the headings to the tables shows that they are, almost without exception, accurate and adequate, though in places perhaps a little pedantic. It is, however, submitted that the style of type adopted leaves much to be desired. There is indeed about the headings a dignity, a freedom from ostentation, accompanied by a difficulty of ready comparison, that one feels has descended to us from the earliest days of the Census. There can be no doubt that a much more judicious use of type has been made in the Scottish Volumes. This may be partly due to a less ambitious tabulation and to the policy of relegation of detail to County Volumes. Where, indeed, as in the Occupation Tables in the Scottish County Volumes, compression has been attempted by putting Males and Females on the same page, the tabulation is not particularly effective. However that may be, the Scottish tabulation is undoubtedly more attractive as a whole, and it would distinctly relieve the over-burdened table headings if, as in the Scottish Census, instead of printing "Census of England and Wales 1911" at the head of each page in each volume, the heads of alternate pages at any rate were utilized to indicate, in a word or so, the subject of the tabulation in that opening. It is perhaps not too great a strain on the imagination to leave out the heading referred to, say, on every other page.

47. One other printing defect may be mentioned, that is, the practice, to which reference has already been made, of printing the very necessary notes on the tables in almost microscopic type. Important notes should be in print of such a size and character that nobody, not even the most energetic reformer, could possibly miss them. It seems beside the point

to consider saving a few inches of space in connection with a tabulation extending over several thousand pages.

48. Further minor suggestions are that *all* volumes should be numbered, *i.e.*, including the Summary Volume and the General Report, and that some definite indication that they are so included should be given at the head of tables which are embodied in the Summary Volume. The Summary Volume is of great interest in that it brings together in one volume all the tables relating to England and Wales. It does not appear that there is the same case for including in the volume in question further tabulation in relation to administrative and other areas, but, if this is deemed to be essential, a volume relating to England and Wales only would appear to have at least as much justification as a County Volume.

Duplication. 49. There is undoubtedly a considerable amount of overlapping and duplication of statistics in the Census Volumes, arising partly from the complexity of local government areas, partly from the number of attributes in respect of which tabulation has to be made. It will be found, however, that a large proportion of this duplication is unavoidable, and that, while differences of opinion must exist as to the best way in which duplication can be minimized, the compromises made in the tabulation are in most cases eminently reasonable.

General Conclusion. 50. But when all has been said the fact remains that a close perusal of this very large piece of tabulation gives the impression of an undertaking finely carried out by people who have evidently given very close consideration as to what shall be tabulated and who have clearly taken extraordinary pains to do their work as thoroughly as possible. This is possibly not in accordance with the somewhat vague idea that is prevalent that in some respect or other, not very precisely defined, the presentation of the results of the Census is defective or perhaps that a good deal more could be tabulated on the basis of the existing Householder's Schedule. But it is submitted that the view put forward is not remarkable. When a piece of work has been done a considerable number of times extending over 120 years, and has been in charge of a series of extremely able Registrars-General, at least one of whom contributed to the development of our methods of dealing with population statistics, it is only reasonable to expect that a tradition and technique would grow up. If certain points of view, such as the actuarial, have been



overlooked, more especially, it must be insisted, in the Householder's Schedule, the omissions need not necessarily be ascribed to "official inertia", but rather to a Victorian tradition that the less the State interfered in matters the better for all concerned, with its natural consequence of a slow development of public opinion in matters that affect the actuary and the statistician.

(iii) TABULATION OF NEW DATA AS TO ORPHANHOOD AND  
DEPENDENCY.

*Orphanhood.*

51. The information that will be available for each child under 15 will be whether, Both parents alive, Father dead, Mother dead, or Both parents dead. It will, of course, be impossible to deal with orphan children from the point of view of recording particulars of complete families, as all such children in each family will not necessarily be included in the same Householder's Schedule. It appears, therefore, that the only tabulation that can usefully be made is one showing the total number of children in each of the above classes for each age from 0 to 14. Tabulation will presumably be made for England and Wales and for the administrative areas for which condition as to marriage is tabulated. The form of tabulation is obvious.

*Dependency.*

52. The new particulars in regard to dependency are also of great interest to the actuary in view of their importance in relation to pension and other cognate problems. In the treatment by the collective method of such problems as the insuring of an allowance to a widow and children, we require to know, in addition to the data that the last Census provided as to proportion of married men at each age and relative ages of husbands and wives, the average number of children of each age in relation to the age of father (married men and widowers being separately dealt with) and similar information for widows. In the case of a non-insurance scheme, the annual cost of providing a widow with an allowance for each of her children could, of course, be obtained from a mere tabulation of numbers without reference to ages, unless such elements as the duration of pension entered into the problem. For example, the widow's own allowance might cease when the youngest child reached age 16, in which case some knowledge of the ages of the children would be necessary. Again, if the pension varied according to the size of family a tabulation according to size of family would be required. A further possibility is the amendment of the law



relating to Workmen's Compensation by the substitution of pensions for lump sums in cases of fatal accidents. The Departmental Committee on Workmen's Compensation recommends in pars. 56, 57 and 58 of their Report (Cmd. 816) that when the total dependants of a deceased workman include a widow and children, the widow's compensation should be a lump sum, but that compensation to children under the age of 15 should take the form of individual weekly allowances. It is proposed that the liability in respect of children should be dealt with through a central fund to which a capital payment of £500 should be paid in respect of each death where a family is left, regardless of the number of children in the family. It is suggested that the capital payment should be revised triennially. It is clear that an analysis of the new data as to dependency in regard to the occupation of the father would be of the greatest value in connection with the working of the suggested central fund, and in the assessment of the premium required to cover the employer's liability for the capital payment.

Suggested  
tabulation.

53. The data available will be the number and ages of all children and step-children under 16, in respect of each married man, widower or widow,\* and it appears that the

\* The following extract from the Provisional Census Regulations shows the form in which the question will be asked :—

[illegible]

following tables should be prepared for England and Wales :—

(a) For the married men of each age there should be shown the total numbers having 0, 1, 2, . . . . children respectively under age 16, and for each size of “family” there should be given the total numbers of children of each age, and also of the youngest children of each age. Similar tables should be prepared for widowers and widows.

(b) The age of the wife has an important bearing on the problems mentioned above, and it is therefore suggested that, where husband and wife can be identified on the same schedule, the above particulars as to dependent children should be further tabulated in relation to the ages of husband and wife jointly. In order to reduce the number of tables it would be sufficient if this tabulation were made for quinquennial age-groups.

(c) Where husband and wife can be identified on the same Schedule, it would be of great value if the data could be tabulated in relation to the ages of the mothers, in the same way as is suggested in (a) above for married men.

The following form, with appropriate variations in the heading, indicates the tabulation suggested under (a), (b) and (c) above.

Age of

Number of Children under 16 years of age	Number of Families	NUMBER OF YOUNGEST CHILDREN AGED				NUMBER OF CHILDREN AGED				Total number of Children
		0	1	—	15	0	1	—	15	
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										
10 or over.										
Total										

(d) There is further the problem of dependency in relation to occupation. It is suggested that it is unnecessary to consider the age-distribution of the children in this connection as it is unlikely that this will vary for different occupations where a

father of given age has a family of a given size. It would therefore appear to be sufficient to prepare tables for certain occupational groups showing the number of "families" of each size according to the age of the father (in the case of married men). Probably "sub-orders", especially if modified as suggested in par. 30, would be the most suitable basis for the occupational grouping.

54. In this connection reference may be made to the subject of relative ages of husbands and wives. At the 1911 Census this information was tabulated in Table 9 of Volume XIII, Pt. 1, which dealt with Fertility. The question relating to Fertility has, as explained above, been taken out of the schedule, but it is of course very essential that the relative ages of husbands and wives by individual ages should still be tabulated where they can be identified on the same schedule. An important question arises as to these relative ages in relation to occupation, and it would be of great interest if an experimental tabulation were made for a few selected homogeneous occupational headings, in addition to the tabulation for England and Wales as a whole.

#### CONCLUSION.

55. It is not of course claimed that the foregoing remarks exhaust the subject of the tabulation of the Census statistics for England and Wales. The paper is submitted, as explained at the outset, as a basis for discussion for the purpose of eliciting actuarial opinion on the general subject, and on the special subject, which is of course of direct interest to the actuary, of the tabulation of the new data in relation to orphanhood and dependency. On the general question it should be explained perhaps that the subject was originally approached with the traditional prejudice appropriate to any examination of official statistics, and without any particular reverence for existing institutions. As will be seen, the conclusion arrived at is that the tabulation has been carried out in a thorough and painstaking manner, and that with slight exceptions the most has been made of the data provided by the Householder's Schedule. The subject, for example, of the improvement or modification of the precise form in which the questions have been put in the past has not been touched upon, the object having been to take the Schedule as it was in 1911 and to see whether the tabulation based upon it could be criticized rather than to explore the whole

Relative ages  
of husbands  
and wives.

subject of census taking. It is, of course, very desirable that the tabulation for Scotland and Ireland should be co-ordinated as far as possible with that for England and Wales. A full discussion of this point, however, would require some account of the volumes relating to Scotland and Ireland, and would involve a considerable addition to a somewhat lengthy paper. It was accordingly decided not to deal with this subject. It is hoped that a due proportion of any dullness in the mode of treatment of the subject of tabulation will be ascribed to the subject itself.

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#### ABSTRACT OF THE DISCUSSION.

MR. S. J. GUNNINGHAM said that questions relating to the Census had been considered by the Institute on two or three previous occasions—although naturally at rather long intervals since the Census occurred only every ten years—and it might not be out of place to note that three at least of the suggestions made in Sir Gerald Ryan's paper had been adopted, namely, the publication of ages in detail, the permanent provision for the Census, and (not the least important) the provision of tabulating machinery for getting out the results.

With regard to the Householder's Schedule he thought that the employment question was one of the least likely to be satisfactorily answered, because it might be regarded as being of a somewhat inquisitorial nature. The fertility question had disappeared, but in its place there was a question designed to bring out the number of children under sixteen years of age, and the number of parents living. Passing to the volumes, he believed that—speaking generally—actuaries were not interested in the first four volumes. With regard to the Parliamentary areas, the author had perhaps forgotten the effect of the new Representation of the People Act. On reference to the schedule to that Act it would be found that the Parliamentary areas for the boroughs were based in every case on wards, which were administrative areas, but the Parliamentary area for the county was based on parishes, and in several cases the Rural District Council was divided, one part going into one Parliamentary area and another part into another. That made a distinction between Registration and Parliamentary areas. Nor was any provision made in the Act for keeping abreast of alterations which might subsequently be made in administrative areas, and perhaps it would have been well if some provision had been made for that. The volumes dealing with buildings of various kinds were, perhaps, the least interesting to most people, and yet had the information been given in a somewhat fuller form in the introduction to the volume it might have served a more useful purpose. In the introduction to that volume was a table giving the number of houses, and also the number of houses building at the particular moment when the Census was taken

# NOTES ON TYPES OF

Administrative Parishes and Wards, and certain aggregates of these

Registratio

Judicia

E. & W., &c

Ad. Co. & C.E

ing.

*General view of the tabulation, showing the more important tables*

In Cols. (6)-(10) a cross (x) denotes the tabulation of the information indicated in the column heading.

General Subject	Special Classes (if any)	Type of Area	Table No.	POPULATION						Other Information	Paragraph in Paper
				No. of Families or Separate Occupiers	CONSTITUTION AS TO MARRIAGE						
					Sex	Age	Sex	Age			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	
Population	—	Administrative	1-10	—	—	—	—	—	—	Population only	11
	Persons enumerated in	Judicial	14, 15	—	—	—	—	—	—		
	Institutions ..	Administrative	16, 17	—	—	—	—	—	—	Patients, &c., and Officials separated	
	Barracks ..	"	18-20	—	—	—	—	—	—	No. of Vessels and No. of Persons	
	Yards ..	"	21	—	—	—	—	—	—		
	Barra, Sheds, &c.	"	22	—	—	—	—	—	—		
	—	Registration	—	—	—	—	—	—	—	Total Births, Deaths and Marriages in inter-censal period in each Reg. Div. and Dist. (Table S)	12
	—	Parliamentary	—	—	—	—	—	—	—		13
	—	Ecclesiastical	—	—	—	—	—	—	—	Population only	14
Area in Island ..	—	—	—	—	—	—	—	—	—	Index to areas mentioned in Vols. 1-IV	15
Area ..	—	Administrative	1-3	—	—	—	—	—	—	Population is given for each area	16
Area ..	—	Administrative	1-7, 9	—	—	—	—	—	—	Average No. of persons per dwelling and per family (Tables 4 & 5)	18
Area ..	—	Registration	12	—	—	—	—	—	—		
Area ..	—	Administrative	8	—	—	—	—	—	—		
Area ..	—	Registration	11	—	—	—	—	—	—		
Area ..	—	Administrative	10	—	—	—	—	—	—		
Area ..	—	Registration	13	—	—	—	—	—	—		
Area ..	—	E & W only	14	—	—	—	—	—	—		
Area ..	—	Occupants	15	—	—	—	—	—	—		
Area ..	—	Persons	16	—	—	—	—	—	—		
Area ..	—	—	—	—	—	—	—	—	—		
Area ..	—	Administrative	1-3	—	—	—	—	—	—	Average No. of persons per acre in the wards of different sizes (Table 7)	19
Area ..	—	Administrative	1, 2	—	—	—	—	—	—		20, 21
Area ..	—	"	3	—	—	—	—	—	—		
Area ..	—	"	4	—	—	—	—	—	—		
Area ..	—	"	5	—	—	—	—	—	—	Occupations	
Area ..	—	E & W ..	1	—	—	—	—	—	—		
Area ..	—	E & W, &c.	3-6	—	—	—	—	—	—		
Area ..	—	E & W ..	7	—	—	—	—	—	—	Economic status, i.e., Employers, Work- ing for Employers, Working on own account, (Totals only, are given by ages)	22, 26
Area ..	—	E & W, &c.	8-11	—	—	—	—	—	—		
Area ..	—	M & G, & C.B.	12	—	—	—	—	—	—		
Area ..	—	E & W, &c.	14 A, B, C, D	—	—	—	—	—	—		
Area ..	—	E & W ..	17	—	—	—	—	—	—		
Area ..	—	Parquet	18	—	—	—	—	—	—		
Area ..	—	Persons	19	—	—	—	—	—	—		
Area ..	—	E & W ..	26	—	—	—	—	—	—		
Area ..	—	"	28	—	—	—	—	—	—		
Area ..	—	Certain Areas	29	—	—	—	—	—	—		30, 31
Area ..	—	E & W only	—	—	—	—	—	—	—		
Area ..	—	Administrative	13	—	—	—	—	—	—	The Volume consists of Table 13 only	32
Area ..	—	—	—	—	—	—	—	—	—		
Area ..	—	E & W ..	1	—	—	—	—	—	—		
Area ..	—	M & G, & C.B.	2	—	—	—	—	—	—		7, 11
Area ..	—	Administrative	3, 4	—	—	—	—	—	—		
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What one would like to have would be the number of houses built during the preceding decennium. Every local authority had to pass plans, and presumably had some record, and it should not be a very difficult matter for those records to be collected. For the condemned houses it would only be necessary to apply to the large towns for the numbers. He thought if that were done, and some correction applied for any variation in distribution of ages at marriage, a fairly reasonable estimate might be made of the number of houses required. He supposed few would imagine that a million new houses were required, or possibly even 500,000. In Vol. vii, the first volume which actuaries were keenly interested in, was given the age-by-age distribution of marriages, male and female; that was done for the first time in 1911, and one or two questions arose which for want of further information it was not possible to answer. One was the distribution of the numbers living in the female table, ages 20 to 30. Two suggestions had been made to account for that—one, migration into this country from Scotland and Ireland, and the other incorrect statement of age. In 1914, the Royal Commission on the Resources of the British Dominions when investigating the question of how many females of a marriageable age were available for emigration from the country, in view of the deficiency of females in the Colonies, assumed that the abnormal distribution at ages 20 to 30 was due to migration. It seemed to him that in the present Census it would be possible to go some way towards determining whether the distribution was due to migration or understatement of age, or to both.

The question of fertility had been treated at some length recently. The Scottish authorities published for numbers of families the age of the wife and husband at marriage and the duration of the marriage, but were unable to publish the number of children born, owing to tabulation difficulties. The English authorities gave the number of married couples, the number of children born and the number of children dead, according to ages of wife and husband and duration of marriage. In considering mortality, one did not think of the number of persons alive ten years ago and the corresponding number alive at present, but rather of the rate of mortality. The corresponding function for fertility, he thought, would be the issue rate. In 1911 the duration of marriage was given, and for that year at all events it should have been possible to calculate the issue rate. For the present Census it could not be done, because the duration of marriage was not one of the questions on the schedule. Had that question been asked in addition, the issue rate would have been obtained this year and could have been compared with that of ten years ago. In connection with the duration of marriage (English figures), out of the total of about 6,000,000 marriages the duration under one year was considerably less than the duration between one or two, and the duration between ten and eleven more than the duration on either side. A similar feature occurred in the age statistics.

MR. E. H. BROWN agreed with the author's conclusion that the

tabulation had been carried out in a very thorough manner, and that, with slight exceptions, the most had been made of the data afforded by the Householder's Schedule. During the past year he had been engaged in tabulating new business by means of the Power's cards, and in dealing with occupations he had found it necessary to decide on some method of grouping that would give the best results and yet be easy to work. The 23 orders into which the population was divided in the Census return appeared to be the most suitable, as he had the advantage of the index of occupations to guide him in the classification. So thoroughly had that index been prepared that he had only about 160 queries, which was little more than 1 per-cent of the total number dealt with. Another advantage of the Census classification was the limitation of the number of the orders to less than 29, which was the largest number of groups that could easily be recorded in one column of the Power's card. As there was room on the card for six additional orders, he would venture to suggest a slight re-arrangement in the tabulation. Orders 6 and 20 dealt with about 1,400,000 persons each, and he would suggest that Order 6 should be divided into three new orders—mechanical transport on railways and on roads, horse transport and water transport. He thought it would be found that those three orders would experience light, normal and heavy mortality respectively. In a similar manner he hoped that Order 20 could be split into two new orders by separating the makers of and dealers in spirituous drinks from dealers in food and tobacco. If that could be done there would be then 26 orders, of which six would be subject to light mortality, four to heavy mortality and the remainder to normal mortality. He might explain that he had taken light mortality to be a rate not greater than 80 per-cent of the mean death rate for all occupations, and heavy mortality a rate at least 20 per-cent greater.

The new method of asking for age in years and months was to his mind a step in the right direction and should have the effect of correcting the tendency to the use of numbers ending in 0 and 5 in giving the "age last birthday", but there could be no doubt that sooner or later the question would be recast so as to ask for the year and month of birth.

The author gave the welcome information that the question as to infirmity was to be omitted in the coming Census. He thought that the Institute could claim some of the credit for that deletion, since Sir Joseph Burn in the course of his Chadwick Lecture on Vital Statistics in 1914 strongly urged that such questions should be omitted, leaving the information to be obtained from other and more reliable sources. For that reason it was to be regretted that the question as to marital condition had been enlarged by the inclusion of divorce. Probably many divorced persons would continue to describe themselves as unmarried or widowed according to their age and inclination.

MR. S. P. VIVIAN thanked the members of the Institute for having invited his colleagues and himself to come that evening, and

the President for the cordial manner in which he had welcomed them. He had read the paper with very great interest, particularly as the author had evidently made himself familiar with the large number of volumes comprised in the Census reports, and the natural suspicion with which he, like other critics, had approached them had given place to a more just appreciation that on the whole they were thoroughly and carefully prepared. As he himself had had no part in the preparation of those volumes, he might perhaps be pardoned for endorsing that opinion. On the other hand, it was not pretended for a moment that the Census reports were perfect, and the Office would value the contents of the paper and anything which was said in the discussion if it would help the Office to make the Census reports of 1921 of more use to the actuarial profession.

The paper dealt with tabulation, which was the last stage in the preparation of the Census, and it was due to the Institute to explain some of the previous operations, because he took the view of his office that he was a trustee of the important Census machinery for all those who were consumers of Census statistics, and he would like to indicate some of the difficulties that had been encountered and some of the progress that had been made. In the first place, reference had been made to legislation. An Act of Parliament had always been necessary because a Census placed certain obligations on certain members of the population to reply to certain questions, and up to recently a separate Act of Parliament was passed for every Census; it came into being a short time before the Census and concluded when the Census was finished. The Census Act of 1920 was a permanent Act providing for all future Censuses, and that was an important step forward, as it enabled successive Censuses to be regarded from a statutory point of view as links in a successive series of statistical operations and not as a series of isolated and independent operations. The next thing was that the Act of 1920 provided for a quinquennial Census, which was again, he ventured to suggest, an object that had been sought for some time. As the Act was a permanent one it followed that the Act itself could not lay down what was necessary in every Census, for the reason that it was impossible to foresee what sort of enquiries would be necessary in the future. Some elasticity was therefore necessary, and the Act provided that the particulars of the enquiries to be included in the Census schedule on any particular occasion should be such as were prescribed by an Order in Council, and the Act gave considerable powers as to the limits within which enquiries could be included by Orders in Council. There were specific items which might be included without the ordinary opportunity of Parliament to challenge the Order, and an important sweeping clause provided that enquiries of a very large range dealing with the social and civil condition of people might be included, subject to confirmation by Parliament.

With regard to the quinquennial provision, he thought the Office could not be expected to avail itself of the power of taking a Census five years from the present time unless either the

opinion of Parliament and the public became far more convinced of the importance of statistics to the extent of facing the very heavy expenditure involved, or the expense of Census taking, which was very great, was reduced. That appeared to him to be the line of concentration, and every effort would be made to secure that the machinery for Census taking was reduced by organization to a cost which would make it possible for the Office to avail itself in future of the power to take a Census every five years. There was now full legislative equipment but the Office had yet to obtain full administrative equipment. With regard to the 1921 Census, the paper dealt really with the outcome of a long series of processes which it might be convenient to explain as showing the steps which were taken. Reference was made to the question of co-ordination. The Census authorities were national—there was one for England and Wales, another for Scotland and another for Ireland, and it had been a frequent and not unnatural ground of complaint that the Census reports of those authorities were not always in as close uniformity and co-ordination as they might be. The Office had done the best it could. A joint committee was appointed consisting of one representative of each to take general supervision of the Census arrangements, and that body had been the basis of all subsequent operations. Sub-committees were formed, which examined for the United Kingdom as a whole the various suggestions falling within a certain subject matter. By way of an example of the work which the sub-committees did, he would like to refer to the difficult subject of occupation and industry, which had been touched upon that evening. He made no apology for saying that he did not think the Office was now satisfied with the results of the occupation and industry side of the last Census reports. The work was experimental and was a subject of extreme difficulty. For that reason an attempt had been made to deal with it as thoroughly as possible, and there had been collected a small body of persons, representing certain Government Departments, which was in a position to make use of very considerable resources, to set to work and make classifications on the one hand of occupations and on the other hand of industries. That work was now completed and the Census Office would follow and subsequently publish the two classifications. In addition there would be published a very comprehensive glossary of occupations which he thought could not but be of great advantage to anyone who dealt with the subject. What was even more important, the great Departments that subscribed to the work, the Board of Trade, the Home Office, &c., had, with trifling exceptions, undertaken to use the classifications in their own statistics. So that it might be taken that on those important subjects all the Government statistics issued in the future would be directed to a common system of classification which would enable them to be collated and used together. On that point he would like to refer to a matter which, perhaps incorrectly, was noted in the paper. He thought the author assumed that the enquiry as to occupation and industry was directed particularly to unemployment, which was bulking very



largely at the present time. That was not really so. The matter was included at a time when unemployment was not what it was at present. At any particular time there were certain persons unemployed, and as the industrial classification of the individual was obtained with his employer's name and business, a man who was not for the time being employed would be omitted from the industrial classification if the question was merely limited to his present employment. It was thought desirable, therefore, to extend the question to a person who was unemployed, so as to obtain particulars of his previous employer and that employer's business, at the same time earmarking the man so that if necessary he could be excluded from any industrial classification in which it was necessary to discriminate between the total man power of the industry for the time being and those who represented in any sense a margin of non-effectives. The question was not directed to eliciting particulars of unemployment, but merely to enabling the Office to complete the industrial classification with that limitation.

There were just one or two points he noted in the speeches of previous speakers to which he would like to refer. With regard to age in years and months, the matter was debated very long and earnestly as to whether an attempt should be made to get the age by reference to the year and month of birth, but it was found that instructed opinion was not quite so unanimous as to the superiority of that as compared with the other. The year and month had been included in order to get people to address themselves more particularly to what they were talking about and to be more careful than they had been in the past. The opener of the discussion had referred to the difficulty in regard to areas. The Redistribution Commission of 1918, established in connection with the Representation of the People Act, so arranged that electoral areas were co-terminous with the administrative areas at that time. The registration officer was the officer of the local authority for the Council or the Parliamentary Borough. In the period that had elapsed since then they had departed from grace. The usual process was that the administrative borough enlarged its boundary and the Parliamentary constituency stayed where it was. So that once more, notwithstanding the evidence of 1918, there was the difficulty of overlapping areas. It was only to be hoped that, in the course of re-construction in connection with local government, those things might be put straight—at any rate, it would not be for want of appreciation of the difficulty or desire on the part of the authorities.

The Office was very grateful for the help of the Institute, and was very anxious to make the Census as useful to everybody as it could be made. It would not be an epoch-making Census, except in the sense that it came at a time which was a distinct epoch after so great a war. It comprised some important new enquiries, the dependent's enquiry which the actuarial profession had put forward, and another enquiry, which in his view was very important, the place of work. Those were two big enquiries and entirely new. With the improved arrangements with regard to occupation and industry, he thought

it was possible to look with complacency upon the efforts which had been made to secure that the questions, if not the results, were as good as could be arranged.

MR. W. P. ELDERTON said that while he agreed with the author that the tabulation of the data in the published volumes was on the whole very satisfactory, he hoped arrangements would be made whereby that tabulation was not altered from Census to Census, as such changes made comparisons between successive Census periods very difficult. With regard to the omission of the fertility question and the inclusion of a new question, he imagined that the reason for that was that for some financial purpose the new question was essential and that there was not room for both. At any rate, he hoped that was the reason, because it implied something temporary which could be changed at another Census. He admitted that the old question required modification, but felt sure that some question of the kind ought to be in each Census. The excuse indicated by the author for the omission of the question was: "It must be remembered that the effects of the war on statistics of this character would undoubtedly tend to vitiate any conclusions that might be drawn from similar data collected at the coming Census." He should have thought that the effect of the war was one of the reasons why the fertility question should be in. After all, fertility lay at the basis of almost every social statistical study. He might suggest to the author that perhaps a slight improvement of the wording might be made, as follows: "It must be remembered that the effects of the war on statistics of this character might tend to vitiate conclusions that might be drawn by careless statisticians."

MR. H. H. WOLFENDEN expressed his appreciation of the courtesy of the Institute in extending to him, as a Fellow of the Actuarial Society, an opportunity of taking part in the discussion of Mr. Menzler's paper. The general subject of population statistics was of great interest to him, in view of a paper on the work of the United States Census Bureau, which he read before the Actuarial Society of America a few years ago, and which appeared in vol. xviii. of the *Transactions* of that Society. The conditions under which the Census was taken in both the United States and Canada, however, were of course very different from those of this country, and in any comparison of British and American methods it was most important to bear those differences in mind. For example, the author, in discussing the Householder's Schedule, as used in Great Britain, expressed an apparently general disapproval of a Census taken "by enumerators upon the American model." The less expensive householder method was, of course, to be preferred where, as in Great Britain and similar countries, it could be so operated that serious omissions of information did not occur; but in Canada and the United States the enumerator system had been retained after very careful consideration and experiment, because of the scattered nature of some portions of the population and the large numbers of foreign and coloured stock. As a further illustration of the difference in conditions, it might be of interest to state, in connection with



the author's reference to the subject of a quinquennial Census, that in Canada, by reason of the waves of immigration and the consequent violent fluctuations in the population movement, it had been found necessary to take a quinquennial Census, since 1901, in the three Prairie Provinces of Manitoba, Saskatchewan and Alberta. He had had reason to appreciate the value of those quinquennial figures in connection with an investigation upon which he had recently been engaged, for the Dominion Statistician, upon the subject of post-censal population estimates; and that work had also emphasized the desirability of extending the quinquennial system to British Columbia, where again the population growth was very difficult to estimate over a ten-year period.

It was not at all surprising—although, at the same time, it was reassuring—to find that, after a detailed examination, the author reached the conclusion that the tabulations in the past had been carried out in a complete and generally satisfactory manner. Many of the changes which the author suggested had to do with matters—such as style of type, &c.—which were relatively unimportant, and he thought that the value and interest of the paper would have been enhanced had the author devoted less space to those matters and had he given instead some consideration to the closely allied problem of the annual reports on vital statistics. The complete utilization of Census results depended so largely upon their ultimate employment in conjunction with properly collected annual returns of births and marriages that it seemed to narrow one's outlook when those two closely related sections of population data were not considered together.

In Canada, Mr. Coats, the Dominion Statistician, as a result of a vital statistics conference which was held in Ottawa in June, 1918, had inaugurated a scheme which promised to remedy, as far as Canada was concerned, the present reproach that neither in Canada nor in the United States was it possible to produce vital statistics for the entire country. In the work which was still being carried out to that end, the methods of the British Registrars-General had frequently been held up as a model, and for that additional reason it would be of great interest if the Institute at some future time would give its attention, along the lines of the paper, to the possible betterment, from the actuary's point of view, of the annual returns of births, deaths and marriages.

The PRESIDENT said that one of the difficulties which had beset Census takers in the past, and he was afraid would beset them in the future, was the constitution of the United Kingdom, dating back, as it did, over 1,000 years of chequered and active history. He was interested some time ago, in reading the Reports of a very old Committee on Official Statistics, to come across a phrase by an eminent civil servant of that day which seemed to him to go to the root of the matter. The Report said: "In discussing certain defects in official statistics, Mr. Farrer pointed out that it was important to recognize the causes, among which he included "differences in the laws, customs and circumstances under which

“ ‘the different statistics are collected by different departments  
“ ‘and in different parts of the kingdom.’ He went on to say, after  
“ ‘mentioning certain examples: ‘These differences and many  
“ ‘others lie deep in the laws and customs of the three Kingdoms,  
“ ‘and it would be sanguine to hope that they could be removed  
“ ‘in order to render statistical records uniform. Human life and  
“ ‘habits can seldom be altered in order to make records perfect.’ ”  
That was the difficulty and a difficulty which actuaries met in their daily work, quite apart from the discoveries which they came across every now and again in studying the Census volumes and other forms of official statistics. If the Registrar-General gave the kind of Census that the author had outlined that evening, in the course of the next two or three years, he thought the actuarial profession would have every cause to be grateful to him.

Mr. MENZLER, in reply, doubted whether enumerators who had to range over a large number of subjects, as on the Continent and in America, could give a very large amount of accurate information. They were not a permanent staff and necessarily received only a brief training. With regard to the omission of the question as to the duration of marriages he thought that the right way of dealing with the question of fertility was by means of issue rates. It appeared from what Mr. Vivian had said that the question of co-ordination had been grappled with successfully, and that the hope expressed in the paper that the unemployment statistics would fit in with the Census statistics would be fulfilled.

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### *National Health Insurance.*

IN an interim report on the first valuation of Approved Societies as at 31 December 1918 (Cmd. 1130. Price 1d.) the summarized results were published early in the year of the valuations completed up to 24 December 1920. Over 5,000 valuations, being about 40 per-cent of the total number, had been made at that date, and in the large majority of cases substantial surpluses were found to have existed at the date of valuation. The report comments on this satisfactory result as follows :

The outstanding feature of the valuations so far completed is the large amount of the surpluses brought out, the deficiencies being relatively insignificant. The conditions which have prevailed during the war have greatly affected the finance of National Health Insurance, and it is evident that to a material extent the surpluses now declared are due to this cause. To indicate this it is only necessary to explain the more significant of the factors which have entered, generally, into the production of surplus. These are as follows :

- (a) The claims for sickness, disablement and maternity benefits have been considerably below the provision made for the expenditure under these heads in the financial basis of the Acts.

As regards the sickness and maternity benefits much of the resulting gain is traceable to the fall in the claims which set in with the year 1915 and continued until 1918. No previous experience of friendly societies affords a parallel to this phenomenon, which is unmistakably connected with the war.

It is important to remember in this connection that the super-normal experience of men who were invalided from the Forces has been met by an annual grant from the Exchequer.\* The relatively heavy liability of married women, which at one time attracted attention, has also been corrected by an annual grant from public funds. In respect, therefore, of both men and women, the societies have been protected from elements of liability that were outside the scope of the original estimates, and accordingly reap the full advantage of the favourable experience which has prevailed.

In the case of disablement benefit the claims have steadily risen, as they must do for some years to come, since disablement benefit represents the provision for more or less permanent incapacity among a community all of whose members were in employment on first entering into insurance, in or after the year 1912. The increase in the cost of disablement benefit has, however, been considerably less than that anticipated, though it should be added that this feature is much less noticeable in the case of women than it is in the case of men.

- (b) The interest realized on the investments of the accumulated funds has been appreciably in excess of the valuation rate of 3 per-cent. Interest at this rate is credited in respect of reserve values, which at present form the major part of the assets of approved societies, but the great bulk of the funds accumulated since 1912 has been invested in Government securities producing, especially since the war, a much higher rate. Approved societies are exempt from income tax on interest, and therefore obtain the full benefit of the high rates of interest at which they have been enabled to invest.

The element of depreciation had not up to the valuation date become a serious factor, and the valuation regulations provided that Stock Exchange securities held by societies should be taken at their

\* This grant continues up to 31 December 1920, and then ceases.

cost prices. So far as the funds of societies consist, under statutory directions, of credits in the Commissioners' Investment Account (comprising approximately 50 per-cent of the total amount accumulated) the question of depreciation may be presumed, in any case, not to arise, since by appropriate regulations, prescribing the rate of interest allowed, provision may be made for the creation of sufficient reserves to protect the fund. The rate of interest at present allowed in respect of these credits is 4 per-cent, leaving, it is understood, a suitable margin for the purpose here indicated.

(c) The receipts from contributions in many cases have exceeded the "expectation." Contributions are not payable in periods of sickness or unemployment,\* and it follows that in years when sickness is below the average and employment abundant the receipts from contributions show a considerable gain to societies. These conditions prevailed generally up to 31 December 1918.

(d) "War mortality", in the case of men, has added greatly to surplus. The liabilities of societies have been reduced by deaths of which there had been no expectation, while the credits given, and the funds accumulated, to meet those liabilities remain intact.

While, as indicated above, much of the gain derived from light sickness and disablement claims is due to the war, some part of it, in the case of many societies, must be ascribed to the favourable environment of the members and the nature of their occupations. So far as men who, normally, would have been subject to low rates of sickness have been withdrawn from civil life the benefit of these conditions to their societies has been suspended during the period of their naval or military service, but otherwise the conditions indicated have operated with their customary force. Rural societies, and societies whose members are confined to certain special classes, *e.g.*, the staffs of banks and insurance offices, journalists, clerks, teachers and nurses, are noticeable in this respect.

Another element has contributed materially to the valuation results, namely, the quality of administration. It is impossible, in the nature of the case, to trace specifically the results of good administration: where various factors are in operation the influence of careful management, as one of them, can be inferred, but cannot be measured. Indirectly the importance of this element can, however, be established. In a considerable number of cases the Insurance Commissioners

\* This general provision is subject to the qualification that in cases of considerable unemployment the benefits are reduced unless certain modified payments are made in lieu of contributions.

were led, especially before the war, to infer from excessive expenditure on benefits that administration was defective, and to take appropriate action. The results of their intervention are generally apparent in the subsequent course of the claims, and in some instances are remarkable. It would appear that in nearly all cases wherein deficiency has been found on the valuation of societies action of this kind had become necessary, and there is little doubt that the deficiencies are due, at least in part, to the heavy expenditure which caused the official intervention. In other cases of the kind the valuation shows a comparatively small surplus, and it is evident that some of the societies whose surpluses are insufficient to permit of additional benefits are of this class. From these results of lax administration it is a reasonable inference that careful and efficient management, where this has been consistently exercised, has contributed its due share to the surplus disclosed.

As the outcome of the valuations made up to 24 December 1920, 85 per-cent of the insured members of the societies concerned will be able to receive "additional benefits." In most cases these additional benefits will presumably take the form of additions to the normal benefits, and each society which has a sufficient disposable surplus has been informed, in an appendix to the valuer's report, what additional benefits can be given in the proportion of 1*s.* a week sickness benefit, 6*d.* a week disablement benefit, and 2*s.* maternity benefit. No society will be allowed to give more than an additional 5*s.* sickness benefit with the other benefits in proportion. In those cases in which the surplus is more than sufficient for these maximum additional benefits the excess is to be kept in reserve at any rate until after a second valuation. The following extract from the report deals with the general question of the disposal of surplus :

#### *Disposable Surpluses.*

Under Section 37 of the Act of 1911 (as amended by the Act of 1918) a society or branch is empowered to submit a scheme for utilizing in additional benefits a surplus which the valuer certifies to be disposable. The responsibility is thus imposed upon the valuer, in the exercise of his professional judgment, of determining the extent to which any particular surplus which he finds on valuation may be safely distributed.

In coming to a decision he must take into account not only the normal risks of fluctuation in future experience (paying due regard, in this connection, to the provisions of the Act respecting the "Contingencies Funds" and other protective measures) but



must give full weight to the uncertainties of the present situation with their possible reactions upon the future financial responsibilities of approved societies. This latter is a point of great importance not only in regard to the future claims, but also in reference to the value of investments. Difficult questions arise, moreover, on the general decline in the death rate, a feature which, if maintained, may hereafter add considerably to the charges on societies, as at present estimated. It would be detrimental to the welfare of approved societies if surpluses acquired under exceptionally favourable conditions were distributed without sufficient regard to such important points as these.

The Acts give no direction as to the period over which a scheme of additional benefits shall operate, but the National Insurance (Joint Committee) Regulations, 1912, provide that in approving schemes the Joint Committee shall be concerned with their actuarial soundness, and to meet this condition, it is necessary that such a term should be set to the operation of each scheme providing money benefits as will enable the liabilities under it to be actuarially measured. Having regard to the points mentioned in the last paragraph it is at least advisable that the operation of the initial schemes should be limited to a period of five years, since at the end of this period the whole situation will be reviewed, in each case, in connection with the second valuation. It has accordingly been recommended that the limitation of the scheme to five years should be secured, in every case, by the inclusion of a provision to that effect among the conditions to be prescribed by the scheme under Section 37 (2) of the Act of 1911. It follows that in each case the amount certified as disposable surplus is the sum which, in the opinion of the valuer, may be safely distributed over the next five years.

In the calculation of the reserves to be made in the process of obtaining the disposable surplus the valuers have, as far as possible, adopted a uniform procedure under which these reserves, while dependent on the ages of the members and other factors special to each case, do not differ widely (save in exceptional cases) from a general average of about 11s. a member (men) and 9s. (women). In a comparatively few cases the surplus is smaller than the reserve so ascertained, and the valuer has been unable to certify that any part of it is disposable. In the great majority of cases the balance available for distribution is considerable, the total amount certified as disposable being £2,171,576 or on the average 62 per-cent of the related surpluses.

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*Valuation of Redeemable Securities with allowance for Tax.*

ON pp. 70-71 of vol. lii, it is stated that the value to yield the net yearly rate  $i$ , with allowance for tax at rate  $t$  per unit throughout, of a loan bearing interest at the gross rate  $g$  payable yearly, and either repayable by an annuity or redeemable by yearly drawings at par, may be found by the formula  $(1+t/\lambda)a_n^g/a_n^g - t/\lambda$ , where  $n$  is the remaining term and  $\lambda = i/g - 1$ .

The formula, which was worked out for  $i > g$  (p. 70) holds also for  $i < g$ , but when  $i = g$  it becomes indeterminate. In this special case it will be seen, on reference to the expression from which the formula was obtained, that the value is  $1 + nvt/s_n - t$ . It can be easily shown that this is also the limiting value of the general formula when  $\lambda = 0$ .

## REVIEWS.

*Elements of Statistics.* By ARTHUR L. BOWLEY, Sc.D., F.S.S.  
Fourth Edition.

[Pp. xi + 459. London: P. S. King & Son, Ltd. 1920. Price, Vol. I, 16s. net: Vol. II, 12s. 6d. net; or in one vol. 24s. net.]

*Lezioni di Statistica Metodologica.* By Prof. F. INSOLERA.

[Pp. 191. Turin: Fratelli Treves. 1921.]

*Zur Ausgleichung von zweifach abgestuften statistischen Reihen.* By Hofrat Prof. E. BLASCHKE, PHIL.D. (Vienna).

[Pp. 26. Extracted from *Zeitschrift für die gesamte Versicherungs-Wissenschaft*.]

THE list given above contains examples of different styles of statistical work and it is even more interesting to consider them in their general relationship than to deal with them as separate entities. The list contains an English text book, a course of lectures in Italian which may, not unfairly, be taken as an example of the point of view of continental text books, and a pamphlet which deals with a theoretical problem by a well-known continental writer. I think if I had to distinguish between the continental and the English points of view I should be tempted to say that the continental writers care more for the mathematics and the English more for the arithmetic. Possibly a personal bias in favour of the arithmetical side may influence this distinction, which sounds strange when we remember that the theory of correlation is due to an Englishman, that much of the mathematical work on which the science depends

was produced by Englishmen and that many English writers (Pearson, Edgeworth, Sheppard and others) are obviously interested keenly in the mathematical development of the subject. Nevertheless, the general distinction is not, I think, far out. The English writers, with rare exceptions, have been looking for something to help them to express the statistics with which they are dealing and their mathematics is a means to that end. Galton's idea of a correlation coefficient—the most recent evidence seems to show that the idea originated with Galton rather than Bravais—arose because he felt that it was necessary to be able to measure relationships on a scale just as we have a scale for measuring probabilities. The large amount of mathematical work that has been done subsequently on that subject by English writers has been necessary because practical statistical work showed definite limitations in the simple coefficient first suggested. The English writers, as a whole, give more numerical examples and, even when they content themselves with few, an arithmetical paper by the same or another investigator follows quickly and indicates a definite numerical study which demanded the preliminary mathematical work. The whole of the first part of Professor Bowley's text book is, in a sense, arithmetical; he shows how the figures can be used to enable us to reach a simple arithmetical result whether it be an average, a mode, or index. The second part, which has been so much altered from the earlier edition that it is almost a new work, deals with the mathematical side but it is frequently arithmetical in intention. On the other hand, Dr. Insolera seems to think algebraically; he expresses his views either in algebra or geometrically, and his arithmetic seems to be used rather to exemplify his algebra than to show the reason for which the algebra was created. Dr. Insolera does not go into very much detail; he has avoided the more difficult parts of the subject and has, we think, been successful within the limits he set himself: he no doubt felt that he was lecturing on general lines and that his lectures could not contain the total amount that he would, presumably, expect a student of statistics to read, but he gives enough to show his point of view and enable us to see the lines which it is his inclination to follow.

Whether, however, there is a real difference in view between English and continental writers may be open to some question, but there can, at any rate, be little doubt that the arithmetical point of view is that which would be most likely to appeal to the actuary. He is, in matters of graduation, and therefore on curve-fitting, an arithmetician and an utilitarian and his test of a mathematical formula for graduation would be to see if it described the particular set of facts he was examining and, if it did, to see if it had a wide range of utility. The derivation of the formula might be empirical; but, if it answered the purpose, the actuary would use it in preference to an alternative which is less useful even if it has some apparent theoretical justification. This may seem an extreme attitude but it has much to commend it: width of range is as good a test of properly applied theory as the development of

mathematical formulæ on chosen lines. Statistics are a matter of practical arithmetic and if a particular theory does not fit statistics with sufficient generality then there is something imperfect in the theory. The attractive theoretical treatment which led to the normal curve of error has provided the temptation of overlooking its narrow applicability: it is reached by approximations which show that it is a special case of some law inviting wider generalization. This has been widely appreciated and Charlier, Bruns, Edgeworth and others have worked out series which are generalizations. Pearson, starting from the hypergeometrical series—of which the binomial is a special case—had worked out a different generalization which has what the actuary would call better graduating power than the other generalizations.

When we turn from the attempts to express frequency by a formula to the more complicated problems where there are two variables, and consider correlation, we find that the analogous surface to the normal curve of error involves the Galtonian coefficient. This coefficient is also a measure when the surface does not describe the facts provided the means of the arrays are on a straight line, but when this is not the case another measure must be used, and Pearson has suggested the correlation ratio. It is interesting to remark that a surface formed from the Bruns-Charlier generalization leads to this function as the measure of correlation, but it would seem that the correlation ratio is more general and would apply to other surfaces as well. In Dr. Blaschke's paper we have an attempt to fit a Bruns-Charlier surface to a correlation table. The paper gives the normal form and that of the Charlier surface; then provides formulæ for the moments of the arrays and shows how these moments can be used to find the constants; an example gives the actual frequency surface. Dr. Blaschke's method is one of steps. He gets a symmetrical surface and the differences from it and then goes to his final table which is, as it were, the graduation of the correlation table from which he starts. Dr. Blaschke points out that the difficulty of the Bruns-Charlier series is that one does not know where to stop: why should we stop at the fourth differential coefficient? This must, I think, always be a difficulty in using a series such as Charlier's for graduation: if we take a sufficient number of terms we can reproduce the data closely, but perhaps so closely that we do not graduate; if we take few terms we may not be close enough to the facts. It comes almost to a question of trial and we must go on till we feel satisfied. There is another difficulty in that the Charlier series may give negative values for frequency when it is used to describe very skew distributions. Dr. Blaschke, in his example, deals with a distribution which is not extremely skew, and a formula for a skew correlation surface will, I think, have its greatest value in cases of extreme skewness. The example is interesting but it must be confessed that it leaves us with the feeling that we have not yet found a skew correlation surface of practical value. Possibly we might reach one in a particular case of extreme divergence from normality by

assuming that both functions are geometrical progressions, but "that is another story."

We may return for a moment to Professor Bowley's book to point out that the actuarial reader will find much of help and interest in it; he may feel that the formula which is called the Generalized Law of Error, second approximation (page 298), is hardly suited to graduation work as it can only be relied on for graduating a few groups in many cases, and he may feel that some of the chapters on correlation are more of the nature of summaries of existing work than chapters of a self-contained text book, but he will, at any rate, be impressed by the comprehensiveness of the summary and helped by many suggestive remarks and practical hints throughout the work.

The productions mentioned at the head of this review may, as has been indicated, be taken as examples of what is going on in statistical work in somewhat different fields: they imply an extending interest in the treatment of statistical data and it would be of advantage if actuaries became more proficient in such work than they are at present; actuarial work rests on a statistical basis; actuarial science and statistics are youthful sciences and should be able to help one another.

W. P. E.

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*Pension Endowment, Life Assurance, and other schemes for Employees of Commercial Companies.* By HAROLD DOUGHARTY, A.I.A., F.S.S., F.C.I.S.

[Pp. 95. London: Sir Isaac Pitman & Sons, Ltd. 1920. Price 6s. net.]

IN the minds of readers of the *Journal* the title of this book may evoke more or less painful recollections of voluminous papers, of complicated symbols bristling with prefixes and suffixes, of life and service tables and salary scales, and—if it has been their lot to deal with the subject practically—of long and anxious wrestling with the problems and responsibilities entailed by the formation or valuation of a Pension Fund on a salary-percentage basis. But the book itself has little to do with these recondite matters. It does indeed just mention—in the short chapter on "Types of Pension Fund"—the salary-percentage method, but only to point out the uncertainty of the special factors which enter into it and the serious effect on salary-percentage funds of the recent great increase in the scale of salaries. It advocates, and is in the main concerned with, the deferred-annuity method of providing staff pensions, *i.e.*, the method of securing a deferred annuity (whether from an insurance company or by means of a Fund) in respect of *present* salary and supplementing it from time to time by additions in respect of future increments; it deals also, incidentally and briefly, with the analogous methods of providing such extra benefits as temporary assurance and disablement allowances, and it includes two articles on "Co-partnership and Pension Schemes", contributed by Mr. A. B. Culley to the official organ of the Labour Co-partnership Association.



It may seem a simple matter to provide for pensions by deferred annuities, and in a sense it is so. But every actuary or insurance official who has had occasion to quote for a pension scheme on a deferred-annuity basis knows that the office deferred-annuity table is a long way from being a pension scheme. There is a distinction between the two things which makes all the difference between a barren quotation and one which is really helpful and may be productive of business. Various special points arise, and then there is the scheme itself to be put into shape. In these matters Mr. Dougharty's practical notes should prove useful. The reader will find in the book suggestions for dealing with the problem of the older members of the staff, some indication of the cost of including death and disablement benefits, a specimen set of rules, several model schemes and a number of illustrative tables. A few pages are given to the subject of group insurance, but we miss any reference to Sir W. Schooling's plan of combining a temporary decreasing life assurance with an endowment secured through a staff savings association (*J.I.A.*, vol. li. p. 331).

On the subject of the relative merits of salary-percentage schemes and deferred-annuity schemes one word may be said. The fact that a general increase in the scale of salaries may create a deficiency in a salary-percentage Fund does not seem to us necessarily conclusive. It might be argued that the deferred-annuity scheme merely shifts the onus on to the employee. It is no doubt unsatisfactory to find a deficiency in the Pension Fund, but it is at least as unsatisfactory to the employee to see his prospective pension becoming a dwindling percentage of his salary—as it must do if he has reached an age at which it is impracticable to supplement his deferred annuity in proportion to his increased salary. The problem after all is to provide for some continuity, as regards income, between official life and life after retirement, and the salary-percentage scheme makes an attempt to do this, even if it breaks down in the process. If the employer is prepared in any case to shoulder the burden, it is merely a question of making up the deficiency in the Pension Fund in the one case or of supplementing the inadequate deferred annuity in the other.

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*Supplement to the 75th Annual Report of the Registrar-General of Births, Deaths and Marriages in England and Wales. Part II. Abridged Life Tables.*

[Pp. xlvii + 65. London: H.M. Stationery Office. 1920. Cd. 1010.  
Price 1s. 6d. net.]

THIS second part of the Supplement to the 1911 Census Report would in the ordinary course have been available five or six years ago, but its publication was held over during the War. It contains abridged (male and female) life tables, based on the rates of mortality experienced in 1911-1912, for certain large areas (*e.g.*, the North, Northern Boroughs, other Northern Urban Districts, Northern

Rural Districts), for the separate administrative counties (also for the Urban and Rural Districts of the more populous Counties) and for County Boroughs with populations exceeding 100,000—some 280 abridged tables in all—together with a report of 30 pages by Dr. E. C. Snow, describing an “elementary rapid method” used in constructing the tables and designed to enable Medical Officers of Health, with the aid of a further 10 pages of auxiliary tabular matter, to construct more tables of the same kind for themselves. The functions tabulated are  $q$  at age 0,  $r$  (the mean annual death rate) for ages 1-2, 2-5, 5-10, 10-15, 15-20, 20-25, and thereafter for decennial periods,  $l-l_{+n}$  for 0-1 and the same periods as  $r$ ,  $l$  and  $e$  for ages 0, 1, 2, 5, 10, 15, 20, 25, 35, &c., also the ratios of the  $l$ 's and  $e$ 's to the same functions for the whole of England and Wales. Dr. Snow distinguishes  $r$  from the  $m$  of the life table—elsewhere he observes that the  $m$  for age 1-2 differs from the crude death rate  $r$  only by the process of graduation—but it seems to us to be a perfectly good  $m$  (regard being had to the length of the interval) and little or none the worse for not being graduated. The abridged life tables are not graduated, but they are none the less life tables, and the  $r$ 's (on which they are based) are their  $m$ 's. One other point of detail which may be mentioned is that the  $l$ 's and  $e$ 's at the several ages are numbered in order of magnitude—or merit. It is to be feared that this useful feature will hardly escape the notice of the house agents and publicity experts of certain suburban boroughs that stand high in the list.

We have never been able to understand why Medical Officers of Health have, or are supposed to have, such a strange craving for abridged life tables in general and expectations of life in particular—still less why they should wish to construct such tables in the scanty leisure afforded by their multifarious duties. *Que font-ils donc dans cette galère?* Owing partly perhaps to the way in which the actuarial profession has always made a fetish of the life table an entirely erroneous notion seems to us to have become prevalent as to its uses. Practically all the functions required for actuarial and statistical work can be calculated directly from the experience  $p$ ,  $q$  or  $m$  without the intervention of the life table, and in investigations based on a special experience—when a life table is not required for general purposes—this is as a rule the most convenient method of procedure. As evidence of the need for life tables Dr. Snow adduces a case in which it was desired to compare the chances of surviving from 15-20 to 25-35 in two suburban districts—the population being mainly of low social status in one and of a much higher status in the other—and he remarks that this could have been done if the appropriate life tables had existed. But since the ratio of the cologs of the  $p$ 's is approximately the same as that of the corresponding  $m$ 's or  $r$ 's, it could have been done just as well without life tables. In the case of an important experience which will form the basis of numerous calculations a life table is undoubtedly a valuable instrument for actuarial and statistical purposes, but it has no obvious bearing on the facts of mortality or



vitality with which a Medical Officer of Health is concerned. In regard to these facts it adds nothing to the evidence of the rates of mortality on which it is based. The tabulated values, except those of the fundamental  $p$  or  $m$ , are wholly artificial and hypothetical, representing not facts but a perspective of human life on certain assumptions. What meaning can be attached to the expectation of life at West Ham, according to these abridged life tables, of boys aged, say, 10? It means, we suppose, that if a large number of boys of similar birth and antecedents to the boys aged 10 who happened to be living in West Ham in 1911-12 were subject throughout the remainder of their lives to the rates of mortality experienced in West Ham during 1911-12 by the different groups of persons, of different antecedents, aged 10-15, 15-20 . . . 85-95, who also happened to be living in West Ham in 1911-12, their average after-duration of life would be the number of years specified. In other words it means just nothing at all. It is pure patch-work. But, it may be said, the "expectation of life" is a conventional expression, and no intelligent person would regard the expectation of life in West Ham at age 10 as being anything more than a kind of panoramic view of the vitality of West Ham residents at ages 10 and upwards. It seems, however, to be a singularly complicated function for the purpose. A more direct comparison between the vitality of the inhabitants of different districts would be afforded by a function based as in the case of the Registrar-General's occupational mortality-rates, on a standard population, and showing for such a population the number of years lived between specified ages. Similarly the mortality in different age periods would, in our view, be more usefully represented by the deaths in a standard population than by the life table function  $l_x - l_{x+n}$ . The latter is an entirely artificial number, involving in the case of each district its own particular stationary population.

The method employed in the construction of the tables and described in Dr. Snow's report has certainly the interest of novelty. The stages in the process of construction are  $r$  (or  $m$ ) the central death-rate,  ${}_t p$  the probability of surviving through the period covered by  $r$ ,  $(1 + p + {}_2 p + \dots + {}_{t-1} p)$  the sum of the probabilities of surviving 0, 1 . . .  $(t-1)$  years, and finally  ${}_t l$  being a by-product from  ${}_t p$ . So far there is nothing new. The novelty of the method consists in the fact that  ${}_t p$  is obtained from  $r$ , and  $(1 + p + \dots + {}_{t-1} p)$  from  ${}_t p$ , by empirical formulas determined by fitting equations of the first or second degree (with one exception) to the England and Wales, 1910-12 Table. Twenty-four such formulas were used for different ranges of the value of  $r$  and  $p$ , and for the different periods of grouping; *e.g.*, for a 10-year group, the value of  $r$  being between .15 and .30,  $p$  is taken as  $= .02194 + 22.0192(.20985 - r)^2$ . Judged by this expression the process, if elementary, could hardly be regarded as rapid, but the values of  $p$  corresponding to values of  $r$  differing by .00001 and the values of  $(1 + p + \dots + {}_{t-1} p)$  corresponding to values of  ${}_t p$  differing by .001 having been tabulated in the General Register Office, the formulas were virtually eliminated, and

the actual construction of the tables became a simple matter. At the same time the process must have entailed from first to last a very considerable amount of laborious research and calculation, and we confess that such an expenditure of time and energy appears to us to have been unnecessary. Since  $\text{colog } p = M_m \left(1 + \frac{a_0}{12}\right)$ , and  $\left(\frac{1}{2} + p + \dots + \frac{1}{2} {}_t p\right) = \frac{1}{2} {}_t(1 + {}_t p) + \frac{t^2 - 1}{12} ({}_t p \mu_t - \mu)$  there is no difficulty in constructing an abridged life table. Such a table could be constructed directly from the  $m$ 's as in the example given on p. 301 of vol. xlviii of this *Journal*. But a simpler plan is to construct by reference to a standard table. We then have

$$\text{colog } p' = \text{colog } p \times m' (1 + a'_0/12) / (1 + a_0/12)$$

and

$$\left(\frac{1}{2} + p' + \dots + \frac{1}{2} {}_t p'\right) = \frac{1}{2} {}_t(1 + {}_t p') + \left(\frac{1}{2} + p + \dots + \frac{1}{2} {}_t p\right) - \frac{1}{2} {}_t(1 + {}_t p)$$

approximately. For practical purposes it would make little difference if  $(1 + a'_0/12)/(1 + a_0/12)$  were neglected, since  $a'_0$  and  $a_0$  are both small quantities except at the older ages when they tend to equality. We give below, as an illustration, the construction of an abridged table for the South of England (excluding London) by reference to the official abridged table for England and Wales. We have taken the latter as our standard table because it is at hand as we write, but as the published values of  $r$  at the oldest ages are not sufficient to give  $a_0$  at age 75 and the values of  $p$  for each year of age are not available we have used the official value of  $e_{75}$ , and have also made use of the official tables to determine the England and Wales values of  $\left(\frac{1}{2} + p + \dots + \frac{1}{2} {}_t p\right)$ . In practice a full table such as the English Life No. 7 would be used as a standard. We may add that we have followed the official table in taking  $e$  as  $= e + \frac{1}{2}$ , although this is not quite accurate at the older ages.

Age	$m$		$1 + a_0/12$		$\text{colog } p$		$p$ S. ex. L.	$\frac{1}{2} + \dots + \frac{1}{2}tP$ $-\frac{1}{2}t(1+tP)$	$\frac{1}{2} + \dots + \frac{1}{2}tP$	$e^2$		
	E. & W.	S. ex. L.	E. & W.	S. ex. L.	E. & W.	S. ex. L.				E. & W.	S. ex. L.	S. ex. L.
0	...	...	...	...	...	...	9028	...	...	951	55.88	55.88
1	3750	2251	...	...	...	...	9777	...	...	989	60.84	60.84
2	947	635	.996	.998	0122	0082	9813	-.008	2.963	61.22	61.22	61.22
5	326	274	.999	.999	0074	0062	9858	-.007	4.957	59.37	59.37	59.37
10	193	157	...	...	0041	0033	9924	.001	4.982	55.20	55.20	55.20
15	293	251	1.001	1.001	0063	0054	9876	.001	4.970	50.61	50.61	50.61
20	369	323	1.001	1.001	0080	0070	9840	.001	4.961	46.21	46.21	46.21
25	482	443	1.002	1.002	0210	0193	9565	.016	9.798	41.92	41.92	41.92
35	803	703	1.004	1.003	0353	0309	9313	.030	9.687	33.58	33.58	33.58
45	1469	1208	1.009	1.007	0651	0534	8843	.059	9.481	25.65	25.65	25.65
55	2976	2403	1.020	1.017	1331	1072	7813	.094	9.001	18.28	18.28	18.28
65	6365	5388	1.045	1.041	2918	2461	5674	.079	7.916	11.88	11.88	11.88
75	13692	12142	...	...	...	...	...	...	...	[6.98]	6.98	6.98

The South of England value of  $p_0$  comes direct from the experience, and that of  $p_1$  is found from  $p = (2 - m)/(2 + m)$ . The values of  $a_0$  at age 10 are not known because the position of the minimum of the  $m$ -curve is unknown. If several tables were being constructed it would not, of course, be necessary to insert the four columns of figures relating to the standard table. These would be the same for all cases, and the only values that would have to be inserted would be those of  $m'(1 + \frac{1}{12} a'_0)/m(1 + \frac{1}{12} a_0)$ ,  $\text{colog } p'$ ,  $p'$ ,  $\frac{1}{2} + \dots + \frac{1}{2} t p'$ , and  $e$ .

The foregoing illustration shows, we think, that abridged life tables, if required, can be constructed quite simply without the use of empirical formulas. We must add, since the values given by the formulas have been tabulated and may probably be used for other purposes, that the assumption on which Dr. Snow's process is based, namely, that  $p$  can be expressed in terms of  $m$  with reference only to the particular value of  $m$  and to the range within which it lies, and without any reference to the ages to which it relates, is open to criticism. The relationship between  $p$  and  $m$  depends to some extent on the slope of the  $m$ -curve; two different mortality curves may have materially different slopes, especially at the old ages, at the points where they have the same  $m$ . Hence Dr. Snow's formulas, obtained by fitting the England and Wales curve, will not necessarily apply to curves representing very different mortality. The following is a comparison of the values of  $_{10}p_{75}$  for County of London males (heavy mortality) and Rural District females (light mortality), as given by the extended tables (supplement to the 75th Annual Report, Part I), with those given by Dr. Snow's empirical formula:

	$m$	$_{10}P$	
		Extended Table	Snow
County of London males ...	14280	2055	2003
Rural District females ...	10892	3015	3069

The differences may not be of any great importance for practical purposes, but they are probably systematic—tending to exaggerate slightly divergences from the average of the standard table. It seems not unlikely, for example, that in the illustration given above the official value of  $_{10}p_{75}$  for the South (excluding London), namely, .2631, is somewhat too high and that the value of  $e$  at age 75 is correspondingly over-estimated.

We note one small misprint—29611 for 2.9611 in Table 2 on p. xlii.

*Reduction of Error by Linear Compounding.* By W. F. SHEPPARD, Sc.D., LL.M.

[Philosophical Transactions of Royal Society of London. Series A, vol. 221, pp. 199-237.]

*On a Class of Graduation Formulae.* By CATHERINE W. M. SHERRIFF, M.A., B.Sc.

[Proceedings of the Royal Society of Edinburgh. Session 1919-20; pp. 112-128.]

ALTHOUGH these two papers differ materially in scope and object, they have a common point of contact with actuarial work. Dr. Sheppard's, it is true, does not refer directly to graduation: it has an altogether wider range, being a re-statement, in a more general and concise (but not, perhaps, to the ordinary student, simpler) form, of the author's earlier work on reduction of error—incorporating the results arrived at in the unpublished paper referred to as "Fitting II" in his articles in the *Journal* (vol. xlviii, pp. 171, 390, and xlix, p. 148) and superseding to some extent the published papers "Reduction" and "Fitting I." Miss Sherriff's, on the other hand, deals exclusively with graduation, with special reference to the relative merits for this purpose of least square formulas of the  $p_0u_0 + \sum p_r u_{\pm r}$  type (i.e., the formulas obtained by fitting a polynomial by least squares to the  $2n+1$   $u$ 's of which  $u_0$  is the central term) and summation formulas. The least square formulas, it may be noted, are identical with those obtained by making  $\sum p^2$  a minimum, subject to the condition that  $p_0u_r + \sum p_r u_{\pm r}$  must  $= u_0$  up to differences of a specified order. This follows from the fact (of which a general proof is given in Dr. Sheppard's paper) that when the errors of the  $u$ 's are all independent and have the same mean square, fitting (by least squares or moments) and reduction of error by means of negligible differences give the same result; or it may be readily established for the particular case by writing down the equations for  $\sum p^2$  a minimum, and combining them suitably with the conditions  $\sum p = 1$ ,  $\sum r^2 p_r = 0$ , &c. A useful feature of the paper is a table giving the values of the  $p$ 's for  $j = 3, 5, 7$  or  $9$  ( $j$  being the degree of the polynomial or the order of differences to which the graduated value of  $u$  is to be correct) and for an extensive range of values of  $2n+1$  (the number of terms included in the formula). A simple algebraical expression for the  $p$ 's, with the constants required for evaluation when  $j = 3$  or  $5$ , has been given by Dr. Sheppard in the *Journal* (vol. xlviii, pp. 178, 181-2) and the constants for higher values of  $j$  can be calculated from the general formulas in "Fitting I" (p. 102) or his present paper (p. 230 and Appendix IV), but it is certainly convenient to have the actual numerical values tabulated for use or reference. In discussing the respective merits of the least square formulas and summation formulas Miss Sherriff shows that the former cannot be expressed in the ordinary summation form. This might perhaps be inferred from the consideration that the coefficients resulting from successive summations do not form a single parabolic curve (as the

coefficients of the least square formulas do) but a broken curve made up of a succession of separate curves, although in the best summation formulas, such as Spencer's, the separate curves are so blended that the discontinuities are hardly perceptible (*cf.* diagram, *J.I.A.*, vol. xli, p. 360). For example, the coefficients 1, 2, 3, 4, 5, 4, 3, 2, 1, given by  $[5]^2$ , form two straight lines, and similarly the coefficients given by  $[5]^3$  form three parabolas. Although, however, the least square formulas cannot be expressed in the convenient  $[n][m][p]$  form, they can be expressed in terms of *sums* (*J.I.A.*, vol. xlviii, pp. 181-2). As a test of graduating power Miss Sherriff graduates by the 21-term least square formula and Spencer's 21-term formula the values of two analytical functions (one of the third degree and the other a function of which the fourth differences are relatively negligible) taken to the nearest integer, so that the values are affected by casual errors varying from 0 to .5, and compares the sums of the squares of the residual errors. The least square formula gives the better results—as might be expected since it has an  $R^2(=\Sigma p^2)$  of .1076 as compared with the .1432 of Spencer's formula (the mean square of error, that is to say, of each graduated value being rather less than one-ninth of that of the ungraduated value as against about one-seventh)—but the test is not necessarily conclusive as to the relative merits of the two formulas. The primary object of modern summation formulas is to obtain a smooth curve (consistently, of course, with substantial adherence to the irregular curve of the observed values), and in this respect Spencer's formula, having a smoothing coefficient of  $1/160$ , is much superior to the least square formula, which has a smoothing coefficient of about  $1/18$ . The distinction between accuracy and smoothness is clearly stated in Dr. Sheppard's paper in a short section on "Smoothing", from which the following are extracts:

"(i) If our object is to obtain as accurate values as possible for the  $w$ 's, consistently with our using only the specified number of  $u$ 's for each, the most accurate values would be the  $v$ 's given by the formulæ considered in this and the preceding papers" (*i.e.*, the least square formulas) . . .

"(ii) If the object is to obtain a smooth table of the  $w$ 's as a whole, we could do this by obtaining as accurate values as possible for the differences of the  $w$ 's of order  $j$  . . ."

Summation formulas do not give the best possible values for the third differences of the  $w$ 's, but they aim at giving an approximation to those values consistently with facility of application. Miss Sherriff mentions that Prof. Whittaker has proposed as a test of merit the smallness of  $\Sigma(u' - u)^2 + \lambda \Sigma(\Delta^3 u')^2$  "where  $\lambda$  . . . measures the extent to which we are willing to modify the original data for the sake of obtaining smoothness." It might be possible, by making  $\Sigma p^2 + \lambda \Sigma(\Delta^3 p)^2$  a minimum, to find a formula (for an assigned value of  $\lambda$ ) which would give good results when tested by this criterion, but it is doubtful whether for actuarial purposes it would be worth while. If smoothness is a requisite, the comparatively slight sacrifice of accuracy entailed by making it the



sole test, *i.e.*, practically by omitting  $\Sigma p^2$  from the criterion, would seem unimportant so far as most actuarial data are concerned. Moreover it might perhaps be argued that since (1) the true curve must be smooth, and (2) the assumption as to the errors being independent and having the same mean square does not apply strictly to actuarial data, a better approximation to the true curve may be given by smoothness (however obtained) than by a theory of accuracy based on the assumption in question.

The present opportunity may be taken of considering the position that has been reached in regard to summation formulas of graduation. Their evolution has been somewhat peculiar. It will be recalled that the earliest formula of the kind was suggested by J. A. Higham (*J.I.A.*, vol. xxiii, p. 335) as an alternative to Woolhouse's interpolation formula (which, however, was almost immediately shown by G. F. Hardy and Ackland to be itself a summation formula), and from that time up to 1894 the method was developed by its originator mainly on algebraical lines and without any special theoretical basis except such as might be inferred from the affinity of the method to that of the average interpolated value, and from the fact that summation had the general effect of distributing the irregularities of the data. As between different summation formulas the test of merit was a comparison of results. The proof of the pudding was in the eating rather than in any theoretical appropriateness in the recipe. It was not until 1896 that Hardy (who at an early stage had apparently felt the necessity for, and had evolved for the purpose of the Friendly Society formula, a theoretical basis) communicated to the *Journal* his note on "Graduation Formulas" (vol. xxxii, p. 371). Experimental work by J. Spencer on the lines suggested by this note led to the well-known 21-term formula (*J.I.A.*, vol. xxxviii, p. 338)—a formula much superior to any previously proposed. Shortly after this, Hardy pointed out the possibility of determining the best formula of given range (tested by the smallness of the mean error of, say,  $\Delta^2 u$ ), and suggested as a practical alternative that a convenient set of summations might be fixed upon and the coefficients of the operand determined so as to make the mean error of  $\Delta^2$  or  $\Delta^3$  a minimum\* (Construction of Mortality Tables, pp. 28-29). This suggestion—which does not seem to have been taken up at the time—leads to such formulas as the following:

$$\cdot 01[5][5][7][\cdot 50661u_0 + \cdot 37671u_{+1} - \cdot 08725u_{+2} - \cdot 25705u_{+3}]$$

with a smoothing coefficient of  $1/168$  and very similar  $p$ 's to those of Spencer's formula,

$$\cdot 01[4][6][7][\cdot 51009u_0 + \cdot 16205u_{+1} + \cdot 30346u_{+2} - \cdot 42294u_{+3}]$$

with a smoothing coefficient of  $1/176$ , and

$$\cdot 01[5][13][1\cdot 9834u_0 + 1\cdot 5054u_{+1} - \cdot 3474u_{+2} - 1\cdot 3804u_{+3}]$$

\* The writer is informed that Karup was the first to suggest this course.



with a smoothing coefficient of  $1/73$ . In 1907-8 the subject was further discussed in G. J. Lidstone's "Rationale of Formulae for Graduation by Summation"—which threw much light on the reasons for the superiority of one formula to another and on the effect of the formulas in reducing errors—and J. Spence's "Illustrations of the Employment of Summation Formulas" (*J.I.A.*, vol. xli, pp. 348, 361, and vol. xlii, p. 106). At this stage the summation-formula method, represented by several excellent practical formulas and provided with a more or less satisfactory theoretical basis, held a higher place than at any previous time in actuarial estimation, and here matters rested until the publication in 1912 of the first of Dr. Sheppard's important series of papers dealing with the reduction of error by linear compounding. These papers do not affect the validity of the summation-formula method or of the theoretical basis given to it by Hardy and Lidstone. But they have unquestionably affected the status of the method (1) by showing incidentally, as particular applications of a general theory of reduction of error, that the "best" formulas of given range referred to by Hardy in the "Construction of Mortality Tables" can be determined and applied practically (although possibly with less facility than summation formulas), and further that they apply not only to  $u_0$  but, with suitable modifications, to  $u_r$  and any linear compound of the  $u$ 's, such as  $\Delta^3 u_r$ , thus admitting of the ends of a table—always a difficulty in the application of the summation-formula method—being dealt with by a method consistent with that employed for the intermediate terms, and (2) by introducing the test of "accuracy", which may in certain circumstances be preferable to the test of "smoothness." They have also contributed materially to clearness of ideas by defining with precision the conditions on which the applicability of the general theory depends. Dr. Sheppard's algebraical work, being adapted to the determination of the  $f$ th even or odd central difference of  $u_r$  for a general value of  $j$ , is difficult, but simpler methods are available for deducing the formulas required for actuarial purposes. The subject has recently attracted the attention of American actuaries, and reference may be made to papers by R. Henderson and J. R. Larus, Jr., and to remarks on Larus's paper by Mervyn Davis (*T.A.S.A.*, vol. xvii, p. 46, and vol. xix, pp. 14, 299). Larus claims that with the aid of a calculating machine the work entailed by a graduation by a formula of maximum smoothing-power is little, if any, more than that required in the case of a simple summation formula, and that it requires fewer checks. If this is the fact, it must be admitted that no very obvious reason can be adduced for using even the best summation formula in preference to the corresponding formula of maximum smoothing-power. We have little doubt, however, that summation formulas will still be used, and we should hesitate to say that at present there is any scientific obligation on actuaries who have been accustomed to use them in private work to give them up in favour of the newer formulas. They are simple in form, convenient in application, and,

at their best, effective instruments of graduation in any case in which the data are sufficiently extensive for a linear compound to give good results; they also admit of some discrimination between "accuracy" and "smoothness" according to the character of the data. And it is to be remembered that the test by which we say that they are not the "best" is, although probably fairly reliable, to some extent arbitrary. The application of the theory of reduction of error to actuarial data has not yet been worked out. There is usually some correlation of error in such data, as Dr. Sheppard has pointed out (*J.I.A.*, vol. xxxviii, pp. 903-4), and certainly a great deal in their third differences; moreover the observations are of varying weights. A test based on the assumption that the errors are independent and have the same mean square does not, therefore, enable it to be said with absolute confidence that a particular formula is the "best" even as regards accuracy.

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## CORRESPONDENCE.

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### A NEW METHOD OF VALUING POLICIES IN GROUPS.

*To the Editors of the Journal of the Institute of Actuaries.*

DEAR SIRS,—In accordance with a suggestion by Mr. Lidstone I have tested the application of the method described in the above-mentioned paper (*J.I.A.*, April 1920), to the calculation of expected claims.

Applying it to the data of Table I, I have obtained the following results:

*Whole-Life Assurances. Expected Claims O<sup>M</sup>.*

*$\alpha$ ,  $\beta$  method.*

Years of Birth	Age Groups	EXPECTED CLAIMS		Percentage Deviation
		Amount	Deviation	
1823-29	90-96	394	- 1	·25
1830-39	80-89	6,337	- 28	·44
1840-49	70-79	18,672	+ 1	·01
1850-59	60-69	18,122	+ 6	·03
1860-69	50-59	12,003	+ 4	·03
1870-79	40-49	7,157	+ 2	·03
1880-89	30-39	3,382	- 1	·03
1890-98	21-29	671	...	...
Total	...	66,738	- 17	·03

I am indebted to Mr. Lidstone for pointing out to me

- (1) that if the  $a, \beta$  expression for the value be written in the form  $a'\Sigma + \beta\Sigma^2$ , where  $a' = a - 4.5\beta$ , the multipliers of  $a'$  and  $\beta$  can be obtained by a double summation, the multiplications by 4.5, &c., being thereby avoided ;
- (2) that if the S's are constant, my method will produce accurate results, whatever the values of the  $u$ 's.

Yours faithfully,

H. L. TRACHTENBERG.

*National Institute for Medical Research,  
Hampstead, N.W. 3.*

11 January 1921.

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OF THE

## INSTITUTE OF ACTUARIES.

*Superannuation Funds. Notes on some Post-War Problems, together with an account of a Pensioners' Mortality Experience (Civil Service Pensioners, 1904-1914). By G. S. W. EPPS, B.A., F.I.A., of the Government Actuary's Department.*

[Submitted to the Institute, 21 March 1921.]

### INTRODUCTORY.

1. THE ordinary member of a pension fund, if asked whether there is any connection between the high cost of living at present oppressing him and the security of the pension to which he looks forward to support him in his old age many years hence, would probably fail to discover that the one is related to the other. The financial position of pension funds is, however, fundamentally affected by the sudden drop in the purchasing power of the sovereign during the last few years and more directly by the enhanced scales of salaries or wages which have resulted from the change in the value of money, and it is to be feared that in many funds a very serious position has to be faced. The actuary who is called upon to advise as to the situation and to formulate remedies by which equilibrium may be restored has not only difficult professional problems to grapple with, but, having made up his mind as to the course to be recommended, he has to carry with him the responsible officers of the fund, who in their turn have to secure the co-operation of the employer and also the assent of the "ordinary member" of the fund, to whom I have referred above, to the remedies put forward.

In the following paper the new conditions which have arisen since the beginning of the war are broadly examined, and it is hoped that those of our craft who have had practical experience of the difficulties to be overcome and of the pitfalls to be avoided will welcome an opportunity of comparing notes for the benefit of the profession as a whole.

2. In the present paper I have limited my remarks to schemes where the contributions and benefits vary with the emoluments received by the members. There are, of course, especially in the case of wages staffs, schemes where the rules provide for uniform contributions and benefits, and in many of these it will doubtless be necessary to revise the benefits in order to bring them into line with the smaller purchasing power of money. In such schemes, however, the increases of benefits and of contributions would not arise automatically under the rules hitherto in force—as is the case in the schemes discussed in this paper—but would have to be secured by something in the nature of a supplementary scheme, special arrangements being required to deal with existing members as, for example, are necessary in the case of a new fund. As an illustration of what has just been said I may refer to the recent revision of National Health Insurance. Under the Act of 1920 the benefits and contributions were increased by roughly one-half, the additional contribution being so calculated as to make a small part of it available as a sinking fund payment to redeem the additional reserve values which were required in the case of existing insured persons; the principle of a uniform contribution payable by entrants of all ages, adopted in the original scheme of 1911, was thus continued.

3. The position of a member of an ordinary pension fund differs from that of a person insured under a life assurance or annuity contract in that the contributions and benefits of the former are not definitely fixed in terms of money when he enters into insurance. He does not pay a uniform contribution year by year, but his contributions are expressed as a proportion of the salary which he may enjoy when each payment falls due. Similarly he does not know at the outset what his pension will be on retirement, the scheme providing for the rate appropriate to his case to be ascertained, when he becomes a pensioner, by reference to the length of his service and to the salary he has earned during the whole or part of the period. It follows that in calculating the proportion of salaries required as a contribution to a fund out of which specified



benefits are to be provided a salary scale varying from age to age has to be assumed, and that the financial equilibrium of the fund at subsequent valuations depends, *inter alia*, upon the maintenance of that scale. During recent years the rates of pay both of salaried staffs and of wages employees have increased very materially mainly as a result of high prices, and the salary scales hitherto adopted are no longer appropriate. The actuarial basis of each scheme has accordingly to be reconsidered *ab initio* ; in the first place to ascertain whether the percentage of salary payable as a contribution to the fund ought to be varied in the case of new entrants, and secondly, to estimate the increased liability falling upon the fund in respect of existing members.

#### POST-WAR LEVEL OF SALARIES.

4. Very full information is given in the *Labour Gazette*, which is published towards the end of each month by the Ministry of Labour as to current rates of wages. Articles are included monthly summarising changes which have been reported in the preceding month, and also awards given by arbitrators under various Acts of Parliament, for example, the Industrial Courts Acts, 1919. Special articles appear from time to time reviewing the position and drawing general conclusions as to the current situation. In many industries agreements have been made providing for the regular and automatic adjustments of wage rates in accordance with variations in the cost of living.\* The general principle of the agreements is that a given amount of rise or fall in the cost of living shall be followed by a definite amount of increase or decrease in wage rates. In nearly all cases, the statistics compiled each month by the Ministry of Labour, which are designed to indicate the average increase in the cost of maintaining unchanged the pre-war standard of living of working class families, have been adopted as providing the most suitable indication of the variations in the cost of living for the purpose of these agreements. It is worthy of note that in most cases the agreements contain provisions under which increases in wages, or bonuses, equivalent to the full amount of increase shown by the index number, are granted only to the lower paid workers.

5. From the point of view of our present enquiry it is regrettable that the official returns deal mainly with wages staffs,

\* Special articles on this subject will be found in the *Labour Gazette* for November 1919 and December 1920.

and that less complete information is available in the case of salaried staffs. With certain important exceptions little is published officially bearing on the present level of salaries, and it would appear that in most salaried services official arbitration is not sought and that private awards are neither reported officially to the Ministry of Labour nor published in the newspapers. In this connection I was interested in looking through the files of the *Labour Gazette* to observe that an award had been made covering the staffs of the principal banks in Ireland; in these democratic days it was not surprising to find that the notice was sandwiched between paragraphs referring to "Skip and basket manufacture (Yorkshire)" and "Shuttle-making Industry (Lancashire)"!

6. In one scheme with which I am familiar the war bonus took the simple shape of a constant addition, together with a percentage addition to pre-war salary, a larger proportionate increase being thus automatically secured to the lower paid workers. In other schemes somewhat similar results are achieved by decreasing the percentage of salary given as bonus as the basic rate rises. Again, some war bonuses have been converted (or part of them) into consolidated pay, others are still merely temporary adjustments of salary, the basic rates remaining unchanged. Further, practice probably differs as to the treatment of war bonus in reckoning superannuation rights; in one instance I noticed that in calculating pensions the "final salary" is normally taken at the pre-war rate with the addition of three-quarters of the war bonus current at the date of retirement. Presumably, as soon as prices become stabilised new scales of permanent salaries will be adopted approximating to the old scales increased by the war bonuses then current, but in the meantime it is evident that the rapidly shifting conditions make the selection of a salary scale for valuation purposes an extremely difficult matter.

7. The cost of living increased gradually but continuously during the early years of the war, the curve becoming steeper from the beginning of 1918; after a slight fall in the middle of 1919 a rapid increase was observed each month until the curve commenced to fall in November last. I do not propose to make any rash speculation as to the level to which prices will ultimately fall, and base new salary scales in relation thereto. In the present enquiry I think the most practical course is to take illustrative examples showing the effect of adjustment of salaries on various bases:—

(a) that all pre-war salaries will be doubled.

(b) that all pre-war salaries will be increased by a level £50 and by a proportionate increase of 50 per-cent.

(c) that all pre-war salaries will be increased by a level £100.

These adjustments have been applied to a pre-war salary scale commencing at £50 at age 18 and rising to £347 at age 65. Specimen figures are as follows :—

Salary Scale	I		II		III		IV	
Percentage addition to pre-war salary .. ..	}	...	}	100%	}	50%	}	...
Constant addition to pre-war salary .. ..		...		...		£50		£100
Age	Salary	Rate of Increase	Salary	Rate of Increase	Salary	Rate of Increase	Salary	Rate of Increase
18	£ 50	% ...	£ 100	% ...	£ 125	% ...	£ 150	% ...
20	62	11	124	11	143	7	162	4
30	138	8	276	8	257	6	238	4
40	225	4	450	4	388	3½	325	3
50	300	2	600	2	500	2	400	1½
60	338	1	676	1	557	1	438	1½

The progression in the rate of increase is the same in Scales I and II, but is very much flatter in Scale IV, Scale III being between the two.

8. In adjusting salaries by a percentage or a constant, or a combination of both, the salary scale, being based on averages, is arrived at from the old scale by the same adjustment, and the financial effect of the variation is readily obtained by calculating two sets of commutation columns (i) based on the old scale and (ii) based on a constant scale; by combining these any number of examples can easily be worked out. The position is of course more complicated where it is not practicable to express the adjustment in this simple form, as the ultimate effect on the salary scale would depend on the proportions of the several rates of salary expected at individual ages.

9. For simplicity I have assumed in the examples given in a later stage of the paper that the change of salary basis has occurred immediately before the valuation; in practice, of

course, the new level will only be reached gradually, but for illustrative purposes refinement on this point does not seem necessary.

#### INTEREST.

10. In common with other insurance institutions pension funds, as a general rule, have the very serious problem of depreciation of existing investments to face. Reserves which were accumulated when the rate of interest yielded was no more than 3 or  $3\frac{1}{2}$  per-cent have dropped to a level corresponding with the high rates of interest at present obtainable in respect of new investments, and, apart from other sources of loss, serious deficiencies have doubtless accrued from this cause, especially in the case of old established funds. It is, of course, true that materially higher rates than those assumed in past valuations can be looked for in the case of the accruing funds to be invested for many years to come; but the question as to how far it is prudent to meet past depreciation by raising the valuation rate of interest is one of extreme difficulty, the solution of which throws a great responsibility on the actuary concerned. Different problems arise where the fund has the benefit of a guaranteed rate of interest and the risk of depreciation is absent. In such cases the interest granted was often in excess of that ordinarily obtainable on trustee investments and the members have derived a substantial benefit from the guarantee of a liberal rate without fear of depreciation of capital during the troublous years of changing values. Proposals will doubtless be made by the members of many of these funds that the guaranteed rate of interest, at any rate for future savings, should be brought into line with the enhanced rates now prevailing; but it is evident that in considering a request of this kind all the circumstances of the arrangement should be taken into account, due regard being had to the relations of the guaranteeing authority to the beneficiaries of the fund, and to the advantages which have accrued under the guarantee in the past.

In a later stage of this paper examples are given illustrating the weight of the interest factor.

#### *Income Tax.*

11. Before leaving the question of interest it is fitting to refer to the liabilities of pension funds in respect of income tax. Under existing law a fund is required to pay tax on all interest received, but is allowed as a set-off (not exceeding the tax paid

on interest) the tax which is recovered (or would be recovered if its members were not wholly or partially exempt) from the pensions payable to superannuated members. The present position is set out in a memorandum\* prepared by the Board of Inland Revenue, at the request of the Royal Commission on the Income Tax, with regard to the position of superannuation funds of public companies and corporations in relation to the income tax. By the courtesy of the Controller of His Majesty's Stationery Office this memorandum is reprinted as an appendix to this paper.

12. The effect of the present arrangement is that during the early years of a scheme interest payments are subject to the full rate of tax, and the rapid accumulation of funds which is required at this stage is seriously hampered by the heavy demands for income tax. It is not until the pension liabilities begin to mature that the fund can get any relief from this strain, and the full relief is not reached until the pension commitments are equal to the gross interest receipts.

13. This subject was examined recently by the Royal Commission on the Income Tax, their conclusions being set out in Part III, Section XVI of their Report.† In the case of Superannuation Funds the Commission recommended unanimously that these funds should be exempt from income tax and that all pensions should be regarded as income and assessed in the hands of the recipients, with the proviso that the managers of the various funds, as a condition of exemption, should be required to give all necessary information for the due assessment of the pensions. This recommendation is, of course, of great importance especially at the present time, having regard to the high rate of tax now in force.

The strain of the present practice is clearly seen from the evidence submitted to the Commissioners. A witness speaking on behalf of a number of funds with a total membership of over 58,000 and assets of nearly seven millions (*Mitchell*, Q. 4465 *et seq.*) stated that their present commitments for pensions were less than one half of their income from investments and that, quite apart from the losses due to past depreciation, many funds were unable to earn even on their new investments the net rate assumed in the actuarial basis.

\* Royal Commission on the Income Tax; Appx. 48(a) of the Seventh Instalment of the Minutes of Evidence (Parl. Paper, Cmd. 288-7).

† Parl. Paper, Cmd. 615 of 1920.



Those who are responsible for existing funds or who are proposing to initiate new schemes will no doubt be keenly disappointed that legislative sanction for the above recommendation was not included in the Finance Act of 1920 which gave effect to the main recommendations of the Commissioners.\*

14. Although it is not strictly cognate to the subject under discussion I think it will be of interest to refer briefly to certain other recommendations made in the same section of the Report in regard to Provident and Thrift Funds. The scope of Superannuation Funds and Provident Funds respectively was broadly defined by the Commissioners as follows :—

*Superannuation Funds.*

- (a) deductions are made from the salaries or wages of the employees and paid over by the employer to the fund ;
- (b) the employer contributes a further sum to the fund ;
- (c) these contributions from employer and employees are invested, and the capital of the fund consists of the accumulated contributions with interest additions ;
- (d) on retirement, at a specified age or on previous incapacity, the employee receives from the fund a pension, based on the salary or wages received by him during his period of service with the employer ;
- (e) in the event of the employee leaving the service of the employer he receives back his contributions, with or without interest, according to the constitution of the particular fund ;
- (f) in the event of an employee dying before he becomes entitled to superannuation, his legal representatives receive a sum equal to twice the amount of his contributions, with or without interest, according to the constitution of the particular fund.

*Provident and Thrift Funds.*

- (a) deductions are made from the salaries or wages of the employees, and paid over by the employer to the fund ;
- (b) the employer also contributes a sum, usually, but not necessarily, equivalent to the amount deducted from the employees ;
- (c) the total contributions are invested, and the capital of the fund consists of the accumulated contributions with interest ;
- (d) on retirement at a specified age, or on previous incapacity or death, the employee, or his legal representative, receives from the fund a capital sum equal to his own and the employer's contribution with interest ;
- (e) in the event of an employee leaving the service of the employer he receives back his contributions with interest.

The Commissioners arrived unanimously at the conclusion that in either case the employer's contribution (if it is made to a recognized fund and is irrecoverable) could properly be regarded

\* Since the paper was read the law has been amended by Section 32 of the Finance Act, 1921. Subject to regulations to be made by the Commissioners of Inland Revenue the income of superannuation funds will not in future be liable to income tax.



as a business expense and should be allowed as a deduction in arriving at his Income Tax assessment. The majority, however, drew a distinction between the two types of funds and limited to Superannuation Funds their recommendations :—

- (i) that the contribution of an employee should be allowed as a deduction from the income upon which his income tax would otherwise be charged, and
- (ii) that the interest earned by the investment of the employers' and employees' contributions should be exempt.

Mr. Geoffrey Marks in a Reservation to this Section of the Report expressed the view that there is no difference, either in their principles or practice, which would warrant the application of different principles of taxation to these funds. He drew attention to the fact that the method by which the two classes of funds are built up (set out in parallel columns above) and the benefits which they give are identical, except to the extent that there is no direct provision in a Provident and Thrift Fund for payment of pensions. Since, however, a capital sum received from such a fund can be converted into an annuity Mr. Marks concluded that this power places the beneficiaries in each class of fund on exactly the same footing and recommended that the same principles of taxation should be applied.

#### EXAMINATION OF MAIN PROBLEM BY ILLUSTRATIONS.

15. Returning to my main theme I think the best plan will be to proceed in its examination by a series of examples. Having regard to the number of variables which have to be taken into account in pension fund problems it is, of course, inevitable that each case has to be considered on its individual merits and caution must be exercised in drawing general conclusions from any particular fund. As has been pointed out in previous papers, anything in the nature of a "model office", with which we are familiar in our discussions on life office business, is inappropriate to the examination of superannuation questions, but it is thought that a series of examples using different salary scales applied to the same service table will give some general indication of the change from the old to the new conditions.

#### *Active Service Table.*

16. For my examples I have employed a hypothetical service table which is assumed to relate to a well established and efficient

service recruited by a regular flow of selected entrants at age 18 and exhibiting low rates of mortality and of withdrawal. A large proportion of the survivors to age 60 retire near that age, a few dropping out during the following quinquennium, and the remainder being placed on the pension list at age 65. Before age 60 superannuation is limited to cases of ill-health. Specimen rates of mortality, withdrawal and retirement at quinquennial ages are as follows :—

Age	RATE OF		
	Mortality (active list)	Withdrawal	Retirement
18	·001	·018	...
20	·001	·013	...
25	·001	·004	·001
30	·001	·001	·002
35	·002	...	·003
40	·003	...	·004
45	·004	...	·005
50	·007	...	·010
55	·012	...	·024
60	·019	...	·290

### *Pensioners' Mortality Rates.*

17. For the mortality rates of pensioners I have had access to a new table recently compiled from certain published returns relating to retired Civil Servants. As there are certain distinctive features in the experience, a short account of the compilation of the table may be of interest. In order to retain the continuity of the paper I have reserved for an Addendum the examination of the new experience from which both aggregate and select tables have been deduced. In the examples which are given in the following paragraphs the select rates of pensioners' mortality have been employed; but a comparison of the results brought out by using the select and aggregate tables respectively will be found in Part II of the Addendum, together with some general remarks on the use of select rates in valuing prospective pensions.

### EFFECT OF INCREASED SALARIES.

#### *(a) Position of New Entrants.*

18. I now propose to deal with the effect of increased salaries, and consider first the position of future entrants to a scheme. It is, of course, evident that where the salaries

of all the staff have been increased proportionately the percentage of salary required as contribution under a scheme is not affected. Where, however, the increases granted to the lower paid officers have been larger in relation to their pre-war salaries than those given to the more highly paid officers, the progression of salary from age to age is affected and the new salary scale is flatter than the old scale. It follows that during the early years of membership a larger accumulation of reserves

*Table of Contributions (Entry Age 18).*

RATE OF INTEREST	4 PER-CENT				5 PER-CENT			
Salary Scale	I	II	III	IV	I	II	III	IV
Percentage addition to pre-war salary ...	...	100%	50%	...	...	100%	50%	...
Constant addition to pre-war salary	...	...	£50	£100	...	...	£50	£100
Contribution :— (Final Salary)	%	%	%	%	%	%	%	%
4% Returns on Withdrawal ...	·02	·02	·03	·04	·03	·03	·03	·04
8% Returns on Death ...	·70	·70	·69	·67	·61	·61	·59	·58
Pension (Final Salary) ...	10·72	10·72	9·75	8·57	8·16	8·16	7·35	6·38
(a) Total* ...	11·44	11·44	10·47	9·28	8·80	8·80	7·97	7·00
Reduction from Scale I (4%)†	...	...	·97	2·16	2·64	2·64	3·47	4·44
Relation to Scale I ...	100	100	92	81	77	77	70	61
Contribution :— (Average Salary)								
2½% Returns on Withdrawal ...	·02	·02	·02	·02	·02	·02	·02	·03
5% Returns on Death ...	·44	·44	·43	·42	·38	·38	·37	·36
Pension (Average Salary) ...	6·56	6·56	6·32	6·02	4·99	4·99	4·76	4·49
(b) Total* ...	7·02	7·02	6·77	6·46	5·39	5·39	5·15	4·88
Reduction from Scale I (4%)†	...	...	·25	·56	1·63	1·63	1·87	2·14
Relation to Scale I	100	100	96	92	77	77	73	70

\* Referred to below as "minimum contribution," *i.e.*, the contribution required to support the benefits in the case of a new entrant.

† Referred to below as "surplus contribution."

can be made, and consequently a smaller percentage of salary is needed to support the benefits, the drop being more marked where pensions are based on terminal salaries.

19. To illustrate this point the contributions needed under the various assumptions as to salaries given above (para. 7) have been calculated for pensions of one-sixtieth of salary for each year of service (without limit) based on (a) final salary or (b) average salary. In the former case returns of 4 per-cent of salaries (without interest) on withdrawal and 8 per-cent on death have been assumed, and in the latter  $2\frac{1}{2}$  per-cent and 5 per-cent respectively. In order that the examples may not be unduly complicated the contribution required for expenses of administration has been omitted.

20. It will be seen that under the new scales of salary (III and IV) considerably reduced contributions suffice to support the benefits, especially where pensions are based on final salaries, and that the proportionate reduction of contribution is practically the same at either rate of interest. Thus the change from I to IV reduces the contribution (final salaries) almost as much as the variation in the rate of interest, and the combined effect of salary adjustment and of increased interest is to diminish the contribution by 39 per-cent. Even in the less extreme case (III) the contribution can be abated by 30 per-cent of the old scale if interest is raised to 5 per-cent.

#### (b) *Valuation Reserves.*

21. Passing to the effect of the changes on valuation reserves I give summaries of results on the several bases. In each case the aggregate liabilities in respect of the individual benefits are shown, and the reserves which would be required if the new percentage contributions applicable to new entrants were payable only. Smaller reserves are also shown taking credit for the full percentage contribution assumed to be payable under the existing rules of the Fund (namely, Salary Scale I, 4 per-cent interest)—for convenience these are called “valuation reserves.” A stationary membership of 3,673 on the active list is taken, the numbers at individual ages corresponding with the service table, and a fund of £2,209,345 is assumed to be in existence. The change of salary scale is deemed to have taken place immediately before the valuation.

## (A) SUMMARIES OF VALUATIONS.

*Pensions based on Final Salaries.*(i) *Interest 4 per-cent.*

Salary Scale	I	II	III	IV
	£	£	£	£
<i>Present Value of</i>				
4% Returns on Withdrawal				
(a) Past Contributions...	240	240	240	240
(b) Future „ ...	256	512	540	568
Total ... ..	496	752	780	808
8% Returns on Death				
(a) Past Contributions...	84,120	84,120	84,120	84,120
(b) Future „ ...	83,560	167,120	142,172	117,224
Total ... ..	167,680	251,240	226,292	201,344
Pension				
(a) Back Service ...	2,123,510	4,247,020	3,502,231	2,757,442
(b) Future „ ...	1,219,337	2,438,674	2,011,646	1,584,619
Total ... ..	3,342,847	6,685,694	5,513,877	4,342,061
Total Benefits ... ..	3,511,023	6,937,686	5,740,949	4,544,213
“Minimum” Contributions (See Table para. 19).	1,301,678	2,603,356	2,030,311	1,487,287
Reserve ... ..	2,209,345	4,334,330	3,710,638	3,056,926
“Surplus” Contributions... (See footnote to Table, para. 19).	...	...	188,099	346,179
Valuation Reserve ...	2,209,345	4,334,330	3,522,539	2,710,747
Deficiency ... ..	...	2,124,985	1,313,194	501,402
Surplus ... ..	...	...	...	...

(ii) *Interest 5 per-cent.*

Salary Scale	I	II	III	IV
	£	£	£	£
<i>Present Value of</i>				
4% Returns on Withdrawal				
(a) Past Contributions	236	236	236	236
(b) Future „	240	480	510	540
Total ... ..	476	716	746	776
8% Returns on Death				
(a) Past Contributions	78,720	78,720	78,720	78,720
(b) Future „	69,056	138,112	117,384	96,656
Total ... ..	147,776	216,832	196,104	175,376
Pension				
(a) Back Service ...	1,789,257	3,578,514	2,950,956	2,323,397
(b) Future „ ...	907,727	1,815,454	1,497,697	1,179,940
Total ... ..	2,696,984	5,393,968	4,448,653	3,503,337
Total Benefits ... ..	2,845,236	5,611,516	4,645,503	3,679,489
“ Minimum ” Contributions	899,017	1,798,034	1,389,482	1,010,492
(See Table, para. 19).				
Reserve ... ..	1,946,219	3,813,482	3,256,021	2,668,997
“ Surplus ” Contributions	269,705	539,410	604,956	640,941
(See footnote to Table,				
para. 19).				
Valuation Reserve ...	1,676,514	3,274,072	2,651,065	2,028,056
Deficiency ... ..	...	1,064,727	441,720	...
Surplus ... ..	532,831	...	...	181,289

22. It is seen that in the case of pensions based on final salaries the revaluation on a revised salary scale brings about a very severe deficiency which even in the most favourable case (IV) amounts to  $18\frac{1}{2}$  per-cent of the reserves needed. The position is improved by increasing the rate of interest, but even on this basis only two valuations show surpluses. Again, in arriving at the valuation results no account has been taken of depreciation of existing securities, and the position shown would be realised only where the fund had had the benefit of a guarantee against capital depreciation. In the case of other funds not so happily placed in this respect, the heavy strain of depreciation would, of course, have to be faced in addition to the burden arising from change of salaries and a large part—probably in many instances the whole—of the relief secured by increasing the valuation rate of interest would be swallowed up in writing off the capital losses of the last few years.



23. Comparing the several deficiencies with the aggregate liabilities in respect of benefits it is found that, in default of other methods of meeting the deficiency, the benefits of a scheme in respect of which the existing assets are assumed to be equal to the reserves needed under Scale I at 4 per-cent, would have to be reduced by the following percentages :—

Rate of Interest				4 per-cent	5 per-cent
Scale	I	...	...	nil	...
"	II	...	...	31	19
"	III	...	...	23	10
"	IV	...	...	11	...

(In the case of Scale I the surplus resulting from a valuation at the higher rate of interest is equivalent to nearly 20 per-cent of benefits, the corresponding figure for Scale IV being 5 per-cent.)

24. The deficiencies, however, arise mainly from the fact that pensions are to be assessed with reference to final salaries in accordance with the new scales, whereas the funds have been accumulated from contributions derived from smaller salaries (Scale I). This is broadly indicated by a comparison of the deficiency (arising on revaluation at 4 per-cent) with the increased liability in respect of back-service.

Scale	Increased Liability in respect of Back Service	Deficiency
	£	£
II ... ..	2,123,510	2,124,985
III ... ..	1,378,721	1,313,194
IV ... ..	633,932	501,402

This similarity of results suggests a possible method of meeting the deficiency which might be practicable in certain cases, where, for example, the new salaries bear a well-marked relation to the old salaries. The pension might be calculated in two portions, service after the change of scale ranking for sixtieths of the actual salary at retirement, whereas back-service would secure only a pension based on the salary which would have been payable had the old conditions continued. Under this plan the strain would, in effect, be apportioned according to years of membership before and after the change respectively, and the

older members would not reap an unfair advantage at the expense of the younger. Other considerations arise, however, where the change of salary scale is accompanied by an alteration of the rate of interest. Here the strain in applying the new scale to back-service is not strictly comparable with the valuation result since the increase of the rate of interest has itself brought substantial relief. Having regard to what was said above with respect to depreciation it might, nevertheless, be found that after allowing for this factor the resulting deficiency could equitably be dealt with by reducing the back-service rights of existing members on the lines suggested.

25. An alternative method of liquidating the deficiency without involving a reduction of benefits would be by conversion into an annuity payable by the employer or shared between him and the members. For example, taking a forty-year term, the payment would work out as follows :—

			Annual payment for 40 years	Expressed as percentage of all salaries
<i>Interest 4 per-cent.</i>			£	%
Scale II	...		105,250	7.1
„ III	...		65,050	5.0
„ IV	...		24,840	2.2
<i>Interest 5 per-cent.</i>				
Scale II	...		60,540	4.1
„ III	...		25,120	1.9

26. There are, however, certain difficulties in the above methods of rearrangement since the rate of contribution strictly required in the case of new entrants is less than the old rate and a valuation which takes credit for the latter contribution includes negative values at the younger ages. This applies not only to some of the existing members, in whose case the present value of the liabilities is less than the value of the full contribution, but also to all new entrants. Thus so long as the basic rate of contribution remained unchanged a negative value would be created on the entry of each new member, amounting to over £50 in the case of Salary Scale III if interest is assumed at the rate of 4 per-cent, or £150 on a 5 per-cent basis. A steady flow of one hundred new entrants a year would be represented by a potential profit of £5,000 or £15,000 a year respectively. In the latter case this profit, if realized, would go a long

way to cover the interest (namely, £22,000 a year) needed on the deficiency produced by the change of valuation basis as regards salary scale and rate of interest. In other words, if the old rate of contribution were retained the surplus contributions paid by new entrants would cover about two-thirds of the deficiency without the application of any special remedies, provided that the regular influx of new entrants was continued.

27. Such a method of attempting to carry a deficiency without taking specific steps to secure equilibrium might, however, be open to criticism, and it would appear preferable to make definite arrangements to liquidate the deficiency within a reasonable period. One method of securing this result would be to adjust the valuation and take credit for the minimum contribution only in the first instance, negative values being thus eliminated. The adjusted deficiency (further increased by the burden due to depreciation of securities) could then be converted into an annuity payable over a selected term. This could be expressed as a percentage of contributions and divided between the employer and the members according to a plan to be agreed upon. At the end of the term of 40 years the minimum contribution only would be required, and the basic rate could be reduced or the benefits revised in the light of the conditions then obtaining.

Thus, applying this method to Salary Scale III, we get :—

Rate of Interest	4 per-cent	5 per-cent
Valuation Deficiency ...	£ 1,313,194	£ 441,720
Present Value of " Surplus " Contributions ... ..	188,099	604,956
Adjusted Deficiency ...	1,501,293	1,046,676
Equivalent Annuity for 40 years ... ..	74,360	59,510
Annuity expressed as Per- centage of Salary ...	5·7	4·6
" Minimum " Contribution	10·5	8·0
Total contribution for 40 years (all members) ...	16·2	12·6
Present contribution ...	11·4	11·4
Increase ... ..	4·8	1·2

It will be noticed that apart from any loss due to depreciation of securities, the percentage contribution (on the 5 per-cent

basis) would have to be more than 1 per-cent above the present rate.

28. I now give the corresponding summaries of valuations where pensions are based on average salaries. Here a fund of £1,365,332 is assumed to be in existence.

# (B) SUMMARIES OF VALUATIONS.

## *Pensions based on Average Salaries.*

### (i) *Interest 4 per-cent.*

Salary Scale	I	II	III	IV
	£	£	£	£
<i>Present Value of</i>				
2½% Returns on Withdrawal				
(a) Past Contributions...	150	150	150	150
(b) Future „ ...	160	320	338	355
Total ...	310	470	488	505
5% Returns on Death—				
(a) Past Contributions...	52,575	52,575	52,575	52,575
(b) Future „ ...	52,225	104,450	88,858	73,265
Total ...	104,800	157,025	141,433	125,840
Pension—				
(a) Back Service ...	1,060,832	1,060,832	1,060,832	1,060,832
(b) Future „ ...	998,147	1,996,294	1,679,860	1,363,427
Total ...	2,058,979	3,057,126	2,740,692	2,424,259
Total Benefits ...	2,164,089	3,214,621	2,882,613	2,550,604
“ Minimum ” Contributions (See Table, para. 19).	798,757	1,597,514	1,312,818	1,035,331
Reserve ...	1,365,332	1,617,107	1,569,795	1,515,273
“ Surplus ” Contributions... (See footnote to Table, para. 19).	...	...	48,479	89,750
Valuation Reserve ...	1,365,332	1,617,107	1,521,316	1,425,523
Deficiency ...	...	251,775	155,984	60,191
Surplus ...	...	...	...	...

(ii) *Interest 5 per-cent.*

Salary Scale	I	II	III	IV
	£	£	£	£
<i>Present Value of</i>				
2½% Returns on Withdrawal				
(a) Past Contributions...	148	148	148	148
(b) Future „ ...	150	300	319	338
Total ... ..	298	448	467	486
5% Returns on Death—				
(a) Past Contributions...	49,200	49,200	49,200	49,200
(b) Future „ ...	43,160	86,320	73,365	60,410
Total ... ..	92,360	135,520	122,565	109,610
Pension—				
(a) Back Service ...	908,165	908,165	908,165	908,165
(b) Future „ ...	752,853	1,505,706	1,265,387	1,025,067
Total ... ..	1,661,018	2,413,871	2,173,552	1,933,232
Total Benefits ... ..	1,753,676	2,549,839	2,296,584	2,043,328
“Minimum” Contributions	550,648	1,101,296	897,846	704,457
(See Table, para. 19).				
Reserve ... ..	1,203,028	1,448,543	1,398,738	1,338,871
“Surplus” Contributions...	166,522	333,044	326,014	308,922
(See footnote to Table,				
para. 19).				
Valuation Reserve ...	1,036,506	1,115,499	1,072,724	1,029,949
Deficiency ... ..	...	...	...	...
Surplus ... ..	328,826	249,833	292,608	335,383

29. It will be observed that where the average salary governs the pension the strain caused by change of salary scale is much less severe than that observed in the case of final salary pensions. More than one-half of the total liabilities for pensions is in respect of back-service which are, of course, not affected by the change of scale, and the enhanced contributions to be received in the future are to a larger extent available to meet the burden of pensions accruing in respect of future service. With the rate of interest unchanged a deficiency is, however, brought out in each example, ranging from 4 per-cent of the reserves needed in the case of Scale IV to 15½ per cent (Scale II). If interest is increased to 5 per-cent a surplus is found in each example: the amount of the surplus in the several examples varying within fairly narrow limits; in Scale IV indeed the change of salary brings out a slightly larger surplus. In this connection it should be noted, however, that the favourable position dis-

closed by increasing the rate of interest assumed is to a large extent illusory when the fund has itself to bear the burden of depreciation, and that the considerations to which I referred when dealing with final salary pensions apply equally to the present case.

#### ILLUSTRATION OF A NEW FUND.

30. In the previous examples it was assumed that the scheme had been in operation for a long time and that the Fund had reached a stationary condition. I pass now to consider some special problems which would arise in connection with a new scheme. As before, I assume that the staff consists of 3,673 members recruited by 100 new entrants a year at age 18. The benefits to be provided are :

(1) Pensions based on final salary :

(a)  $\frac{1}{60}$  for each year of service after the commencement of the scheme.

(b)  $\frac{1}{120}$  for each year of back-service before the commencement of the scheme.

(2) Returns of contributions 4 per-cent on withdrawal and 8 per-cent on death.

The percentage contribution required for new entrants at age 18 are, accordingly, the "minimum" contributions shown in the table in paragraph 19 since the benefits for future entrants are the same. In the case of existing members the normal benefits are provided but only in respect of future service, and half pension rights for back-service.

31. The following table shows the several contributions according to age at commencement of the scheme for future service and back-service separately under the various assumptions as to salary scale :—

*Table of Contributions.*

Age at commencement	FUTURE SERVICE (ALL BENEFITS)						BACK SERVICE (HALF PENSIONS ONLY)					
	4 per-cent			5 per-cent			4 per-cent			5 per-cent		
	I & II	III	IV	I & II	III	IV	I & II	III	IV	I & II	III	IV
18	11.4	10.5	9.3	8.8	8.0	7.0	...	...	...	...	...	...
25	11.0	10.4	9.6	8.6	8.2	7.5	1.1	1.0	1.0	.9	.8	.8
30	10.9	10.5	9.9	8.7	8.4	7.9	2.2	2.1	2.0	1.7	1.7	1.6
35	11.0	10.7	10.3	9.0	8.8	8.5	3.7	3.6	3.5	3.0	3.0	2.9
45	11.7	11.6	11.5	10.1	10.0	9.9	10.0	9.9	9.8	8.6	8.6	8.5
55	13.3	13.3	13.3	12.0	12.0	12.0	37.3	37.2	37.1	33.8	33.7	33.6



It will be observed that the progression of contribution from age to age is affected not only by the change of salary scale, but also by the interest factor; the contribution tending to increase more rapidly with age where the salary scale is flatter, and the rate of increase being accelerated when the higher rate interest is assumed. It would appear, however, that the relation of one scheme to another at any given age is hardly affected by the interest factor. These deductions are more easily seen if the first part of the preceding table is restated in the following form :—

*Progression of Percentage Contribution.*

Salary Scale	AGE AT COMMENCEMENT					
	18	25	30	35	45	55
<i>(a) With reference to Age 18.</i>						
<i>Interest 4 per-cent</i>						
I & II	100	96	95	96	102	117
III	100	99	100	102	111	127
IV	100	104	107	111	123	143
<i>Interest 5 per-cent</i>						
I & II	100	98	99	102	115	137
III	100	102	106	110	126	151
IV	100	107	113	121	141	171
<i>(b) With reference to Scale I.</i>						
<i>Interest 4 per-cent</i>						
I & II	100	100	100	100	100	100
III	92	95	96	98	99	100
IV	81	88	91	94	98	100
<i>Interest 5 per-cent</i>						
I & II	100	100	100	100	100	100
III	91	94	96	97	99	100
IV	80	87	91	94	98	99

32. In the case of back-service the figures suggest that the change of salary scale has little effect on the percentage contributions required at individual ages; the progression from age to age under each assumption as to salary being similar.

33. In my illustration I assumed that membership would be compulsory for all existing officers, and accordingly in computing the contributions at the several ages I provided for an immediate title to benefits. If membership were optional and a qualifying

period of (say) 10 years before a pension could be taken on retirement was imposed in order to safeguard the scheme, the progression of the rate of contribution from age to age would be flatter than the figures in the foregoing table. Another practical point is that by assuming unlimited pensions the fall in the contribution between ages 18 and 25 is accentuated. If the usual arrangement of limiting pensions based on final salary to a maximum of two-thirds of salary is adopted, and a waiting period of 10 years is imposed, the combined effect of the two adjustments is indicated below in the same form as in the preceding table :—

					AGE AT ENTRY				
					18	25	30	35	45
Scale	I	4%	...	...	100	100	98	97	97
..	III	..	...	...	100	104	103	104	105
..	I	5%	...	...	100	102	101	103	108
..	III	..	...	...	100	106	107	112	119

It may be mentioned that the recent Departmental Committee on the Superannuation of Local Government Officers recommended a uniform contribution irrespective of age at entry.\* The scheme proposed by the Committee provided for a waiting period, and, to meet the case of existing officers who become invalided before completing ten years of contributory service, suggested that back-service rights might in these cases be assessed beyond the date of the commencement of the scheme and up to the date of retirement. The percentages of salaries representing back-service rights which I give above agree broadly with those suggested by the Committee in their Report.

34. Another method of examination is by comparing the liabilities arising under the scheme at its commencement under various assumptions, and in the following table I work out the initial deficiency which would arise in each case (Scales I, III and IV) if all members were called upon to pay only the "minimum" contributions.

\* Report (Cmd. 329).

## Summaries of Valuations at Initiation of New Scheme.

RATE OF INTEREST	4 PER-CENT			5 PER-CENT		
	I	III	IV	I	III	IV
Salary Scale						
<i>Present Value of</i>	£	£	£	£	£	£
Future service :—						
4% Returns on With- drawal ... ..	256	540	568	240	510	540
8% Returns on Death	83,560	142,172	117,224	69,056	117,384	96,656
$\frac{1}{100}$ Pensions ... ..	1,219,337	2,011,646	1,584,619	907,727	1,497,697	1,179,940
Total (future service) ...	1,303,153	2,154,358	1,702,411	977,023	1,615,591	1,277,136
$\frac{1}{100}$ Pensions (back ,, )	1,061,755	1,751,116	1,378,721	894,628	1,475,478	1,161,698
Total Benefits ... ..	2,364,908	3,905,474	3,081,132	1,871,651	3,091,069	2,438,834
"Minimum" Contributions*	1,301,678	2,030,311	1,487,287	899,017	1,389,482	1,010,492
Initial Deficiency :—						
(i) including Back- service ... ..	1,063,230	1,875,163	1,593,845	972,634	1,701,587	1,428,342
(ii) excluding Back- service ... ..	1,475	124,047	215,124	78,006	226,109	266,644
Conversion of Deficiency.						
(a) <i>Perpetuity</i> :—						
Annual Payment—						
(i) Future service ...	nil.	5,000	8,600	3,900	11,300	13,300
(ii) Back service ...	42,500	70,000	55,100	44,700	73,800	58,100
Total ... ..	42,500	75,000	63,700	48,600	85,100	71,400
% Salaries :—						
(i) Future service ...	nil.	·39	·78	·53	·87	1·20
(ii) Back service ...	5·73	5·40	4·97	6·03	5·70	5·24
Total ... ..	5·73	5·79	5·75	6·56	6·57	6·44
(b) <i>40 year term</i> :—						
Annual Payment—						
(i) Future service ...	nil.	6,100	10,600	4,400	12,800	15,200
(ii) Back service ...	52,700	86,700	68,300	50,900	83,900	66,000
Total ... ..	52,700	92,800	78,900	55,300	96,700	81,200
% Salaries—						
(i) Future service ...	nil.	·47	·96	·59	·99	1·37
(ii) Back service ...	7·10	6·70	6·16	6·87	6·48	5·95
Total ... ..	7·10	7·17	7·12	7·46	7·47	7·32
"Minimum" Contributions*	11·44	10·47	9·28	8·80	7·97	7·00

35. It will be observed that under Scale I (4 per-cent) the "minimum" contribution would suffice to cover all liabilities except back-service,\* but that this relation ceases to obtain

\* The approximate balance of "minimum" contributions and future service liabilities is to a certain extent fortuitous, being due to the decrease in the rate of contribution for several ages above the youngest age.

where a flatter salary scale is adopted or the rate of interest is increased. On the other hand the relative strain of back-service is less where the salary scale is flattened. Taking future service and back-service together the figures indicate, at any rate in the illustration taken, that the annual payment, required to support the total initial deficiency in perpetuity or to liquidate it within a term of 40 years, if expressed as a percentage of salaries, is fairly constant whichever salary scale is assumed, and that the percentage is not much affected by the increase of the rate of interest.

36. The foregoing consideration would, at first sight, suggest that an actuary called upon to advise as to the initiation of a scheme at the present time would not be unduly troubled as to the ultimate shape of the salary scale. This, however, is unfortunately not the case since, as has been pointed out above (para. 19), the "minimum" contribution is governed to a very marked extent by the particular scale adopted. Further, the present uncertainty as to the future position of superannuation funds in regard to income tax—the recommendations of the Royal Commission being of special importance to a new fund—makes the choice of the rate of interest to be assumed a matter of great difficulty. A few weeks ago the initiation of a new scheme for the benefit of the employees of a large firm was widely reported in the Press; it would be interesting to know how the difficulties to which I have drawn attention were overcome, especially having regard to the fact that the newspaper reports emphasized the point that the calculations had been based upon the post-war rate of wages.

#### CONCLUSION.

37. In bringing this paper to a conclusion I am conscious that I have touched upon only a few of the disturbing influences which affect the actuarial position of pension funds at the present time. I have, for example, ignored the problems arising in the case of existing pensioners, on whom the high cost of living presses very hardly. To meet such cases of hardship proposals will doubtless be made with a view to increasing the existing pensions, and the question will arise from what source the additional liability is to be met. Other subjects, to which Mr. Marks drew attention in his address two years ago (*J.I.A.*, vol. li, p. 202 *et seq.*), and on which those who follow in the discussion may wish to express their views, are the effect of the high

death rates suffered by the members of military age during the war, the possible strain to be expected in respect of early retirements in the future in the case of invalided soldiers, and the effect of the postponement of superannuation which may have often occurred among the older members during the war.

38. In the preparation of these notes I have received great help, which I have pleasure in cordially acknowledging, from several of my colleagues in the Government Actuary's Department, more especially from Mr. F. A. A. Menzler, F.I.A.; without their assistance this paper would never have seen the light.

#### ADDENDUM.

##### PART I.

##### CIVIL SERVICE PENSIONERS' EXPERIENCE, 1904-1914.

1. Every four years a return is presented to Parliament giving a list of Superannuation and Retired Allowances to Civil Servants, published as an appendix to the Estimates for the Civil Services (Class VI, Vote 1, Superannuation and Retired Allowances); for example, the Return appended to the Estimates for 1908-1909 related to pensions in force on 30 November 1907, *i.e.*, the end of the preceding pension year.\*

The particulars given for each Department (with certain exceptions) are as follows :—

- (i) Name.
- (ii) Rank.
- (iii) Age at retirement.
- (iv) Period of service.
- (v) Salary at retirement.
- (vi) Yearly allowance granted.
- (vii) Cause (age, ill health, &c.).
- (viii) Date of commencement.

In addition to particulars of Superannuation Allowances similar information is given relating to Compensation Allowances and Compassionate Allowances, payable for example where an office has been abolished. The present enquiry was, however, limited to Superannuation Allowances payable to men where retirement took place (*a*) before age 60 owing to ill-health, or (*b*) at or after age 60 owing to age. Further, certain services, *e.g.*, Consular and Diplomatic Services, Prison Warders, in whose case special

\* The financial year in the case of Estimates runs from 1 April to 31 March; the pension year from 1 December to 30 November.



circumstances obtain, were excluded from the investigation ; the data relating to women was too small to investigate and was omitted.

2. To bridge over the intervals between the four-yearly Returns described above, recourse was had to the particulars given annually in the Estimates of increases and decreases of commitments during the preceding pension year. The details given in the Annual Estimates are as follows :—

(i) Department.

(ii) Name.

(iii) Rank,

and in the case of new pensions,

(iv) Age at retirement.

(v) Period of service.

(vi) Salary at retirement.

(vii) Cause (age, ill-health, &c.).

(viii) Yearly allowance granted.

In the case of cessations the cause and the amount of allowance only are given in addition to (i)–(iii). It is unfortunate that in the case of cessations the date of death is not stated ; in the absence of this information an average date (31 May) had to be assumed. In a few instances (18 in number) pensions ceased by being unclaimed ; under the circumstances these were included among the deaths. Again, in the case of new grants the annual statement does not give the date of retirement ; in most cases this was obtainable from the next ensuing Return, but in a few cases where retirement and death both took place between two successive Returns average dates of each had to be assumed.

3. The period of the investigation was limited to the last ten years before the war, the survivors being brought into the experience on the anniversaries of their retirement in the year 1903–04. The close of observation was similarly taken as the anniversary of retirement in 1913–14, new entrants being exposed for integral years from the date of retirement. In the absence of information as to dates of birth, the ages at retirement were taken as ages last birthday. This assumption is probably sufficiently accurate even in the case of retirements due to age, since it is not unusual for superannuation to be deferred until the anniversary of entry is reached, the pension being based on the number of completed years.

4. A summary of the data in 5 year groups together with the crude death rates is as follows :—



Age attained last Birthday	AGGREGATE YEARS OF PENSION		EXCLUDING FIRST 5 YEARS OF PENSION		EXCLUDING FIRST 10 YEARS OF PENSION		CRUDE DEATH RATES		
	Exposure	Deaths	Exposure	Deaths	Exposure	Deaths	Aggregate Rate	Rate excluding first 5 years	Rate excluding first 10 years
30-34	17	1	...	...	...	...	·059	...	...
35-39	73	4	27	...	9	...	·055	...	...
40-44	172	13	71	4	35	3	·076	·056	·086
45-49	297	21	169	9	56	1	·071	·053	·018
50-54	596	35	322	9	176	5	·059	·028	·028
55-59	984	47	557	16	313	8	·048	·029	·026
60-64	2,806	103	680	30	389	15	·037	·044	·039
65-69	3,911	188	1,672	94	429	25	·048	·056	·058
70-74	2,679	189	2,502	174	967	75	·071	·070	·078
75-79	1,595	162	1,519	152	1,285	129	·102	·100	·100
80-84	799	99	780	96	705	83	·124	·123	·118
85-89	278	59	274	59	255	56	·212	·215	·220
90-94	55	14	55	14	51	13	·255	·255	·255
95 or over	17	7	17	7	17	7	·412	·412	·412
Total	14,279	942	8,645	664	4,687	420	...	...	...

5. For comparison with other tables an aggregate mortality table was first prepared and graduated graphically; the graduated values at every fifth age are shown in the following table together with the corresponding figures in other tables.

Age	Retired Civil Servants	Scottish School Teachers	Elementary School Teachers (Deferred Annuity Fund)†
30	·120	...	...
35	·084	·100	·182
40	·075	·079	·149
45	·069	·065	·116
50	·063	·055	·096
55	·052	·049	·092
60	·039	·030	·083
65	·040	·033	...
70	·064	·052	...

\* Appendix B, Table III, Actuarial Report to the Scotch Education Department on the preparation of the Superannuation Scheme for Teachers, 1911. (Cmd. 5982).

† Aggregate rates deduced from the Table on p. 14 of Report on Superannuation of Teachers, 1914 (Cmd. 7364).

It will be observed that the new rates are usually somewhat greater than, but agree generally with, the rates of pensioners' mortality adopted by Sir George Hardy and Mr. King in their Report to the Scotch Education Department on the preparation of the Superannuation Scheme for Teachers. In their Report (para. 18) it is stated that the rates "were derived by comparing "the rate of mortality amongst the teachers who had received "disablement allowance with the rates prevailing amongst "Elementary School Teachers, and with the rates of mortality "amongst pensioners in some other large funds . . . ." It would appear from the foregoing extract and from the corresponding figures deduced from the Elementary School Teachers Deferred Annuity Fund that the rates in question were considerably lighter than the actual experience of the past would have warranted. I gather, however, from the several reports on the Deferred Annuity Fund that the breakdown pensions in force, before the improved scheme was adopted, were so small that teachers were tempted to remain on the active list even after they were really too ill to work. The heavy rates of mortality due to these special circumstances are not, therefore, comparable with those deduced from a more normal experience. I return to this point later in comparing the select rates taken from the same investigation.

6. The next stage in the Civil Servants' investigation was the preparation of select and ultimate tables. It was found that in the case of men superannuated owing to ill-health the selection worked off in about 5 years, similarly the selection of pensioners retiring at 60 or over continued for about the same period, the two ultimate tables coalescing near age 70. Separate ultimate tables were accordingly adopted for the after 5 years experience of the invalid pensioners and age-pensioners respectively running into a combined table at age 70. A comparison of the actual deaths with the expected deaths according to the graduated ultimate tables is as follows :—

Age Group	Expected Deaths	Actual Deaths	Difference		Accumulated Difference	
			+	-	+	-
35-39	1.9	...	...	1.9	...	1.9
40-44	3.8	4	.2	...	...	1.7
45-49	6.3	9	2.7	...	1.0	...
50-54	9.3	9	...	.3	.7	...
55-59	17.6	16	...	1.6	...	.9
60-64	28.3	30	1.7	...	.8	...
65-69	90.6	94	3.4	...	4.2	...
70-74	183.6	174	...	9.6	...	5.4
75-79	145.1	152	6.9	...	1.5	...
80-84	104.0	96	...	8.0	...	6.5
85-89	50.9	59	8.1	...	1.6	...
90-94	14.8	14	...	.8	.8	...
95 and over	6.3	7	.7	...	1.5	...
	662.5	664	23.7	22.2	...	...

7. The select rates during the first five years after retirement were subsequently taken out, and graduated values running into the smooth ultimate tables were obtained by the graphic method.

8. As far as I am aware the only other published table of select mortality of invalid pensioners is that deduced by Sir George Hardy and Mr. Fraser from the experience of the Elementary School Teachers' Deferred Annuity Fund.\* For comparison the respective rates of mortality at quinquennial ages are shown in the following table:—

*Select Mortality Rates (Invalid Pensioners).*

Age at retirement [ $x$ ]	CIVIL SERVANTS					
	$q_x$	$q_{x+1}$	$q_{x+2}$	$q_{x+3}$	$q_{x+4}$	$q_{x+5}$
30	.120	.096	.087	.084	.082	.080
35	.118	.094	.083	.075	.069	.065
40	.115	.086	.067	.056	.049	.044
45	.111	.083	.059	.046	.038	.032
50	.131	.091	.060	.046	.036	.028
55	.149	.100	.066	.052	.042	.036
	MALE ELEMENTARY SCHOOL TEACHERS (HARDY AND FRASER)					
	$q_x$	$q_{x+1}$	$q_{x+2}$	$q_{x+3}$	$q_{x+4}$	$q_{x+5}$
30	.200	.160	.146	.140	.138†	...
35	.197	.157	.138	.126	.118	...
40	.192	.145	.116	.095	.081	...
45	.184	.134	.097	.072	.057	...
50	.176	.121	.082	.058	.046	...
55	.169	.111	.073	.053	.046	...
60	.162	.105	.070	.056	.053	...

\* Report on Superannuation of Teachers, 1914 (Cmd. 7364). † Ultimate  $q_{x+4}$ .

As explained above (para. 5) the Teachers' and Civil Servants' experiences are not strictly comparable in view of the different conditions affecting retirements in the past, and in order to bring the two tables into line for the purpose of testing the selection the ratio of  $q_{[x]+n}$  to  $q_{[x]}$  has been taken out for comparison.

$$100 \ q_{[x]+n} \div q_{[x]}.$$

Age at retirement	CIVIL SERVANTS						ELEMENTARY TEACHERS (MALES)				
	Duration						Duration				
	0	1	2	3	4	5	0	1	2	3	4
30	100	80	72	70	68	67	100	80	73	70	69
35	100	80	70	64	58	55	100	80	70	64	60
40	100	75	58	49	43	38	100	76	60	49	42
45	100	75	53	41	34	29	100	73	53	39	31
50	100	69	46	35	27	21	100	69	47	33	26
55	100	67	44	35	28	24	100	66	43	31	27
60	...	...	...	...	...	...	100	65	43	35	33

Having regard to the marked difference between the actual mortality rates, both select and ultimate, the close similarity of the figures in the above table is remarkable.

9. Passing to the mortality experience of the pensioners who retired by reason of old age, I give in the next table the select rates for ages 60 and 65 and compare them with the corresponding rates taken from other tables.

*Select Mortality Rates (Males).*

Duration	AGE AT COMMENCEMENT 60			
	Civil Servants	Elementary Teachers	British Offices Annuitants	Government Annuitants
0	·023	*	·017	·016
1	·026	*	·022	·023
2	·029	*	·027	·029
3	·034	*	·033	·034
4	·038	*	·037	·038
Duration	AGE AT COMMENCEMENT 65			
	Civil Servants	Elementary Teachers	British Offices Annuitants	Government Annuitants
0	·033	·031	·024	·023
1	·040	·033	·032	·033
2	·047	·036	·040	·041
3	·053	·039	·048	·048
4	·059	·043	·054	·055

\* In the Teachers' experience the rates at age 60 relate to invalid lives and are not comparable, but will be found in the table in para. 8.

As would be expected the above table indicates that a purchaser of an annuity is more select than a civil servant who retires on the grounds of age. It is probable too that the apparent closeness of the pensioner's rates (Civil Servants) with the British Offices and Government Annuitants tables is somewhat illusory. In comparing civil servants in 1904-1914 with Government annuitants 1875-1904 and British Offices' annuitants 1863-1893, the progressive improvement in mortality has, of course, to be borne in mind; and it is not unlikely that an up-to-date experience of annuitants would show rates of mortality appreciably lighter than those of the Civil Servants' table.

10. A similar comparison of the ultimate rates is shown below. In the case of the Teachers' experience separate tables were given for lives who retired disabled before attainment of age 65, and for those who were superannuated at that age. A single table from age 70 onwards embraces both classes in the Civil Service investigation, the experience not being sufficient to warrant separate tables.

*Ultimate Mortality Rates (Males).*

Age Attained	Civil Servants	ELEMENTARY TEACHERS		British Offices Annuitants	Govt. Annuitants
		Grantees of Disable- ment Allow- ances	Grantees of Super- annuation Allowances		
70	·064	·068	·046	·059	·060
75	·084	·090	·074	·087	·088
80	·114	·129	·118	·129	·129
85	·161	·190	·180	·190	·190
90	·232	·275	·262	·276	·275
95	·348	·390	·371	·392	·390
100	·531	·534	·507	·535	·534

It is probable that the comparison of the new table with the annuitants' tables is vitiated for the reasons advanced in the preceding paragraph. At the older ages the Civil Service rates are generally lower than those in either of the Teachers' tables, but having regard to the small numbers exposed it is not, I think, wise to lay much stress on this feature.

11. The following tables give the graduated mortality rates for each section of the experience :—

CIVIL SERVICE PENSIONERS' EXPERIENCE, 1904-1914 (MALES).

*Rates of Mortality.\**

Age at Retirement [ <i>x</i> ]	YEARS ELAPSED SINCE RETIREMENT						Age attained <i>x</i> + 5
	0	1	2	3	4	5 or more	
	<i>q</i> [ <i>x</i> ]	<i>q</i> [ <i>x</i> ]+1	<i>q</i> [ <i>x</i> ]+2	<i>q</i> [ <i>x</i> ]+3	<i>q</i> [ <i>x</i> ]+4	<i>q</i> [ <i>x</i> ]+5	

(a) Invalid Pensioners.								Both Classes.	
30	·120	·096	·087	·084	·082	·080	35	Age attained <i>x</i>	<i>q</i> <sub><i>x</i></sub>
1	·120	·096	·087	·083	·081	·078	6		
2	·120	·096	·086	·082	·079	·076	7		
3	·119	·095	·085	·080	·076	·073	8		
4	·119	·095	·084	·078	·073	·069	9	70	·064
								1	·068
35	·118	·094	·083	·075	·069	·065	40	2	·072
6	·118	·093	·081	·072	·065	·060	1	3	·076
7	·117	·092	·078	·068	·061	·056	2	4	·080
8	·117	·090	·074	·064	·057	·052	3		
9	·116	·088	·070	·060	·053	·048	4	75	·084
								6	·088
40	·115	·086	·067	·056	·049	·044	45	7	·093
1	·114	·085	·064	·053	·046	·041	6	8	·099
2	·113	·084	·062	·050	·043	·038	7	9	·106
3	·112	·083	·061	·048	·041	·036	8		
4	·111	·083	·060	·047	·039	·034	9	80	·114
								1	·122
45	·111	·083	·059	·046	·038	·032	50	2	·131
6	·113	·084	·059	·046	·037	·031	1	3	·140
7	·116	·085	·058	·045	·036	·030	2	4	·150
8	·120	·086	·058	·045	·035	·029	3		
9	·125	·088	·059	·045	·035	·028	4	85	·161
								6	·173
50	·131	·091	·060	·046	·036	·028	55	7	·186
1	·138	·094	·062	·047	·037	·029	6	8	·200
2	·145	·097	·064	·049	·039	·030	7	9	·215
3	·150	·100	·066	·051	·040	·032	8		
4	·152	·102	·067	·052	·041	·034	9	90	·232
								1	·251
55	·149	·100	·066	·052	·042	·036	60	2	·273
6	·143	·097	·064	·051	·043	·038	1	3	·296
7	·134	·093	·062	·051	·044	·040	2	4	·321
8	·122	·087	·060	·050	·045	·043	3		
9	·107	·080	·057	·049	·046	·045	4	95	·348
								6	·378
60	...	...	...	...	...	·048	65	7	·411
1	...	...	...	...	...	·051	6	8	·447
2	...	...	...	...	...	·054	7	9	·487
3	...	...	...	...	...	·057	8		
4	...	...	...	...	...	·061	9	100	·531

(b) Age Pensioners.							
60	·023	·026	·029	·034	·038	·042	65
1	·024	·028	·032	·037	·041	·046	6
2	·026	·030	·035	·040	·045	·051	7
3	·028	·033	·038	·044	·050	·055	8
4	·030	·036	·042	·049	·054	·059	9
65	·033	·040	·047	·053	·059	·064	70

\* NOTE.—In the above table the rates are carried down to age 30, but it must be remembered that at the younger ages the number of deaths was not sufficient to indicate definite features, and the rates should, accordingly, be taken



## PART II.

## VALUATION OF PENSION LIABILITIES BY SELECT TABLES.

12. It will be of interest to examine the effect of using the new select table in place of the aggregate table based on the same experience. At 4 per-cent interest the annuity values are as follows :—

*Retired Civil Servants, 1904-14.*

*Select and Aggregate Annuity-Values, 4 per-cent.*

Age	$a_{[x]}$	$a_x$	Difference	
			+	-
30	8.14	7.35	.79	...
35	9.04	8.29	.75	...
40	10.00	8.85	1.15	...
45	10.21	9.37	.84	...
50	9.38	9.85	...	.47
55	8.28	10.08	...	1.80
56	8.23	10.06	...	1.83
57	8.22	10.01	...	1.79
58	8.25	9.92	...	1.67
59	8.32	9.79	...	1.47
60	9.89	9.62	.27	...
61	9.59	9.41	.18	...
62	9.28	9.16	.12	...
63	8.97	8.88	.09	...
64	8.67	8.58	.09	...
65	8.35	8.26	.09	...

The select values rise steadily, reaching a maximum at age 44, and then decline until the end of the invalid section of the table. A sudden rise is shown at age 60 when age retirements commence. The aggregate values at the youngest ages are considerably below the select values owing to the preponderance of new invalids experiencing a heavy death rate in the first few years after retirement. The aggregate curve rises continuously to age 55, passing the select curve between age 47 and 48, and during the last few years of the invalid section the aggregate values are more than 20 per-cent in excess of the select values, being distorted by the coming influx of lives retiring under the age rules at 60 or later. The sudden jump in the select table now reverses the position and the aggregate annuity is nearly 3 per-cent smaller at age 60, but this difference gradually narrows until the two tables coalesce at age 70.

13. Assuming the service rates of mortality, &c., shown in paragraph 16 of the paper and the pre-war salary scale, the cost

of a pension based on sixtieths of salary (without limit) is represented, at 4 per-cent interest, by the undermentioned percentages of salary in the case of an entrant at age 18 :—

Percentage of Salary required as Contribution	PENSION BASED ON	
	Final Salary	Average Salary
Aggregate Table ... ..	10·92	6·67
Select Table ... ..	10·72	6·56
Decrease ... ..	·20	·11

14. Again, applying the new tables to a hypothetical fund (based on the service table) which is assumed to be recruited by 100 entrants a year at age 18 and to have reached a stationary condition of 3,673 members on the active list, I find that the value of the contingent liabilities for pensions (based on final salary) is decreased from £3,392,650 to £3,342,850, or  $1\frac{1}{2}$  per-cent by substituting the select table for the aggregate. The corresponding amounts where pensions are calculated on average salaries are £2,087,300 and £2,059,000 respectively. In each case the aggregate table produces larger liabilities up to age 57 and smaller thereafter; for example, in the case of pensions based on final salary the present value of the aggregate liabilities is as follows :—

	Under age 57	Over age 57	All ages
Aggregate Table ...	£2,852,500	£540,150	£3,392,650
Select Table ... ..	£2,796,300	£546,550	£3,342,850
Difference ... ..	+ 56,200	— 6,400	+ 49,800

15. The use of select annuities may possibly be considered of somewhat theoretical interest only in the example cited, *i.e.*, where all members enter at the same age. The invalid retirements are assumed to take place before age 60, and all pensions taken above that age are deemed to be awarded to men retiring on the ground of age. No overlapping of the two types of retirement thus arises.

16. The position, however, would be different if retirement between 60 and 65 was limited to cases of ill-health in certain sections of the membership. Take the case of a fund consisting of two classes of members, entering at age 20 and at age 25

respectively, and assume that retirement is not allowed until after 40 years' service except where a breakdown in health occurs. In the former group the invalid table would apply to retirements under age 60 and the age table thereafter. This table would not be appropriate to the latter group until age 65, and lower annuity values based on an invalid table should properly be used to measure the cost of pensions taken if retirement occurs between ages 60 and 65.

17. A similar point arises in connection with the rates of retirement assumed at the older ages where the conditions governing superannuation are not the same. In a recent case, which came before me, it was found necessary to vary the rates according to age of entry in order to obtain a true measure of the respective contributions required from different classes. The additional work involved was not, however, serious, as it appeared that sufficient accuracy could be secured by adopting one table for invalid retirements to age 59, and thereafter select tables.

18. The main defect of an aggregate table appears to me to be that the rates are unduly disturbed by the weight of new retirements, and that the comparison of one aggregate table with another gives little indication of the true mortality experience unless it is known that the retirement rates are fairly similar. It would be possible, for example, to have two experiences showing identical select rates of mortality but widely divergent aggregate rates. This point is of special importance at the initiation of a new scheme, when the experience of other funds has often to be borrowed.

In the examples given above in the paper the calculations are based on the select table in all cases.

#### APPENDIX.

*Memorandum prepared by the Board of Inland Revenue at the request of the Royal Commission on the Income Tax with regard to the position of Superannuation Funds of Public Companies and Corporations in relation to the Income Tax.*

[Reprinted from Parliamentary Paper (Cmd. 288-7), by permission of the Controller of His Majesty's Stationery Office.]

#### I. BRIEF STATEMENT OF POSITION.

1. The ordinary Superannuation Fund is built up of :—

- (a) contributions by the employer ;
- (b) contributions by the employee ;
- (c) interest or dividends on the invested funds.

There are, however, numerous funds where either the employers or the employees make no contributions.

*(a) Contributions by the employer.*

2. These are generally regular contributions equal to, or bearing a fixed proportion to, the employees' contributions. Such contributions are allowed to be deducted, as a working expense, in computing the liability to Income Tax of the profits of the business under certain conditions, namely :—

- (1) that the amounts in question are definitely alienated by the employer and do not represent a reserve fund over which he retains control ; and
- (2) that it is provided that the employees shall ultimately benefit to the full extent of their contributions.

*(b) Contributions by the employees.*

3. The question whether or no the employees' annual\* contribution can be allowed as a deduction in assessing his earnings depends on the nature of his prospective benefit from the fund.

If that benefit is to be paid in the form of a pension or annuity on retirement (in which case it is taxable income) the contributions to the fund are allowed.

Conversely, if the benefit is to be paid in the form of a lump sum, that is, of a payment which is not taxable, the contribution to the fund is not allowed. (If it were, the employee's savings would be relieved from tax altogether.)

Where the benefit is to be paid in the form of a pension or annuity on retirement, but provision is also made for payment of a lump sum on death or retirement before pension age, it is made a condition precedent to the allowance of the employee's contribution that tax shall be accounted for to the Revenue on so much of the lump sums paid on retirement (but not on death) as represent the returned contributions of employees liable to Income Tax.

\* Initial contributions, whether of the employer or of the employee, made in order that an employee with several years' service may come into the fund on a satisfactory footing are disallowed as being in the nature of capital expenditure.

(c) *Interest or dividends on the invested funds.*

4. The income from the investments of a fund is liable to tax in full like any other income of this nature. On payment of pensions, however, the fund is entitled to deduct tax at the full rate from the pensions\* and (so far as the pensions are paid out of the investment income of the funds which has borne tax) to retain the tax so deducted. Thus it suffers tax, if at all, only on its undistributed income.

5. In the case of funds which have been in existence for a number of years, and have not substantially increased their membership, the pensions either exceed the taxed investment income or approximate to it; correspondingly, the tax deductible from the pensions either exceeds or approximates to the tax paid on the investment income, with the result that little or no tax falls upon the fund. (This represents the position of most of the railway funds.)

6. In the case of funds which have only been in existence a short time or have increased their membership substantially, the converse holds good; that is to say, the investment income exceeds the pensions paid and the tax on the excess falls upon the fund.

## II. STATEMENT AS TO RELIEF SOUGHT BY THE FUNDS.

7. As just explained, the Superannuation Funds—

- (a) pay Income Tax at the full rate on all their incomes from investments; and
- (b) afterwards recoup themselves to the extent of tax at the full rate on so much of that income as is distributed in pensions.

In other words, the funds ultimately bear tax at the full rate on their undistributed income. This is the subject of their complaint.

\* Where a fund has deducted tax from a pension at the full rate, the pensioner can of course claim any repayment to which he may be entitled by reference to the amount of his total income. In order to save separate claims for repayment by the individual pensioners, the Inland Revenue Department has made an arrangement with many pension funds under which the fund deducts tax from the pensions only to the extent of the pensioner's net liability, the amount of which is notified by the Department to the fund. The Department subsequently repays or allows to the fund an amount representing the difference between the tax at the full rate of the amount of taxed income distributed in pensions and the aggregate liability of the pensioners in respect of their pensions.



8. The funds seek relief from the tax in whole or in part, putting forward as reasons—

- (a) that they encourage thrift, and that certain other funds which do the same are accorded exemption ; and
- (b) that their individual members have small incomes and consequently are either exempt or liable at the lower rates of tax only.

9. The burden of the complaint is that the effect of the present high rates of Income Tax is to diminish the accumulations to such an extent that the funds may ultimately become insolvent, thereby necessitating either increased contributions by the employer or by the employee or both. On the other hand, recently established Superannuation Funds, which are those mainly affected by the burden of tax on undistributed income, are making additional investments as their funds increase, and on these new investments they are earning a higher gross rate of interest than that anticipated at the time of inception of such funds.

10. As regards 8 (b), the various kinds of relief which the Income Tax Acts provide for individuals\* by reference to the amount of their total income do not extend to any bodies or associations of persons.

11. As regards 8 (a), there is no general relief from Income Tax in favour of funds which encourage thrift.

There are, as the Superannuation Funds point out, exemptions in favour of registered Friendly Societies, savings banks, and Co-operative Societies. As will be seen, however, these exemptions are very limited in character. It has, of course, to be borne in mind that in each case the member or depositor whose income exceeds £130 is liable to direct assessment in respect of any income of a taxable nature which he receives in full from the society or bank.

12. *Registered Friendly Societies* are exempted from tax if they do not assure more than £300 or grant annuities in excess of £52 a year. The limit attached to this exemption seems to indicate that the intention was to confine it to societies all or practically all of whose members are exempt. The exemption certainly could not be justified if the limiting conditions were removed.

\* The only bodies to which a similar relief (*i.e.*, a relief by reference to total income) is granted are *unregistered* Friendly Societies. There is a specific exemption in favour of such societies, where their income does not exceed £160.



The special Income Tax provisions relating to *savings banks* do not affect their undistributed income except so far as it is derived from investments with the National Debt Commissioners. These investments are only made by the banks' "general departments" in which the limit for deposits is £200. This exemption is, therefore, similar to the last, being based on the assumption that small depositors would not be liable to Income Tax.

The general exemption of the banks' funds under Schedules C and D is expressly confined to the income "applied in the payment or credit of interest to any depositor." This does not give the savings banks any advantage over the Superannuation Funds.

13. The exemption under Schedules C and D in favour of *Co-operative Societies* is not very closely in point in connection with the question of thrift funds. A better analogy could be drawn with the Life Insurance companies who have in fact raised a similar claim. These companies build up large reserves out of their investment income in order to meet future liabilities. The policyholders who are eventually to benefit are for the most part people liable to the lower rates of Income Tax, but the undistributed income of the companies is, nevertheless, taxed at the full rate.

14. The question of the incidence of Income Tax on Superannuation Funds is merely a part of the general question of the treatment for taxation purposes of undistributed income of various classes of public and private bodies, and is one of the matters for the investigation of the Royal Commission.

#### ABSTRACT OF THE DISCUSSION.

MR. R. G. MAUDLING said that the great change in the level of wages had placed existing superannuation funds in a very serious position, and Mr. Epps had shown approximately what the effect of the change would be. He (the speaker) had had to deal mainly with railway superannuation funds, and in the absence of any reliable statistical basis he had found considerable difficulty in forming a new salary scale. From consideration of the scale of salaries agreed between the railway companies and the Railway Clerks' Association, in the course of the settlement which took place about August 1919, he had constructed a scale which showed very much the same ratio of increase as the existing scale, and in this respect it was like Mr. Epps's scale No. 2. But there were some special features of the settlement which materially affected the scale.

With regard to depreciation of investments in pension funds he thought that unless the rate of interest were raised accordingly, a serious hardship would fall upon existing members. As to the general treatment of depreciation, he might be accused of heresy, inasmuch as he could not bring himself to write securities down to their present market value irrespective of the actual description of the security. He preferred to make a reserve for possible loss on realization, and to give considerable weight to the fact that securities might ultimately recover. If the capital had not been lost and was repayable at a definite date, there was really not much need for writing down. He thought also that account might be taken of the fact that with the increased salaries some of the superannuation funds would rapidly increase. The money that was obtained thereby could now be invested at a very high rate of interest, and he would be inclined to create a reserve fund out of existing earnings in order to provide for possible loss on old investments. This must involve the immediate removal from the balance sheet of any security that was irretrievably bad. One would have to take very drastic steps with some Harbour Trust investments, for instance. Otherwise he was not inclined to give much credit to present market values, because they were more or less fictitious, being the values which a man who was compelled to realize was prepared to take, and he thought it would be extremely unwise to write everything down to that level.

He hoped that the Institute would be able to make its influence felt on the subject of income-tax. Life assurance companies obtained relief in respect of expenses and the least that superannuation funds might expect would be to have such relief as would place them in as good a position as life assurance companies.

In some cases contributions were charged on bonuses as well as on salary. It seemed to him that in such cases if the pension were based on final salary—which might be no more than the present salary and bonus—possibly the men who had been paying for the last two or three years would get no return for their extra payments.

Passing to the question of valuation reserves, he had taken out some figures for railway superannuation funds based on average salary. If the reserves on the basis of old salaries at 4 per-cent were taken as £1,000, on new salaries at 4 per-cent they would be £1,113; at 5 per-cent on the basis of new salaries the reserve was only £806—that is, a reduction of 20 per-cent on the figure for the old salaries at 4 per-cent.

In discussing possible methods of putting the funds into a solvent position, the author of the Paper gave the uniform percentage reduction of benefits that would be necessary, but presumably, he would not be in favour of that method. It would be distinctly inequitable to apply a uniform reduction of benefits all round. The ingenious method of dealing with the deficiency by means of varying the pension, or rather the amount of salary allowed as pension in respect of future service and past service, might be a very valuable method in many cases.

He would conclude with one or two remarks on the Civil Service Pensioners' Experience appended to the paper. Were any additional climatic risks included in this experience? The India Office was of course included, but did the Colonial Office come in? This would have some bearing probably on the heavy rate of mortality in the first year after retirement in the case of those members who retired before 60. On the whole the rates of civil servants compared fairly closely with those of elementary school teachers (male), but the practical value of the adoption of select tables did not seem to be very great. The differences were comparatively small, and perhaps would hardly be such as to justify the adoption of select tables, even if there were a sufficient volume of data to enable them to be constructed.

MR. D. C. FRASER said there was one point of particular personal interest to him in the paper because Mr. Epps had referred to an investigation in which he (the speaker) had the great privilege of being associated with Sir George Hardy. He well remembered the circumstances in which the table mentioned by Mr. Epps was produced. It was the table giving the select rates of mortality for disabled male teachers, and he would like to say that the full weight of Sir George Hardy's name and experience could be attached to that table. In the winter of 1913, Sir George Hardy was ordered to France, and he (the speaker) was in communication with him and kept him informed of what was going on, communicating figures and crude results to him, and just before Christmas he received a communication from Sir George containing the full graduation of the mortality rates of male disabled teachers, with diagrams showing expected and actual deaths, and a complete table which could be printed just as it stood. That communication was one of his treasured possessions. Mr. Epps had referred to the heavy rates of mortality shown in the table. They represented the experience almost exactly. It was a very noticeable feature that while the mortality of male disabled teachers was extremely heavy, the mortality of female disabled teachers was very light in comparison. The circumstances of the two sexes were apparently quite different as regards disablement. It seemed that the male disabled teachers did not retire unless and until they were absolutely compelled to do so; the female teachers gave way more easily and retired to more congenial conditions, domestic life coming more naturally to women than to men.

Mr. Epps had made a very useful comparison of the effect of using select and aggregate rates of mortality for Pensioners. In his own practice he had been in the habit of giving approximate effect to selection by using a uniform rate of mortality over a wide range of ages for the first year of superannuation, another uniform rate for the second year, and so on up to a duration of perhaps five years, after which an ordinary aggregate table of mortality would be employed. Mr. Epps had, however, shown that an aggregate table might be employed at all durations without much error, provided the aggregate experience on which it was based had been sufficiently

extensive. Where the danger lay was in using an aggregate table in which the ultimate experience had not had time to show its full effect, with the result of giving undue weight to the heavy mortality which was a usual feature of the early years of superannuation.

The memorandum which Mr. Epps had printed as an appendix on the subject of income tax was very valuable in this convenient shape. The subject was one which was particularly perplexing at the present time, and he would like to put in a very strong plea for the question to be definitely settled one way or the other. When dealing with pension funds one was continually having it brought to one's attention that the Royal Commission had unanimously recommended that pension funds should be relieved from the burden of income tax, and the actuary had regretfully to point out that he could not give effect to recommendations which had not become law. With regard to Mr. Maudling's suggestion that pension funds should be put on the same footing as insurance companies, the deduction of expenses for the purpose of calculating the income tax would give a very trifling relief indeed to pension funds, and he hoped that effect would be given to the recommendation that interest on pension funds, as interest, should be relieved from tax. At the present time that would be a most valuable measure in the way of encouraging and stimulating the formation of new funds.

On the question of depreciation and interest his own disposition was to give full effect to depreciation, and also to the higher rate of interest which one would then be justified in assuming. With regard to negative values, he had one case of a fund, which was hopelessly insolvent as it was brought to him, but in which very drastic arrangements had been made by high rates of contribution to put the fund in a proper position. After full investigation of the circumstances, and being satisfied on the point that membership of the fund was compulsory by statutory arrangements and therefore that the contributions were practically guaranteed, he had no hesitation in retaining negative values in the fund. With reference to the use of select rates in valuations, he agreed that there were very few cases in which this was practicable.

Mr. Epps had apparently contemplated the alarming possibility of benefits being reduced, but he (the speaker) thought that the influences which had brought about the great changes in salaries and salary scales were in most cases quite strong enough to secure that the safety of the fund should not be brought about by the method of reducing benefits. The companies and the organizations which had had to increase salaries could not turn round and say that they would reduce pension benefits. In his experience the tendency was to increase pensions and to bring in new benefits. The one method of putting the funds in a sound financial condition was the method to which one was accustomed when dealing with new funds—namely, by the provision of either a capital sum or an equivalent annual charge.

MR. J. BACON said that he would like to emphasize the point which the author made that he was dealing with only a portion of



the whole problem. The difficulty with which the actuary was faced in valuing a pension fund was not so much the effect of a change in the salary scale as of how to deduce the salary scale. He had one case recently in which no fewer than 80 changes in the way of bonus addition had occurred between 1917 and 1920. All these bonus additions counted for contribution to pension. The pension itself was based on final salary, and the salaries were to vary in accordance with the cost of living as determined by the Whitley scheme. Given conditions such as that, the problem was to arrive at a salary scale. Assuming that one had arrived at it—and although the Pension Fund might have a scale of increasing salary from age to age, that was not necessarily the scale to be applied in valuation—one had the effect of special promotions as a result of the war, and of members coming in at a different entry age. He wished to emphasize this in case any deductions were drawn from the curious agreement of the rate of contributions on the 5 per-cent scale with the rate of contributions on the 4 per-cent; such agreement was probably pure coincidence. None of the assumptions which Mr. Epps had made probably would apply in practice, for then instead of having a fixed addition plus a percentage, or a fixed percentage, there would usually be a percentage addition decreasing as the salary itself increased in amount, sometimes with a minimum or maximum. Average salaries were largely fallacious. Even before the war there was a tendency for prices and salaries to increase, and that point had to be borne in mind in deducing salary scales. The scale of salary was only one feature of the whole problem. During the war, rates of mortality and of retirement, of interest and of withdrawal, had all undergone very considerable changes, and these must all be taken into account.

With regard to the rate of interest, he might mention one particular case in which it would certainly be unwise if not quite wrong to take account of depreciation. There were certain funds which made their own investments, but were allowed a guaranteed rate of interest on them. If the investments were written down to market value, the rate of interest on them would have to be raised above the guarantee.

He had compared the rates of mortality which Mr. Epps had used in his paper with the pre-war rates of two railway companies, a bonus fund among railwaymen, a borough municipal fund and a voluntary fund on behalf of a number of ministers of a religious denomination, and he found that in every case the rates of mortality were considerably above those assumed in Mr. Epps's tables. Mr. Epps, for instance, in his active service table had given the rate of mortality between the ages of eighteen and thirty as '001—he gathered that was empirical—and he found the rates for those ages in the funds to which he had alluded to be from '0021 to '0029, the latter figure being nearly three times as high as the one assumed by Mr. Epps. He would like to say that he had compared in several of these cases the expected deaths during the last five years with the actual mortality, and he found that during the war

period the mortality of members on the active list, excluding the deaths on active service, was considerably above the pre-war rate.

He ventured to think that the position when pensions were based on final salary would not be nearly so bad as would appear from the paper because where there was any option at all as regards retirement, a man whose pension had to be based on past salary would be likely to defer his retirement, and they would probably find that the rates of retirement would depend upon how the pension was to be based, whether on past or average salary. That, again, would have some bearing on the rates of mortality among pensioners. He thought there was no doubt that mortality among pensioners had been accentuated to some extent by men remaining in the Service, after they should have retired. They remained during the war partly on account of pressure from employers, and partly owing to the effect of the increased cost of living, which would have made their pensions inadequate. He thought that some of these varying factors would have a compensating effect when they reached a state of equilibrium. He would suggest, in connection with municipal pension funds, that if new legislation were brought in dealing with these funds, the actuary should be allowed to certify as the necessary annual contribution not a fixed amount, but a percentage of salaries, so that there might be some relation between the contribution paid by the municipality and the liability which it was asked to assume.

In conclusion he held that the actuary, knowing that there were so many changed conditions, and that the experience of the past five years had been quite abnormal and was a very unsafe guide for the future, and being aware of the interplay between mortality, retirement, withdrawal and salary, should not burden any fund or municipality to any greater extent than he felt absolutely bound to do. Personally he would certainly not write down investments to their present market price, but would simply set aside an investment reserve and write off any deficiency there might be by means of an annuity, believing as he did that in five or ten years' time, when they had got to a state of equilibrium, the annuity set apart would probably not be required, and by that means he would have succeeded in giving a degree of safety to the fund without unduly crippling the employer.

MR. A. S. HOLNESS, referring to the experience of invalid pensioners, thought that Mr. Epps had erected a somewhat more substantial edifice than the foundation could perhaps bear. In the select table he noticed that only 83 deaths occurred at ages below 60, and that deaths in excess of 10 per quinary group occurred only at ages above 45, and even there the average was less than one death per age and year of duration. It was rather difficult from that to know how much weight to attach to the results. A comparison of actual and expected deaths during the select period would have been useful. That might have made it possible to judge whether much weight could be attached to the remarkable similarity between the rates and those deduced by Sir George Hardy and Mr. Fraser.



It seemed to him to be more probably due to the nature of the graduation than to the facts themselves. There was one interesting point in this connection: if one took the ratio of  $q_{[x]+n}$  to  $q_{x+n}$  very different results were brought out at the older ages. At the age of 35 a comparison between Mr. Epps's figures and those of Hardy and Fraser still showed a remarkable correspondence: in the first year of duration the figures were 147.5 and 147.0 respectively; but the rates became progressively dissimilar as the age increased, until at 50 Mr. Epps's figures were 409 and those of Hardy and Fraser 329, and at a later age the dissimilarity was even greater. The explanation of this peculiarity was probably that the progressions of the rates in the select table and the ultimate table were not similar, that it to say, the comparative slope at the various ages was not the same in the two tables. But at least they could say that the rates which Mr. Epps had produced did clearly show that the effect of this adverse select table was very great indeed. He was rather puzzled to account for Mr. Epps's remark that the effect of selection wore off in five years. Surely the effect of adverse selection was still persisting to a very large degree. There were one or two cases in which adverse selection occurred in life assurance practice; for example, term assurances with the option of effecting a new assurance during the term without fresh medical examination at the rate of premium for the age at conversion. The option premiums were intended to be a measure of the effect of medical selection. But there was something further to be taken into account: in his experience the number of cases in which the option was exercised was comparatively small, and one might safely say that where the life at the time when the option could be exercised was very bad it would certainly be exercised. The effect of this was that the cases which were converted contained an unusually large number of very bad lives. On the other hand, as a set-off against the mortality loss in these cases there was a profit in respect of the option premiums which remained in the fund in the case of all those who did not exercise the option, but dropped their policies. The result would depend upon the relative amounts of this profit and the mortality loss. He could not say that the system was unprofitable, but if it were profitable it was more a matter of chance than of the premium being on a scientific basis.

MR. S. G. WARNER said that the question how depreciation of securities should be dealt with was one of considerable difficulty, and there seemed to be rather varied opinions upon it. The view he had been disposed to take was on the whole that depreciation should be severely dealt with, but that there should be little hesitation about using something nearly approaching the rate of interest which that depreciation revealed as having been secured. It depended very largely of course on the kind of investment and the causes of depreciation, and of those two points the second was the more difficult to assess.

It was easy to say they must assume the investments to be good and sound. But then came the question why it was that they

were at such a low level. That might arise from obvious causes such as the condition of the money market and the price of money, and it might arise from uneasiness and foreboding as to the future. The latter might be particularly the case at the present moment with the majority of railway securities. Still, it might be considered probable that anyone who would not write off depreciation, but would set up a reserve fund instead, would have a fair conviction that a good deal of the lost value was going to return, and that the depreciation was related not so much to the intrinsic value of the security as to the results of general financial conditions which it might be hoped would be fairly temporary. In that case and on those assumptions it certainly had seemed to him that the best and most straightforward course was to face the facts as they stood and not take credit on the one hand for values which they knew were not at the moment obtainable, and set up on the other a reserve the adequacy of which no one examining the account had any means of ascertaining. An objection had been made on the ground that valuation was a thing for all time. But he did not see why the rate of interest should not be quite elastic, varying with money market conditions, just as depreciation of securities varied. If they had securities which were redeemable at fixed rates, and were certain that those securities would be paid in full, and that they were depreciated owing to temporary conditions, then if they took credit for market values in the balance sheet, that automatically gave them a higher rate of interest. As between the two suggestions (1) declining to write down the securities and to face their present depreciation on the assumption that it was temporary and setting up a reserve fund instead of writing them down, and (2) the alternative of writing them down to the absolute market value and dealing more liberally with the rate of interest, he himself would be inclined to favour the second alternative.

MR. L. E. CLINTON said that it had been suggested in the discussion that in practice increases of salary did not take the convenient forms which Mr. Epps assumed in his paper. No doubt the author had made those convenient assumptions in order to save unnecessary labour. He could have taken scales which showed irregular increases from age to age and still have worked out his examples by means of one valuation. He could have done this by using Mr. R. P. Hardy's method of "ages passed through." In one of the President's lectures on friendly societies it was shown how effect could be given to the variation of the standard rates of sickness from age to age. Substituting the salary rate for the rate of sickness it would be seen that a pension fund valuation could be easily thrown into a form which would enable any number of valuations with varying salary scales to be made with very little additional work.

With regard to depreciation, he rather agreed with those speakers who had expressed the opinion that it was not necessary to saddle existing contributors with the whole of the depreciation shown by present market values. He looked at it from this standpoint: pension funds and life assurance companies fell into

totally different categories. Pension funds were established to provide annuities in the future, life assurance funds to make capital payments, and it seemed to him that in the former case whilst one had got to investigate very closely the nature of the securities, yet provided the income was secure there was no necessity to deal so harshly with this question of depreciation. No doubt some irredeemable securities should be written down, but, generally speaking, he felt that the amortisation plan of valuing securities was a suitable one to apply in the case of pension funds, a rate of interest slightly higher than the valuation rate being used in applying the plan.

Mr. Maudling had suggested that pension funds should have at least the same privileges with regard to income tax as life offices. It appeared to him that in regard to the main basis of taxation that was already the case. In pension funds the basis of taxation was interest less pensions, and in life offices having annuity funds it was interest less annuities. Most of them would agree that pension funds, being entirely thrift funds, should be exempt from income tax, but in the present state of the law only the net rate of interest which the funds earned could be taken into account in fixing the valuation rate of interest. He thought that the proper method of dealing with negative values depended upon (1) whether membership was compulsory, and (2) the status of the employer. On the general question of liquidation of deficiencies, it was quite clear that no hard and fast rule could be laid down. He himself regarded the abnormal increases in salaries as being really of the nature of superimposed salaries. The position, therefore, was very much akin to the starting of a new fund in regard to these increases, and in that case the liquidation of the initial deficiency due to the higher attained ages of the existing employees generally depended on the generosity of the employer. Such deficiencies were usually liquidated by means of an annuity over a long period and personally he would advocate that method of basing the valuation on the old scale of contributions.

Mr. Bacon had referred to the fact that Mr. Epps did not deal with some of the more difficult problems arising from post-war effects in connection with pension funds and he referred particularly to the question of the possibility of retardation of retirements. He (the speaker) doubted whether this particular effect was likely to occur. On the contrary, he thought that an acceleration of retirement was more probable. During the war there was a large number of suspended retirements, and when men returned to employment after the cessation of hostilities companies and firms were faced with redundant staffs. One of the simplest methods of dealing with that difficulty was to pension at the earliest possible moment all those that were pensionable. He thought that the matter had been viewed too exclusively from the point of view of the employee, but retirement from service on account of age rested with the employer and not with the employee, and regard must be had to the employer's point of view when visualising what was to happen

in the future. Then again, from the employer's point of view, the enhanced pay-roll and the pension fund commitments were looked at together, and if it were possible to replace men who were in receipt of higher salaries by younger men at lower rates, thereby also reducing the amount of contributions, surely there would be a tendency to do that. It seemed to him that actuaries would need considerable margins to provide for some of the other post-war effects, and for that reason he was in favour of retaining the old scale of contributions rather than the reduced rates of contribution which were obtained by using a flatter salary curve consequent upon the increased salaries.

The mortality investigation which Mr. Epps gave in his addendum was a very valuable addition to the paper; in his view pensioners' mortality was the most important element in connection with the valuation of pension funds. In one very large fund with which he was familiar the rates of mortality amongst pensioners were found to have got very appreciably lighter at three successive valuations, and he doubted whether the general improvement in mortality was the sole cause of this feature. In his opinion the age of the fund had a considerable bearing on the rate of mortality brought out. It seemed quite reasonable to suppose that in the very early years the early deaths represented the heavy mortality, and not until the fund had been established for some considerable number of years did one get a normal sample of pensioners' mortality. The investigation Mr. Epps had made would seem to show that both for the purpose of making valuations and calculating contributions it was desirable to use select annuities rather than aggregate annuities, but although one came to that conclusion from studying the rates of mortality it was rather surprising to find that heavier reserves and higher contributions were deduced by the "aggregate" method. In searching for an explanation of this apparent anomaly he observed that in constructing the aggregate annuities the author had mixed the mortality of the incapacity pensioners with that of the age pensioners, thereby producing much higher values at ages below sixty than those on the select basis. He thought that the two classes of pensioners' mortality should be kept quite separate, and if Mr. Epps had done this he might have found that the valuation reserves and also the contributions based on the select annuities would have been greater and not less.

The PRESIDENT, in proposing a vote of thanks to the author, said that the discussion had revealed differences of opinion on many points, and those differences were bound to exist in the circumstances in which they found themselves to-day with regard to superannuation funds. It was of the highest importance that they should exchange views with one another and try to come to a common opinion on matters of practice so that the world at large would bring against them no charge of inconsistency in the advice which they respectively gave to those who consulted them.

MR. EPPS. in reply, stated that he had omitted to mention,



in regard to the subject of income tax, that the appendix following the one he had given in his paper contained the views of the Inland Revenue on the proposals put forward by Mr. Mitchell. In this appendix the Board of Inland Revenue stated that they would offer no objection to the recommendations which were subsequently made by the Royal Commission. They estimated that what was recommended by the Royal Commission would cost no more than £100,000 a year.

Mr. Fraser and one or two other speakers had drawn attention to some remarks he had made in his paper on the subject of reducing benefits. He quite agreed that any general reduction of benefits was unlikely. Several speakers had suggested that his service table was artificial. He had quite expected that criticism, but, as a matter of fact, his service mortality rates were deduced from a small experience, and he believed them to represent the position in a very select class of life. He agreed, however, that he might have taken heavier rates which would have been a closer representation of the experience to be expected in an ordinary fund. With regard to the criticism that in preparing the salary scale he assumed a constant plus a percentage addition, it might be mentioned that one of the Civil Service awards of war bonus took that form; and in many other cases where the junior members of the staff had a larger war bonus in proportion it appeared that the assumption of a constant plus a percentage addition usually gave a fairly close approximation to the actual facts.

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*Modern Developments in the Methods of Industrial Assurance Valuations.* By CHARLES WILLIAM KENCHINGTON, F.I.A.,  
*Actuary to the Wesleyan and General Assurance Society.*

[Submitted to the Institute, 25 April 1921.]

#### INTRODUCTION.

1. THE Valuation of the policy liabilities of Industrial Life Assurance Companies presents a number of problems which do not arise in the valuation of the contracts of Ordinary Assurance Offices, and as the President indicated in his recent Presidential Address that they are of interest even to those Members of the Institute who are not directly connected with the business of Industrial Assurance, the present paper is submitted in the hope that it may provide the basis for the discussion of some of these special problems.

The recommendations of the Departmental Committee on the business of Industrial Assurance Companies and Collecting Societies (Cmd. 614) on the subject of Valuation, which give

rise to the present general interest in the matter, are chiefly concerned with the bases to be adopted, and it is possible that, as the recommendations involve the vexed question of standardized valuations, a discussion on the bases of Valuation would be of greater interest to the profession than the discussion of valuation methods. The preparation of a paper on the bases of Valuation would, however, have necessitated the expenditure of a larger amount of time and labour than I could have devoted to it, in order to deal with the matter adequately in time for submission during this session, but the discussion of the specialized methods of Valuation applicable to Industrial Assurance Companies may not be an inappropriate introduction to the consideration of the larger subject on some future occasion.

2. The special problems involved in Industrial Assurance Valuations arise partly from the distinctive character of the business, and partly from the mere magnitude of the number of transactions as compared with those of Ordinary Assurance Companies. The distinctive character of the business here referred to is that of fixing the weekly premium as the unit of measurement and the sum assured as the variable function depending upon the age, in contrast to the practice in Ordinary Assurance of taking the sum assured as the standard of measurement and the premium as the function of age. The magnitude of the number of contracts renders it essential to pay more consideration to the methods of grouping and recording the valuation data than is necessary for Ordinary Assurance Companies. This may be illustrated by the statement that, of the Industrial Offices which render Returns to the Board of Trade under the Assurance Companies Act, 1909, nine make Annual Valuations, and the total number of Industrial policies brought into their 1919 Valuations exceeded 40,000,000.

3. It is not known whether the methods of Valuation now described are in general use in Industrial Assurance Offices, but they have in all cases been adopted with some advantage in practical use. The general principles upon which they are based do not differ essentially from those which were described by Mr. Frederick Schooling in his paper entitled "On some methods of Grouping Policies for the purpose of Valuation" (*J.I.A.*, vol. xxxii, p. 301), which was submitted to the Institute more than twenty-five years ago, but the methods differ considerably in matters of detail.

The magnitude of the business makes it essential not only



to group the policies, but also to arrange for continuous recording of data for valuation purposes on the simplest possible lines. It is fortunate in this connection that the general practice of Industrial Assurance Companies is to grant policies for a number of pence per week at the same rate as that quoted for a weekly premium of one penny, so that it becomes unnecessary, in the majority of cases, to tabulate anything more than the number of policies and the aggregate of the penny premiums.

#### MECHANICAL TABULATION.

4. This was the method submitted by Mr. Schooling in the paper already alluded to, but the first point of departure to which I would refer is the use of Power's Machines in obtaining the figures for tabulation. These machines, I believe, have not previously been described in the *Journal*, but Sir Gerald Ryan referred to a somewhat similar system in a paper, entitled "The Case for Census Reform", read before the Institute on the 25 November 1901 (*J.I.A.*, vol. xxxvi, p. 329). The following is one of the footnotes to this paper (p. 352) :

"The applicability of the Hollerith Machine to the  
"classification of insurance statistics in connection with  
"the investigation of mortality experience and actuarial  
"valuations is a subject that demands special and separate  
"consideration."

It is significant that nearly twenty years have elapsed since this statement was made.

5. The Power's system is similar to the Hollerith in that it employs a card in which holes are punched, the value represented by the hole depending upon its distance from the bottom edge of the card. The Power's system differs from the Hollerith, and, at least so far as Assurance Offices are concerned, is to that extent superior in that it not only lends itself to the very rapid sorting and grouping of the cards, but also provides printed lists and totals by an automatic process.

It is not proposed to describe Power's Machines fully; at the moment it is only necessary, in order to allow a proper understanding of the methods about to be described, that the general features of the machines in their application to insurance purposes should be briefly explained.

The use of Power's Machines may be described as the mechanical application of a card system. Each card contains all the information which might be written on one line in a book.

The card is divided into the same columns as would appear in that book, but the card, instead of being wide enough for one line only, has at least ten lines in order to enable values to be designated by position instead of by different symbols. Whatever is required to be indicated in any particular column is represented by punching a hole in the position of the number intended to be represented.

6. The following is a reproduction, 2/3rds true size, of a card designed for Industrial Life Assurance purposes.

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8. The distinctive feature of a Power's Installation is the Tabulator-Printer. The cards pass through this machine at the rate of approximately 3,000 per hour, and as each card passes through, the whole of the information upon that card, or optionally such information as may be required, is printed simultaneously, *i.e.*, not one figure at a time as in the case of a typewriter, but all the figures at once. Moreover, at any desired point, as for example at the end of a sub-group, totals of those columns which it is required to cast are printed automatically.

9. It is not necessary at this point to specify the many different purposes for which these cards are used in providing either general lists or lists of certain specified types of policy. It is sufficient to say, that in addition to the collection of valuation data the cards serve many other useful purposes, of which it is only necessary to mention the Policy Register, printed in strict policy number order, the Sums Assured, Premiums and Number of Policies being totalled.

10. For valuation purposes the cards are first sorted according to the table in the prospectus under which the policies have been issued. The cards for each table are then further sorted, either according to the age at entry, or in the case of endowment assurances according to the year of maturity.

Certain special tables require special treatment, but when the whole of the cards have been grouped in accordance with the method of tabulating the valuation data, they are put through the Tabulator-Printer and listed and totalled in the groups.

11. In the case of certain tables under which a large volume of business is transacted, the groups are further sorted according to the amount of the premium. The group is not divided into smaller batches, this sorting merely affecting the order of the cards.

As an example, the whole of the cards for each age at entry for whole life assurances are listed together, having been arranged previously in order of magnitude of premium. The total of the premiums, the total of the sums assured, and the total number of policies for that age are then printed. The list itself will be arranged so that all the cases of *1d.* premium come first, then all the policies upon which the premium is *2d.*, and so on, but the total is a general total for the whole of the policies for the particular entry age.

When the cards have been punched they are checked against the original information; apart from this, the first step in checking

the valuation data is to compare the total sums assured with the total premiums for each group; for obviously, except in the case of special tables, there should be a definite relationship between the two, and errors are occasionally brought to light in this way. When there is a discrepancy, the fact that the policies are arranged according to magnitude of premium considerably simplifies the work of finding an error which has been passed over previously either in checking the punched card against the proposal form or in checking the proposal form itself in the first instance.

12. Immediately the totals of all the items of data thus prepared for valuation purposes have been obtained, they are combined to give grand totals for the period, and these should agree exactly with similar grand totals which are subsequently obtained in tabulating the Policy Register sheets. The period referred to here may be one week, or it might with certain advantages be increased to, say, four weeks. It does not appear advisable, however, to take advantage of the extraordinarily rapid working of the machines to allow work to accumulate for much longer than this, because the longer the period the greater will be the difficulty in the reconciliation of totals, and the tracing and correcting of errors.

A further check upon the total number of policies occurs in connection with an analysis of the whole of the cards for the period, divided simply according to sex and age, in connection with certain special work which is described below in paragraph 48. Finally there is an additional check upon what is usually considered to be the chief disadvantage of any card system, namely, the possibility of a card becoming totally lost and its loss unsuspected; for the printing of the Policy Register Sheets mentioned above is the last of all the operations, and necessitates the sorting of the cards into strict policy number order, conclusive evidence that no card is missing.

13. In view of the large part which the handling of masses of figures, and the performing of extensive calculations, plays in the practical application of actuarial science, it will not be out of place at this point to mention that the use of the Power's Installation for a period of two years has more and more compelled one to realize the possibility of human error. Inaccuracies have been found by means of the machines in work which has been doubly checked, and which, owing to hidden and neutralizing discrepancies, has been "proved" correct by the ordinary

balancing test which would previously have been considered conclusive. This is not the place to discuss the psychology of human fallibility, but experience proves that errors are overlooked during the checking process, especially when that process is identical with the method of original calculation. This lends added weight to the importance of an independent check such as that described below in another connection (pars. 25 *et seq.*).

14. With regard to the Power's system it is only necessary to add that the cards for policies in force are stored throughout the duration of the policy, arranged primarily in geographical groups, and secondarily in numerical order. By this means it is intended that the cards themselves, besides supplanting the old style valuation cards, shall eventually also take the place of the office copies of the Agents' Policy Registers. At any moment the cards in force for any Agency can be put through the Tabulator-Printer to produce automatically a printed Agent's Register.

15. On lapse, surrender, death or maturity, the card is removed from the in-force files, and at the end of convenient periods such cards are dealt with by methods similar to those described above in connection with the new business, to provide withdrawal figures for entry in the valuation class books. Here again adequate total-checks supplement the ordinary methods of checking.

16. By the usual process of continuous records the data for the Valuation at the end of the year are obtained from similar data as at the end of the previous year, by the addition of analysed figures for new business and revivals, and the subtraction of similar figures for claims, maturities, lapses and surrenders. All these analyses are carried out by means of the Power's Installation, and the total figures are obtained thereby. Each step in this work is checked against the parallel step in dealing, for other office purposes, with the same movements of business, so that at the end of the year it is beyond doubt that the same modifications have been made to the valuation records as to the other office records. It would be entirely a mistaken conception to regard the valuation records as a thing apart from the office system of book-keeping. On the contrary, it is of the greatest importance that the valuation data should be definitely interlocked with the general financial records, and indeed should be regarded as a solid foundation for accurate life office accountancy.



## VALUATION SCHEDULES.

17. Passing from the consideration of methods of recording valuation data in general, we may now proceed to show in detail the arrangements made to carry out the valuation of some of the principal classes of policies. The most important class of assurance in Industrial Assurance practice is Whole Life Assurance by weekly premiums, and it is therefore to this class that attention must first be directed.

The tabulation of the valuation data according to Mr. Schooling's arrangement was primarily in calendar years of entry, and within each year of entry according to ages at entry. The variation which I submit is to make the primary tabulation according to age at entry, and the first subdivision that of calendar year of entry. The type of the valuation schedule is shown on opposite page.

18. The totals obtained from Schedule I are carried to Schedule II, in the form shown below, from which the whole of the particulars required for the Valuation Returns to the Board of Trade (Schedule 4) can readily be obtained.

The total of column (2) gives the number of policies in the class of assurance, and the totals of columns (6) and (7) give the corresponding sums assured and net premiums. The totals of columns (13) and (12) give the values of sums assured and net premiums respectively, and the difference between these totals gives the net liability. The net liability is, however, worked out at each age at entry as this provides a useful check on the work by way of comparison with the figures for previous years.

The office premiums and value of office premiums are deduced by multiplying the totals of columns (3) and (9) in each case by the factor  $\frac{365\frac{1}{4}}{7} \div 240 = \cdot 21741$ .

## FACILITIES FOR STRENGTHENING VALUATION BASIS.

19. There is one outstanding advantage in having Schedule II as a summary of the valuations under ages at entry, in contrast to a summary of the valuation of the business under calendar years of entry. The very rapid development of Industrial Assurance business combined with the inevitably heavy charges involved in setting up agencies, has rendered it necessary in most Industrial Assurance valuations to make an allowance in the premium valued for new business expenses. This is accomplished either by calculating net premiums on the







assumption that the whole of the premiums received in the first  $n$  years, where  $n$  is a small number, are absorbed in meeting current claims and expenses; or by valuing a percentage of the office premiums. The resulting valuation is, in either event, less stringent than that on a net premium basis by the same mortality table and rate of interest, and in most cases the actuary of an Industrial Assurance Company is faced with the problem of making the valuation as stringent as the resources available will allow. The history of the valuations of all the largest offices shows that the actuaries have consistently taken steps to strengthen the bases upon which they have estimated the valuation of the liabilities of the offices they serve.

20. It is therefore important that the method adopted should provide a ready means to enable the valuation to be strengthened when the detailed calculations have reached an advanced stage, without making it necessary to embark upon a process of recalculating a large portion of the work. One of the simplest processes available for the purpose is that of reducing the premium valued. Whether the premiums valued are a percentage of the gross premiums, or are calculated by the exclusion of a number of the early years of assurance, the method suggested of having the valuation data arranged primarily according to age at entry materially assists the process of making the alteration.

21. The question of negative values arises whenever the premium valued does not coincide with the net premium as at date of entry. These negative values are confined to the early years of assurance, and it becomes necessary to investigate their incidence, for each age at entry, by determining the value of  $t$  which makes

$$S_x \times \bar{A}_{x+t} > P'_x \times \bar{a}_{x+t}$$

where  $S_x$  is the sum assured, and  $P'_x$  the premium valued, in respect of an assurance at a weekly premium of 1*d.* effected at age  $x$ . The total of the values of the Premium Function, as shown in Schedule I. for all durations of assurance less than  $t$ , must then be multiplied by the premium valued, and the excess of this product over the corresponding value of the sum assured is the correction for negative values to be carried to column (11) of Schedule II for that age at entry.

## COMPILATION OF SCHEDULES.

22. The nature of the figures to be inserted in the columns of Schedule I is indicated by the headings, but a further explanation may at once be given regarding the processes by which the figures are actually entered. In the first place it should be stated that the years of entry are printed, and that the horizontal lines are printed with precision to correspond exactly with the double spacing of an ordinary typewriter. This enables the entries to be made in columns (3) and (4), from the summary sheets from which the valuation data have been calculated, by means of a Tabulating-Adding machine. Column (8) is automatically completed by the machine, at the same time as the entries in columns (3) and (4) are made, without special mental effort on the part of the operator. The completion of column (8) is further described in paragraph 27 below.

23. It will be observed that columns (2) and (7) are both headed "Valuation Age." Entries are made in column (2) by the Tabulating machine on the first and last lines only for which there are data in columns (3) and (4).

Columns (5) and (6) are not entered by hand or by the machine, but gummed-backed strips printed with the appropriate factors are affixed to the Schedules. These gummed strips are printed with precision in the matter of alignment so that the lines agree exactly with the lines on the Schedule, and it is therefore possible to select the exact portion of the strip that is required. The three columns  $\bar{A}$ ,  $\bar{a}$ , and Valuation Age are printed on the strips, so that it is only necessary to insert the Valuation Ages in column (2) on the first and last lines, as above described, to secure agreement throughout the whole range of Valuation factors for the Schedule.

24. When the entries in columns (2), (3) and (4) have been checked and the factors agreed, the multiplications necessary for the completion of columns (10) and (11) are made on an arithmometer.

## APPROXIMATE COMPUTATION OF PRODUCT TOTALS.

25. The checking of columns (10) and (11) is made by an independent process, being an adaptation of the method of approximate valuation submitted to the Institute by Mr. A. Henry (*J.I.A.*, vol. lii, p. 48).

The theoretical basis of the method may be described as follows: If  $w_t$  be the valuation data for which  $f(t)$  is the

corresponding valuation factor, then the sum of the products for a range of  $n$  terms is represented by

$$w_1 f(1) + w_2 f(2) + \dots + w_n f(n) = \sum_{t=1}^{t=n} w_t f(t),$$

and assuming that for a limited range of terms  $f(t)$  may be written as approximately equal to  $a + bt$ , we have

$$\begin{aligned} \sum_{t=1}^{t=n} w_t f(t) &= \sum_{t=1}^{t=n} w_t (a + bt) \\ &= a \sum_{t=1}^{t=n} w_t + b \sum_{t=1}^{t=n} tw_t \\ &= a \sum_{t=1}^{t=n} w_t + b \sum_{t=1}^{t=n} w_t^2 \end{aligned}$$

where  $\sum_{t=1}^{t=n} w_t$  is written for  $w_1 + 2w_2 + \dots + nw_n$ .

This is really only another way of expressing Mr. Henry's formula, and it indicates that the sum of a number of products may be obtained approximately by making first and second summations of the valuation data and multiplying these sums by appropriate values of the constants  $a$  and  $b$  derived from the valuation factors.\*

26. Mr. Henry suggested that the constants  $a$  and  $b$  should be derived from equations weighted to correspond in general outline with the valuation data to which the approximation is to be applied. While this weighting of the equations may be desirable if there can be any certainty that the valuation data will correspond to the weights applied to the equations, my own

\* It was not pointed out by Mr. Henry that the same formula could have been derived by assuming that the total valuation data for the group were concentrated at the average age and multiplied by the factor for that average age on the hypothesis that the factor could be represented by  $a + by$  where  $y$  is the average age. This proposition can be shown symbolically as follows: Since  $y$  is the average age

$$y = \frac{\sum_{t=1}^{t=n} tw_t}{\sum_{t=1}^{t=n} w_t} = \frac{\sum_{t=1}^{t=n} w_t^2}{\sum_{t=1}^{t=n} w_t}$$

Hence

$$\begin{aligned} \left( \sum_{t=1}^{t=n} w_t \right) f(y) &= \sum_{t=1}^{t=n} w_t \left\{ a + b \frac{\sum_{t=1}^{t=n} w_t^2}{\sum_{t=1}^{t=n} w_t} \right\} \\ &= a \sum_{t=1}^{t=n} w_t + b \sum_{t=1}^{t=n} w_t^2 \end{aligned}$$

as before.

experiments lead me to believe that it is better, as a general rule, to obtain the constants  $a$  and  $b$  from unweighted equations, and to restrict the range of the summations.

27. It was found that a range of twenty terms is not too great with most valuation functions to give a very close approximation, but for several reasons it was finally determined to fix the range at ten values. The reasons were (1) that as the method was intended primarily as a check on the sum of individual products it was desirable to restrict the range in order to be able to localize any error that might have been made, (2) that the shorter the range the closer must be the degree of approximation, (3) that the terminal values not covered by the summations were reduced, and (4) that the valuation data had, in the particular example to which it was being applied, to be separated into groups according to years of entry on account of a change in the scale of the sums assured, and it happened that one of these groups consisted of a period of ten years, namely, 1907 to 1916 inclusive.

Since for each age at entry the valuation data are more numerous at the later years of entry, the summations in column (8) of Schedule I were arranged in groups of ten values in such a manner as to make it possible to obtain values of  $\Sigma^2(4)$  at the years of entry 1866, 1876, . . . 1916.

28. As a matter of detail, but possibly of some interest as indicating the value of modern mechanical methods, it may be stated that as the entries were made in columns (2), (3) and (4) of Schedule I on the Tabulating-Adding machine, a carbon copy was taken on a headed schedule, and it was on this carbon copy that the second summation in column (9) was made. The advantage of this process was that the calculation of the check values could be kept absolutely independent of the detailed calculations, and further that the work could proceed simultaneously.

29. The calculation of the check values demands two multiplications and an addition (in some cases a subtraction) of the products. With the aid of an arithmometer it is unnecessary to write down the values of the products, the operation being carried out by making the product  $a\Sigma$  and, without clearing the product off the machine, adding thereto or subtracting therefrom the product  $b\Sigma^2$ .

#### CALCULATION OF CONSTANTS.

30. As the summations are arranged in groups of ten years of entry for each age at entry, it follows that the valuation



constants  $a$  and  $b$  are required for each age at which there are data for valuation, from the youngest up to ten years before the oldest age. Separate constants are required for Assurance and Annuity factors, but the method of calculation is identical for both types of factor, and it will be sufficient indication of the method if the application to the Assurance factors is described.

The equations are derived as follows :

$$\sum_{t=1}^{t=10} (a + bt) = \sum_{t=1}^{t=10} f(t) = M, \text{ say } \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\sum_{t=1}^{t=10} {}^2(a + bt) = \sum_{t=1}^{t=10} {}^2f(t) = N, \text{ say } \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{From (1)} \quad 10a + 55b = M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$,, \quad (2) \quad 55a + 385b = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$,, \quad (3) \quad a + 5.5b = .1M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$,, \quad (4) \quad a + 7.0b = .018N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$(6) - (5) \quad 1.5b = .018N - .1M$$

$$\therefore \quad b = .06 \{ .18N - M \}$$

$$\text{and} \quad a = .018N - 7b$$

31. The Tabulating-Adding machine was operated with successive values of  $\bar{A}_x$ , thus obtaining the values of  $\Sigma \bar{A}_x$ . The values of  $\Sigma \bar{A}_x$  were then retyped on prepared Schedules, and the column  $\Sigma^2 \bar{A}_x$  was obtained automatically. In both cases the summations were made continuously from the oldest age to the youngest age, and it became necessary to transform the continuous summations into those applicable to successive ten year groups. The formulæ are as follows :

$$\sum_{t=1}^{t=10} f(t) = \sum_{t=1}^{t=\omega} f(t) - \sum_{t=11}^{t=\omega} f(t)$$

$$\text{and} \quad \sum_{t=1}^{t=10} {}^2f(t) = \sum_{t=1}^{t=\omega} {}^2f(t) - \sum_{t=11}^{t=\omega} {}^2f(t) - 10 \times \sum_{t=11}^{t=\omega} f(t)$$

32. From the equations of paragraph 30 in conjunction with those of paragraph 31, it was a simple matter to calculate the constants by a columnar process. The heads of the working Schedule are given below, and sample entries have been made to show exactly the procedure adopted.

Working Schedule for calculation of constants  $a$  and  $b$  for Valuation of Sums Assured.

$x$	$\sum_x \bar{A}_x$	$\sum_x \bar{A}_x^2$	$(2)_x - (2)_{x+10}$	$(3)_x + 10 + 10(2)_{x+10}$	$(3)_x - (5)_x$	$(6)_x \times .18$	$(7)_x - (4)_x$	$b = (8)_x \times .06$	$(9)_x \times 70$	$a = \frac{(7)_x - (10)_x}{10}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
100	.995	.995								
99	1.972	2.967								
98	2.940	5.907								
...	...	...								
91	9.567	53.122	9.567	.000	53.122	9.659	.092	.0061	.429	.923
90	10.495	63.617	9.500	10.945	52.672	9.577	.077	.0051	.359	.922
89	11.419	75.036	9.447	22.687	52.349	9.518	.071	.0047	.331	.919
88	12.338	87.374	9.398	35.307	52.067	9.467	.069	.0046	.322	.915
...	...	...	...	...	...	...	...	...	...	...

Example of use of constants: For the age group 90 to 99 inclusive, the approximate value of the Sum Assured Function is given by the expression  $.922 \sum_{x=99}^x w_x + .0051 \sum_{x=90}^x w_x$ .

33. The figures, though based upon an actual example, have for the purpose of illustration been cut down. In practice it is desirable to obtain values of  $a$  and  $b$  correct to four or five significant digits and the working figures must be arranged accordingly.

It will be observed that the working can readily be checked by the summation of the successive columns, and the check so applied together with the inspection of the differences of the resulting constants was found to be sufficient to ensure accuracy.

34. The constants  $a$  and  $b$  were copied from the working Schedule on strips with the same alignment as Schedule I, so that by the use of these strips the appropriate values should at once appear against the corresponding values of  $\Sigma(4)$  and  $\Sigma^2(4)$  for all valuation ages under each age at entry. By this means also the constants were rendered available for use in future years without additional work in copying.

#### COMPARISON OF RESULTS OF APPROXIMATE AND DETAILED METHODS.

35. The closeness of the approximation is shown by the following statement of the results derived from a valuation recently completed, in which the valuation factors were continuous functions based upon English Life Table No. 6 (Males) with interest at  $3\frac{1}{2}$  per-cent, and in which the annuity factors were temporary annuities terminating at age 75 or as soon thereafter as the policies had been 25 years in force.

##### *Comparison of Results derived from (a) Detailed Multiplications ; (b) Approximate Method.*

Class of Assurance	SUM ASSURED FUNCTION			PREMIUM FUNCTION		
	(a) Detailed Multiplica- tions	(b) Approximate Method	Difference (a) - (b)	(a) Detailed Multiplica- tions	(b) Approximate Method	Difference (b) - (a)
Whole Life						
Single Lives	1,471,960	1,471,949	11	23,472,804	23,474,666	1,862
Joint Lives	23,507	23,506	1	326,743	326,723	-20

36. In the Whole Life class on Single Lives the terminal values, to which the approximate method could not be applied, were less than 1 per-cent in the case of the Sum Assured Function,

and less than 2 per-cent for the Premium Function. In the Joint Life class, owing to the data being smaller and less evenly spread, the terminal values to which the group method could not be applied were rather larger, but even here the proportion not covered of the Sum Assured Function was only 5.5 per-cent, and of the Premium Function 2.6 per-cent, of the respective total values.

37. The closeness of the approximation is the more remarkable in the case of the Premium Function because in the final groups, where the number of years' premiums payable is less than 15, it is clearly a wide assumption to represent  $\bar{a}_t : \bar{y}-t$  as a function of the first degree. In point of fact the assumption leads to a systematic error which could be corrected, if thought to be necessary, the approximate values always being more than the true. The divergence in these particular groups in the Whole Life class led to an overstatement of the Premium Function by 2,846, so that if a correction had been applied the difference of 1,862 shown in the Table in para. 35 above would have been more than counterbalanced, and the resulting net liability by the approximate method would have shown a very small margin on the side of safety.

38. The difference in the total values being so small, it might be thought that the result was purely fortuitous, depending upon a certain balancing of errors. Admittedly some balancing of this character is revealed upon comparing the sub-groups, but as showing the closeness of the approximation it may be stated that a divergence between the individual and approximate values for any sub-group of more than 1 per thousand of the value was always taken as indicating that an investigation should be made to ascertain whether the divergence was due to an error.

The comparison of the totals of the Sum Assured Function and Premium Function for the whole of the business does not indicate the effect upon the net liability, because the differences in the values for each age at entry have to be multiplied by the appropriate sums assured and net premiums. Thus while in the Table in paragraph 35 above, the Sum Assured Function derived by the detailed multiplications is greater by 11 than the corresponding total by the approximate method, the difference between the values of the Sum Assured is 12.3, the approximate values being greater by this amount than the individual values. In total the valuation net liability would have been underestimated by the approximate method by the sum of £329 on a total net liability exceeding £1,400,000.

39. As further evidence that this remarkable agreement is not due to chance, it may be mentioned that the approximate method has also been applied, though not so completely, to a valuation based on  $O^{M(5)}$  Mortality at 3 per-cent interest with similar results.

#### ADVANTAGES OF APPROXIMATE METHOD.

40. The advantages claimed for the approximate method of checking the valuation are (1) its complete independence of the detailed individual multiplications, and (2) the rapidity with which the work can be carried out.

41. The independence of the method is a material gain, because the operation of checking work that has already been done, on precisely the same lines as the calculations were originally made, is frequently the cause of error, owing to the liability of the checker to make the same mistake as the original calculator, or to overlook the mistake by accident in comparing results.

This point is not, of course, a new one, but reference to it is made here because experience of the new methods has proved there is a real danger that a computer may be predisposed towards a figure which he sees written on a paper before him, and which is supposed to represent the result of the calculation he is performing. This inevitably leads to a certain number of mistakes being overlooked when a very large number of calculations are being checked.

42. With regard to the rapidity of the process, it was found that the entry of the particulars in columns (2) and (3) of Schedule I was accomplished on the Tabulating-Adding machine in far less time than that occupied formerly in making entries by hand, in spite of the fact that the continuous totals in column (8) were being obtained at the same time. The fact of the figures being in typewritten characters and not in writing, also facilitated subsequent operations, and eliminated the possible source of error which is found in the bad formation of figures by hand.

By summing the values of  $\Sigma(4)$ , at the points where  $\Sigma^2(4)$  was to be taken, a check by an independent process on the casting of column (4) was provided, the operation being more rapid than that of direct checking.

43. Up to this point the approximate method entails no special mental effort, and has in fact resulted in a saving of time. The calculation of  $\Sigma^2(4)$  is made solely for the purpose



of the check valuation, but as a non-listing adding machine was used, this part of the process was exceedingly rapid. The same values of  $\Sigma(4)$  and  $\Sigma^2(4)$  are used both in obtaining the Sum Assured Function and the Premium Function, and consequently the operation of checking the sum of twenty products is accomplished by the approximate method by the simple process of making the one summation of  $\Sigma^2(4)$  and 4 products, the results of which are only recorded in two items, being in each case the sum of, or difference between, the products  $a\Sigma(4)$  and  $b\Sigma^2(4)$ .

It has already been explained that there were special reasons for the selection of 10 terms as the range of the summation, but experimental calculations indicate that a considerably longer range could have been employed with almost equal accuracy and as the operation is identical in form, no matter what the range may be, it follows that the longer the range the greater is the saving of time and labour as compared with the checking of individual products.

44. The suggested method has other advantages, because it is possible that it might be desired to estimate the effect of a change in the basis of valuation, either by the alteration of the mortality table or the rate of interest or both. The facility with which the work can be done by the approximate method would, in these circumstances, be of great value.

45. In the method so far described it has been assumed that the sum assured remains constant from the outset of a policy for the whole of its duration. It happens, however, that it is the practice of a number of Industrial Assurance Companies to grant policies under which the sum assured is increased after a number of years has elapsed, as for example after 5 years, and again after 10 years. This feature of the business has to be taken into account in the valuation, but it does not disturb the underlying principles outlined above, and effect can readily be given thereto by appropriate modification of the factors for the valuation of the sum assured. The only point which appears to call for comment in this connection is that, if the approximate method is applied, it will be necessary to calculate the constants separately for each age at entry, since the valuation factors will not depend solely on the valuation age. It may, in fact, be the better course to dispense with the Sum Assured Function, and to use factors which involve the sum assured per penny premium, so that the value of the sum assured is given directly, upon multiplication by the number of pennyworths of assurance.



## EXPECTED AND ACTUAL CLAIMS.

46. There is a further use of the method which is thought to be worthy of mention. It seems hardly likely, on account of the work involved, that a comparison of expected claims with actual claims would be feasible if made by detailed individual products for each year of entry under each age at entry; but it is more than possible that selection is operative in Industrial Assurance, although not perhaps in the same way as it is in Ordinary Assurance, and it is therefore desirable to be able to calculate the expected claims, and to compare them with the actual for the business arising under each age at entry, giving due effect to the duration of assurance.

By using the values of  $\Sigma(4)$  and  $\Sigma^2(4)$  already calculated for the valuation, a comparison can readily be made between the number of weekly pennyworths of assurance that were expected to, and the number which did actually, become claims during the year.

In point of fact Schedule I is printed with additional columns, not shown in the draft in paragraph 16. to allow for the calculation of the expected claims without necessitating the recopying of the values of  $\Sigma(4)$  and  $\Sigma^2(4)$ .

47. If the valuation ages are integral, the number of *ld.* premiums in force at valuation age  $y$  under age at entry  $x$ . on any valuation date may be considered as being of the form of  $l'_y$ , and the number of *ld.* premiums in force under the same age at entry on the next following annual valuation at age  $y+1$ , being the survivors of the  $l'_y$  cases, may be considered analogous to  $l'_{y+1}$ . The expected pennyworths of claim can therefore be obtained with very close approximation to accuracy from the expression  $\frac{1}{2}(l'_y\mu_y + l'_{y+1}\mu_{y+1})$ , and it follows that the expected pennyworths of claim for a group can be calculated by multiplying  $\Sigma(4)$  and  $\Sigma^2(4)$  by appropriate values of the constants  $a$  and  $b$  respectively derived from the function  $\mu_x$  according to the mortality table which it is desired to employ.

It is necessary to calculate the mean of the products derived from the valuation data for two successive years in order to obtain the figure desired. The actual number of pennyworths of claim, with which the expected are to be compared, are necessarily recorded in the valuation class books.

If, as may sometimes happen, the valuation ages are really of the form  $(x + \frac{1}{2})$ , the expected claims can be calculated by taking the mean of the values  $\Sigma(4)$  and  $\Sigma^2(4)$  at two successive

valuations, and by multiplying the mean value by appropriate values of constants  $a$  and  $b$  derived from the  $\mu_x$  table at integral ages.

48. A further extension of the comparison of expected deaths and actual deaths is possible by making use of Power's machines to provide figures for the number of policies under which lives are at risk at each age in certain geographical groups. It has been mentioned already that the filed Power's cards for business in force are grouped primarily according to geographical areas, and in adding to, or taking from, these groups of cards, the movements for each group are analysed both according to sex and age, thus providing the means for a continuous record of the numbers exposed to risk. The cards removed in respect of policies which become claims through death are similarly analysed. The approximate method, described in the two preceding paragraphs, is then available for the calculation of the expected deaths for each geographical group, and the comparison with the actual deaths in that area forms an additional actuarial contribution to the "intelligence" side of the Life Office. The advantage for this purpose of using a range of no more than ten ages in the application of the approximate method to the calculation of expected deaths is self evident.

#### POSSIBLE DEVELOPMENTS.

49. It will be seen that the advantages secured by the adoption of mechanical methods for the tabulation of the valuation data are very great. In addition to the important considerations of time saving and accuracy, the use of such methods provides facilities for the investigation of statistical matters relating to the business generally.

50. The possibilities of the approximate method of valuation are so great that it would not surprise me if it should come into general use in the near future, not only for Industrial Assurance Valuations but also for Ordinary Assurance Valuations. It would not surprise me, indeed, if it should become the principal method by which the valuations are actually made, and not be used merely for purposes of checking. Further experiment and trial will, however, be necessary.

#### JOINT LIFE ASSURANCES.

51. In his paper already referred to Mr. Schooling dealt with the valuation of Joint Life Assurances. He discussed various

methods of grouping these for valuation, and recommended that they should be grouped according to two equal ages at entry determined roughly on the basis of Makeham's formula, a process which in all the circumstances he considered to be justifiable. It is interesting to note that since that paper was submitted, it has become the practice of a number of Industrial Offices to quote the sums assured for equal ages at entry only, and to supply a table in their prospectuses whereby the equal ages equivalent to two unequal ages are to be calculated. In this way the actual sum assured depends upon the assumed equal ages at entry, and the method suggested by Mr. Schooling has become even more appropriate than when it was applied to sums assured which were quoted for unequal ages at entry. An advantage derived from this method of quoting in the prospectuses is that it becomes unnecessary to tabulate the sums assured in the class books in order to arrive at the valuation data, since all the information required is available by the tabulation of the number of policies and the number of weekly penny premiums. Apart from this special feature, the valuation of Joint Life Assurances does not present problems differing materially from those which have been dealt with under Whole Life Assurances.

#### ENDOWMENT ASSURANCES.

52. Endowment Assurances are a much smaller class in the general practice of Industrial Assurance business than Whole Life Assurances, but they are nevertheless sufficiently numerous to demand special attention in the consideration of the most appropriate means for their valuation.

53. A method of grouping which has been used is that of arranging the policies under calendar years of maturity, sub-grouped according to years of entry. The data tabulated are (1) number of policies, (2) sum assured, (3) number of one penny weekly premiums, and (4) until recently, Z. It is necessary to tabulate the sums assured as well as the number of weekly premiums, because although the double tabulation according to year of maturity sub-grouped in years of entry gives by implication the original endowment term, effect cannot be given to the variation of the sum assured with the age at entry, for that particular term, by the mere tabulation of the premiums.

The value of Z employed was not based, as is usual in Ordinary Assurance, upon the sum assured but upon the premiums. This was purely for the sake of simplicity, it being easier to

calculate the value of  $Z$ , appropriate to a particular policy, by multiplying the value in the prepared table by the number of pennies in the premium, than it would have been to multiply by the sum assured which, expressed in pounds, would frequently involve two places of decimals as well as whole numbers. The process of finding the average maturity age was therefore to divide  $\Sigma Z$  by  $\Sigma$  (number of *ld.* premiums).

54. It will be observed that net premiums are not tabulated, in spite of the differences due to variation in age. Assuming that net premiums are valued, as contrasted with a proportion of the Office Premiums, the variation with the age is much less than that in the sum assured for any given endowment term, because while the sum assured decreases with the increase of age, the net premium per unit of sum assured increases, and in consequence the net premium per one penny weekly premium tends towards constancy for each endowment term. It was found in these circumstances as the result of experiment, with  $O^{M(5)}$  Mortality and 3 per-cent interest, that an exceedingly close approximation to accuracy was obtained by making use of the average maturity age for the double purpose of fixing the average age at entry, by deducting therefrom the original endowment term, and of obtaining the valuation age by deducting the term to run.

55. A careful scrutiny of the average maturity ages, as calculated by the  $Z$  method, at five successive annual valuations showed that, with the volume of business transacted, there is so close an agreement from year to year towards a regular progression as to render it unnecessary for the calculations to be made every year. The particular data showed a systematic increase of the maturity age with the length of the unexpired term. Keeping the terms to run constant the maturity ages in successive valuation years almost exactly coincided. It has therefore become possible, especially in view of a change of mortality basis used in the valuation from  $O^{M(5)}$  to English Life No. 6 (Males), for which it is clear that  $O^{M(5)}$  values of  $Z$  are not strictly appropriate, to go one step further and assume that the maturity ages will correspond to the following plan for several years to come, namely :

Number of years to run.				Maturity Age.
1 to 24	...	...	...	55
25 to 40	...	...	...	60
41 and upwards	...	...	...	65

Sample statistics will be taken at intervals to test the continued accuracy of the assumption, but in the meantime it becomes unnecessary to tabulate Z.

#### CONCLUDING REMARKS.

56. There are many other types of Industrial Assurance policies, such, for instance, as old age endowments, policies securing guaranteed bonuses, and others which introduce special problems in the valuation, and necessitate the preparation of elaborate valuation factors. The systems already mentioned are adhered to as far as possible even for these special classes, the aim being to reduce the figures to be tabulated per policy, though by so doing it may be necessary in some instances to increase the number of sub-groups. The problems presented by these special classes are in all the circumstances not thought to be of sufficient general interest to warrant attention at the end of a paper already of considerable length.

57. I would only wish to add in conclusion an expression of my indebtedness to my Assistant, Captain E. W. Phillips, O.B.E., F.I.A., for the help he has ungrudgingly rendered in the preparation of this paper, and particularly in the description given of the use of the Power's Machines. His technical knowledge of the intricate mechanism of these machines has been of the greatest possible service in their application to Industrial Assurance purposes, and I consider myself fortunate in having had his assistance in submitting the use of the machines to the Institute.

#### ABSTRACT OF THE DISCUSSION.

MR. R. LEVEY said that the paper dealt almost entirely with methods rather than principles, but in paragraph 19 the author stated that "in most cases the actuary of an industrial company is faced with the problem of making the valuation as stringent as the resources available will allow." To many that would seem a negation of the functions of a valuation. By general acceptance a valuation should be made on bases predetermined to be a proper standard by which to measure the liabilities of the company under its assurance contracts, and the results should not be lightly varied because they did not compare favourably with the assets. Presumably the author did not mean that the resources available should always determine the stringency or otherwise of the valuation, but rather that there were factors affecting industrial assurance which made it impossible to adopt in every case the time-honoured



principles of valuation applicable to ordinary assurance. In early days—and also subsequently, when new business increased so rapidly as to upset the ratio that previously existed between new and old business—the question of a proper allowance for initial expenses exercised the minds even of actuaries to ordinary life offices. This question, together with other factors, still concerned actuaries to industrial companies, and necessarily left more to their discretion. Each had to decide for himself the point beyond which his valuation basis must not be modified. To declare an office insolvent because it failed to conform to some arbitrary standard of valuation which did not take into account the peculiar circumstances of that office would probably result in unnecessary hardship to a large number of people whose claims would be met in full if a somewhat less stringent basis of valuation were permitted. On the other hand, he thought that it was not only advisable but essential for the future welfare of the business that every effort should be made by all industrial offices to strengthen their reserves. Whilst it was impossible for all to hope to attain to the same standard at once, to have some such standard in view and to endeavour to reach it in the course of a few years must be to the advantage of all concerned.

It was evident that in the Power's installation they had a powerful instrument for dealing with the immense volume of data involved not only in the valuations of industrial assurance offices but also in the valuations of large ordinary offices and pension funds and in mortality investigations. In addition to the saving of labour there was a great saving of time, and a gain in accuracy.

In the application of Mr. Schooling's method of valuation—in which the classification was primarily according to year of assurance—cards were prepared giving the usual premium factor, the present value of the sum assured by a weekly premium of one penny, the net annual premium for the sum assured by a weekly premium of one penny, and the net liability for a penny policy. Those cards once prepared were available for use so long as the valuation basis remained the same. Mr. Kenchington, classifying according to age at entry, employed two schedules entailing the completion of twenty-five columns in all and involving several multiplications and summations, and, regarded simply as a method for making a valuation where no change in the premium valued was necessary, that method might be the longer. The relative merits of the two methods could, however, only be properly gauged by considering the respective circumstances in which they were intended to be used. Classification according to year of assurance and the use of prepared factor cards as outlined were, he thought, the better method when, as in the case with which Mr. Schooling was dealing, a stable net premium basis was in use. Mr. Kenchington's method was designed to meet the special circumstances of a gross premium valuation in which it might be desired to vary the premium valued during the actual process of valuation, and it had the necessary elasticity to meet such variations without great additional labour, although if



the interest and/or mortality basis of the valuation were altered he did not think it had much advantage. For the purpose of an approximate check by Mr. King's method, the classification by year of entry was obviously the better, since the first and second summations of the number of penny premiums for a group could at once be used as factors with which to multiply the *a* and *b* constants. The results for a whole group could then be summed, and finally the required liability obtained by multiplying the respective totals for sums assured and premiums by the sum assured and net premium for the particular age at entry, as those were constant for that group. Under the method of classification by years of assurance, the sum assured and net premiums varied in the group and the approximation could not be obtained with the same facility. A re-classification of the data according to age at entry could, however, readily be made by means of the Power's installation.

There could be no doubt as to the usefulness of the approximate method for checking purposes, and still more for making valuations on different bases for trial purposes. The result aimed at in the latter case was the aggregate difference resulting from the change, and any error introduced by the approximation was practically eliminated by differencing. He was, however, doubtful whether, as the author seemed to think, such approximate methods would ever be generally adopted as standard methods of valuation. If so, they would either have to rely on a check by means of a second operator doing the same work—a process on which the author did not seem to place much reliance—or to check one approximate method by another. In such circumstances some future actuary might perhaps re-discover exact methods in order satisfactorily to check his approximation.

MR. C. F. PETERS remarked that his company had had to deal with the problem of a rapid valuation of whole-life weekly premium assurances. Owing to the war and to the depletion of the staff, it had been necessary to abandon a continuous record and to start afresh with between five million and six million assurances. Obviously the work could not be done without mechanical aid, and the Power's machine was adopted as the best for the purpose. The object in view was to get through the valuation very quickly, without regard to anything else. The data were classified in the way described by Mr. Schooling, in years of entry, for each year of entry and age at entry. The individual cards were then abandoned entirely and a system of summary cards adopted. The reason for abandoning the individual cards was that they had served their purpose, and the storing of five million or six million cards in these days was a serious matter. Moreover, before they could be stored they would have had to be re-sorted into some order for reference and extraction when lapse or death occurred. There were seventeen geographical areas in the valuation, for facility in handling, and those areas were chosen with special reference to the particular circumstances of the company—remuneration, supervision and other incidents. The geographical distribution had been the subject of

another investigation, not part of the valuation, relating to the yield from the expenditure of the company upon its business. He hoped at some future time to describe the most interesting results which he had obtained in that investigation. The totals of the seventeen areas were amalgamated into one total on what was called a master card, thereby reducing the five million or six million cases to about 6,000, namely, 60 years of entry and 100 years of age. Obviously that was a great advantage. Then in sorting the cards they had combined the two variables, the year of entry and the age at entry, so as to bring together all cases at the same attained age. They had thereby avoided all the trouble of valuing in separate years of entry. He would like to add that at every stage of the work checks were possible. His company had found, by using the Power's machine in the way he had described, that it got through its valuation in a quarter of the time that it generally took by hand, and in view of the magnitude of the operations in industrial assurance, that meant a substantial saving.

MR. A. HENRY was particularly interested in the results which the author had brought out by the method of approximate valuation which he (the speaker) had had the privilege of bringing before the Institute. In paragraph 26 the author expressed the opinion that as a general rule it was better to use unweighted equations and to deal, in applying the method, with smaller groups. He (the speaker) had devised the method as a check to be applied over a wide range, and from that point of view it was important that the number of essential operations should be cut down. He fully appreciated, however, the different point of view of the author. The author gave four reasons for preferring ten-year groups, but he (the speaker) doubted whether the method, as a check, would be improved by dispensing with weighting and cutting down to ten-year groups. He agreed, however, that the sum assured probably would not vary very considerably within a ten-year group, and that if this grouping were adopted the necessity for weighting was minimized. In dealing with a wider range—and personally he thought this was desirable in order to get the fullest benefit from the method as a short method of check valuation—some form of weighting was essential, or the approximation would be rather rough. A point about which he was not quite clear arose in paragraph 37, where the author referred to the valuation of limited premiums with a short unexpired term. The author said "In point of fact the assumption leads to a systematic error which could be corrected, if thought to be necessary, the approximate values always being more than the true." He (the speaker) did not understand why this should be so, because if a straight line were fitted to a ten-year group it followed, he thought, that some of the values would be higher and some would be lower; that was in fact the basis of the approximation. If the author meant that all the groups showed errors in excess that was probably due to the fact that he had not weighted his data, and this would confirm his (the speaker's) view with regard to the necessity of weighting in some cases.

One other point he would mention was that if ten-year groups were employed there might be much to be said for applying to each of those groups the factor approximating to the mean age for the group. The double summation method might in such circumstances be less simple than a method which employed an approximate valuation factor for the group.

SIR JOSEPH BURN congratulated the author on having given the Institute a most valuable paper. Papers on the subject of industrial assurance had been in the past very rare, owing mainly perhaps to the fact that actuaries who were engaged in industrial business, and in dealing with its various problems, became so intimate with them that they found it rather difficult to believe that any of their methods or any description of their difficulties could be of any practical importance or advantage to their fellow actuaries.

Referring to paragraph 19 of the paper, he thought it was only fair, in reading the sentence which had been quoted, to read also the following sentence, which he thought made the author's meaning a little plainer: "The history of the valuations of all the largest offices shows that the actuaries have consistently taken steps to strengthen the bases upon which they have estimated the valuation of the liabilities of the offices they serve." The matter might be made a little plainer still in a very few words. The valuation bases of ordinary offices were not as rule adopted for the purpose of valuations for solvency. They were bases of valuation adopted with a view to present and future distributions of profit. Industrial assurance companies' contracts were not with-profit policies. At the same time, as was well known, many of the companies during recent years had at various times made voluntary increases in the benefits granted, which increases could not have been given had their valuation bases been merely tests of solvency.

At the present time he thought there was no doubt that all actuaries connected with industrial companies were most anxious to improve, in every possible way, the character of industrial assurance. There were several directions in which improvements might be made. For instance, there was the subject of lapsing and surrendering, as well as the fundamental problem of giving greater benefits for the premium paid. The latter subject had been rather forced on them by the very sudden increase in endowment assurances, an increase which had begun in the ordinary companies and then during recent years had spread to industrial companies. In his view, at the present moment, the national demand was for cheap industrial endowment assurances. In order to provide those, the immediate necessity was to reduce the expense, and concurrently with that reduction in expense he thought every actuary would agree that it was advisable that the reserves should be increased. He believed every industrial assurance actuary was anxious not only to decrease the rate of expense at which industrial assurance was carried on, but, by means of that decrease, to increase his reserve, in order to enable him safely to increase the general benefits which

he was able to recommend under that class of assurance business. He thought, if he understood the author correctly, that the passage which he had read from the paper was an indication of the author's great desire that the utmost possible attention should be given to this important question of increasing reserves.

Mr. H. H. AUSTIN thought there was no doubt that the Power's machines opened up possibilities of handling large masses of statistics in a way that would give actuaries much more powerful analyses than they had ever had before—for instance, in such a manner as the author's classification of his data in geographical areas.

The advantages in the arrangement of the author's whole life data were very marked in certain respects. It afforded, as he had pointed out, a very ready and powerful means of strengthening the valuation-basis. He (the speaker) doubted, however, whether the schedules would as a rule be as simple as the specimen schedules given by the author. If the sum assured had been varied at any time, as was not uncommonly the case in industrial offices, or if it had increased during the first 10 years, the schedules would have to be subdivided. The method adopted in his (the speaker's) office was the usual one of classifying in the first instance by years of entry and then by valuation ages. For check purposes the full valuation based on the number of penny policies was compared with an entirely independent valuation based on the liability per penny policy. The data in the second case were taken direct from the classification books, so that any error in extracting the data would be detected.

As regards endowment assurances, which were now a very important section of the business, his office had adopted a classification by years of entry and original terms, the entry age being found by dividing the premium by the sum assured. That seemed to give, in the case of a large volume of endowment assurances, extraordinarily accurate results, which compared quite favourably even with the Z method of weighting.\*

The first requisite of any method of valuation in industrial assurance was the saving of skilled work. There were such vast masses of material to be dealt with at times of great pressure that he thought the only methods which could be permanently adopted were those which so organized the work that it could be dealt with by unskilled labour. The methods of valuation and check valuation which he had described, worked very automatically. They produced two sets of totals which should be identical, and it was not necessary to consider whether the check-result was sufficiently near, and even perhaps to investigate whether the difference indicated an error or was due simply to the method of approximation. The author anticipated the possibility of the approximate check method being adopted for ordinary branch valuations. In the ordinary branch valuation endowment assurances were the main difficulty and he doubted whether the approximate method could be applied to the valuation of endowment assurances, at any rate without weighting.

\* See paragraph in small type on p. 491.—EDS. *J.I.A.*



MR. H. L. TRACHTENBERG remarked that the previous speaker's doubts as to the efficacy of the approximate check method, when applied to the valuation of endowment assurances, were he thought unfounded. The method employed by Mr. Kenchington differed very little in accuracy from the method suggested by him (the speaker) some time ago and he had found that the latter gave only a small error.

MR. S. G. WARNER said that the author had contributed a very interesting and important paper to the proceedings of the Institute. It might be true, as the author suggested, that the attention of the actuarial world at the moment was directed towards the principles underlying industrial valuations rather than the mechanism of valuation. Still, a thorough grasp of the one was of help in considering the other. The author had, as it were, invited his readers into his actuarial workshop and had given them such a detailed and lucid account of the processes as must interest them all, whether they were directly concerned with industrial assurance or not. The first thing that struck anyone who had not had to do with industrial assurance was the entirely different nature of the problems that presented themselves to the practical actuary in dealing with that branch of actuarial work—the adoption of the premium instead of the sum assured as the unit, the enormous number of small contracts, and the great amount of labour involved. In that connection they had had a very interesting explanation of Power's machines. So much had been heard about these machines lately in connection with actuarial work that it was time the Institute had recorded in its Transactions some such account of them as had been given that evening.

He was very glad to notice that in paragraph 21 the author spoke quite clearly and uncompromisingly on the subject of negative values. It was important that in an official paper read before the Institute by one keenly interested in and actively engaged in industrial assurance, that position should be so unequivocally taken up.

A very interesting section of the paper dealt with the approximate valuation method which they owed in the first instance to Mr. Henry. It was instructive to see how the subject was approached from rather a different point of view with substantially the same result. It must be remembered that the method had been introduced in exceptional circumstances. The fact that the department with which Mr. Henry was connected had to check a very large number of valuations had undoubtedly made such a method most useful and most desirable. Similar considerations applied to industrial assurance in view of the mass of facts to be handled. The question naturally arose whether the method would be useful in connection with ordinary assurance work. He remembered being present when the late Mr. Woolhouse, on his last public appearance in that Hall, propounded his well-known method of making a check-valuation. He would like to bear witness to the fact that, considering the extreme simplicity of Woolhouse's

method, he had found it to give surprisingly good results. He had employed it on many occasions and had always found it answer the practical purpose for which he had used it. Therefore he did not know that for the purposes of ordinary life assurance more elaborate methods were likely to be very generally adopted. The crux of an ordinary life assurance valuation was the data, so far as labour was concerned. When one had the data, the valuation itself was not a peculiarly lengthy or formidable process. The author suggested that possibly in future approximate methods might be adopted for actual valuation purposes instead of merely as a check. It must certainly be admitted that the results which the author had shown were wonderfully close in some cases to those obtained by the detailed method. Moreover, all valuations were approximate: assumptions of some kind necessarily underlay them all.

He had been interested in what the author had said about the  $Z$  function in endowment assurance valuations. In a case within his (the speaker's) experience a section of a large fund had to be separately valued. In the valuation year of the entire fund there was no difficulty, but the valuations of the section had to be made annually, while the valuation of the fund of which it formed a part had not, and in intermediate years the  $Z$  function presented a difficulty. Time was of importance, and an assumption similar to the author's was made. In course of time the entire fund was again valued as a whole, the  $Z$  functions up to date were again available, and it was found there was a perfectly regular progress in the sectional valuations during the interval. This confirmed the author's opinion.

The PRESIDENT, in moving a vote of thanks to Mr. Kenchington, said that in his view the machines which had been referred to, with the improvements that were constantly being developed in them, had added immensely, if indirectly, to the interest of actuarial work in the sphere in which such machines could properly be employed. In industrial assurance practice he had found the utmost interest in putting the results of a valuation under the actuarial microscope, but owing to the magnitude of the numbers involved and the complexity of the data, it was in years gone by an exceedingly difficult task to get a complete conspectus of the working of an industrial assurance fund. These modern tabulating machines had added enormously to the facilities available for such a purpose.

He had been very much interested to notice in the paper that the author laid emphasis on the importance of the valuation data being definitely interlocked with the general financial records. He entirely agreed with him. He thought that in industrial business, as well as in ordinary life assurance business, that point ought constantly and continuously to be kept in view. For example, he had had in the last few days to consider the question of the bearing of the item, in the annual balance sheet of industrial offices, of "outstanding premiums" in relation to valuation bases. It was very important, Mr. Kenchington evidently agreed, that negative



values should be excluded from the assets of an industrial assurance fund. It was very difficult, however, to find whether that principle was carried out, in all cases, to the exclusion from the assets of the outstanding premiums on policies with negative values. This point required attention, of course in a practical way, equally with that of the exclusion from the asset "outstanding premiums" of any commissions to which, in the ordinary course of payment, these premiums would be subject.

He desired principally to refer to the much-debated paragraph 19. All actuaries would understand perfectly the point which the author had intended to make. If the sentence which had been referred to several times, namely, "In most cases the actuary of an industrial assurance company is faced with the problem of making the valuation as stringent as the resources available will allow", were taken in isolation and apart from its context, it was liable to create a false impression and one certainly that the author did not intend to convey. It would create an impression as to the purpose of valuation which Parliament did not intend in framing the valuation requirements of the Act of 1870, now re-enacted in the Act of 1909. He was so sure of what was in the author's mind, and so sure that it was confirmed by the opinion of all actuaries, that he would have deemed it unnecessary to refer to the matter, but for the uncomfortable discovery that he had made more than once that the doctrine which that sentence, taken in isolation, seemed to embody, was in fact the doctrine acted upon in certain cases. He noticed that in the returns for the year 1919, there were two cases in which companies had valued the gross premium, less ten per-cent for expenses of management, in circumstances where the renewal commission alone was probably on the average about 20 per-cent, and that some attempt was made to give the valuation a better appearance by adding to the amount reserved for management expenses, as though it might be treated as available for the purpose, the amount which had been struck off as representing negative values. In cases such as that it would be found, without surprise, that correspondence arose between the company concerned and the Board of Trade, and that in due course the correspondence was presented to Parliament. That was not a satisfactory state of affairs; he deplored the existence of valuation methods which could lead, even indirectly, to criticism of actuarial discretion and actuarial strength of purpose in any return submitted to Parliament. He thought the difference of opinion, so far as there was a difference between them in that room as to the proper basis of valuation of an industrial company, was one of degree and in itself not very large. He thought all present would endorse the view that there must be some minimum standard below which no actuary, in making a valuation of a company, ought to permit himself to go. In certain cases open to criticism the companies concerned were small. But the policyholders were comparatively numerous, their interests were at stake and they were entitled to the protection of a reliable valuation. There seemed to be an idea, however, in those

cases, that the valuation was comparatively unimportant, and the companies seemed, in fact, to assess their liabilities on a basis determined by the amount of assets that they had in possession. That meant that between the strongest and the weakest bases of valuation there was a very wide gap and, possibly in the hope that that gap would be closed, there was a good deal of discussion going on among actuaries as to what lines of policy should be adopted by the actuary for his guidance in undertaking the responsible work of valuing an industrial assurance company. But he feared that, while they were discussing the matter and, perhaps, coming slowly to a definite professional opinion to which, as he had had the privilege of suggesting some months ago, the actuary who found himself in difficulty could appeal for guidance and support, there were other interests at work, and operating in a way gravely to increase the difficulties by which the actuary was confronted. It had been his privilege—if it had been a privilege: sometimes he was inclined to doubt it—to sit upon the Departmental Committee which had enquired into the system of industrial life assurance. Among those who had come to give evidence before that Committee were representatives of the outside staffs of some of the companies. Their attitude was as follows: the larger the reserves were made the smaller was the sum available, out of the premiums, for rewarding the industry of the agents. One witness had gone so far as to say: “I distinctly say that the reserves put aside by assurance companies are too large. It is not necessary.” That witness then went on to compare the valuation reserves of two offices with their respective premium incomes. He said in effect: “Here is one office that makes a reserve equal to six times its annual premium income and here is another, and a very large office, which makes a reserve equal to one and a-half times its annual premium income. It might be argued that the latter office is insolvent. Not at all. We all know that it is not insolvent”, and the witness proceeded to draw the inference he desired. After other questions, the following was put to this witness: “If more money is required by the agents, you propose to get some of that money by diminishing the sum which is now put by the societies into reserve to meet their liabilities”? and the answer was: “I have answered that—yes.” It did seem ludicrous that such ideas should obtain credence and should be seriously advanced before a Committee of Inquiry set up by a Government Department, but there was the fact, and they should realize that those who put such views forward were highly organized and were able to exert great strength in promoting their objects. He wanted to suggest once more to his colleagues of the Institute of Actuaries that they should be prepared to express definite professional views in regard to the valuations of industrial assurance companies, and views of such a character as to lessen existing differences of practice. Only in this way, he thought, could actuaries effectually maintain their professional authority against such subversive influences as those to which he had referred.

MR. C. W. KENCHINGTON, in reply, said that if there was any ambiguity in what he had said in paragraph 19 he did not feel that

he need apologize for having expressed his meaning in ambiguous terms, as it had brought forth an expression of opinion which he felt sure would be of advantage to the profession as a whole.

The use of mechanical methods in industrial assurance valuations had been referred to as the introduction of factory methods into actuarial work, but actuarial science had expanded so much that it was necessary to take every available means to bring the actual work into such a form as would render it possible for it to be done both easily and economically. One of the special advantages to be derived from the use of machines, besides those already referred to in the paper, was that of saving expense. Mr. Peters had referred to the use of the Power's machines in a different form from that in which he himself had used them. Mr. Peters, however, was dealing with the tabulation necessary for the purpose of making the valuation of the whole of his data at the outset. He (Mr. Kenchington) could not contemplate that such a process could be gone through regularly year after year, and he considered that it was absolutely necessary to carry on the valuation data by means of class books from valuation to valuation. He (Mr. Kenchington) felt it right to make full acknowledgment of the valuable services that had been rendered by his assistant, Captain Phillips, both in the actual work of adapting Power's machines to industrial assurance purposes, and in the description of the machines that had been furnished in paragraphs 4 to 16 of the paper.

With regard to the approximate check method, Mr. Henry had suggested that as the grouping extended over such a short period as ten years the check valuation might possibly have been made equally well by using valuation factors dependent upon the mean age. but he (Mr. Kenchington) would point out that for the purpose of obtaining the mean age it would be necessary to make the second summation of the valuation data and to divide that by the first summation. The process would involve the individual calculation of the mean age at least to one and possibly to two places of decimals and then it would be necessary to interpolate to obtain the valuation factors appropriate to the mean ages. He (Mr. Kenchington) preferred to take the course of making the first and second summations and then, with the prepared constants, to obtain the values straight away.

Mr. Austin had raised the question whether the schedules provided all that was necessary, and had referred particularly to the variation in sums assured. In drafting the schedules for the paper he had wanted to make the matter as simple as he could and therefore had not referred to the possibility of having to sectionalize the work according to groups of years of entry. In the valuation with which he was concerned there had been variations of sums assured from time to time, and personally he had to deal with three groups of schedules such as those to which Mr. Austin had referred : but this presented no difficulty ; it only became necessary to make a summary of the groups in order to obtain the grand total.

With reference to the President's remarks regarding the exclusion of outstanding premiums upon policies with negative

values, he thought that from an administrative point of view there would be great difficulties in making such an exclusion. He was not sure whether in fact it would be possible, except by way of estimate. The outstanding premiums as they figured in the accounts of the offices were generally reduced in various ways—for example by the exclusion (as the President himself had assumed) of commission. Whether it was possible to go further and exclude outstanding premiums upon policies which had negative values—or would have negative values if the negative values were not excluded—was a matter of some doubt.

*A Group-check for Endowment Assurance Net Premiums, by means of mean ages based on the Z method.*

1. IN the theory of the Z method it is shown (*J.I.A.*, vol. xxxviii, p. 3) that if the mortality table follow Makeham's Law,  $l_x = k s^x g^{c^x}$ , we may express  $a_{x:n}$  in the easily-remembered symbolic form  $e^{c^x \gamma \Delta a_0}$ ; where  $\gamma \equiv \log_e g$ ,  $a_0, a_1, \dots$  represent the values of annuities-certain for  $n$  years calculated at successive rates of interest, and  $\Delta$  is the symbol of differencing, acting on  $a_0$  only. It is evident that the corresponding expression for  $a_{x:n}$  will be  $e^{c^x \gamma \Delta a_0}$ , and expanding we shall have

$$a_{x:n} = a_0 + c^x \gamma \cdot \Delta a_0 + c^{2x} \gamma^2 \Delta^2 a_0 / 2 + \dots \quad (1)$$

2. Taking the reciprocal of (1), either by ordinary algebraical division or by using formula (V), *J.I.A.*, vol. li, p. 41,

$$a_{x:n}^{-1} = P_{x:n} + d = \frac{1}{a_0} \left[ 1 - c^x \gamma \Delta a_0 / a_0 \left( \frac{\Delta^2 a_0}{2 a_0} - \frac{(\Delta a_0)^2}{a_0^2} \right) - c^{2x} \gamma^2 \dots \right] \quad (2)$$

This is of the same form as (1), and therefore the mean value of  $P_{x:n} + d$  or of  $P_{x:n}$  in a group in which  $n$  is constant and  $x$  varies may be found approximately by means of the mean age derived by the Z method. The success of that method depends upon the rapid convergence of the series\* and particularly on

\* More convergent series for  $a$ ,  $\mathbf{a}$  and  $P$ , leading to the same mean ages, may be obtained by reckoning the ages from the mean age of the group, or from any age near the central age of the group, instead of from age 0. Putting  $l_x = k' s^{x-\bar{x}} g^{c^{x-\bar{x}}}$ , we shall get series similar to (1) and (2) but proceeding by powers of  $\gamma c^{x-\bar{x}}$  instead of powers of  $\gamma c^x$ . In these series,  $a_0, a_1, a_2, \dots$  will represent the values of life annuities at age  $\bar{x}$  (instead of annuities-certain) at successive rates of interest, and the ratios of  $\Delta^2 a / \Delta a$  will be smaller than before, a condition which will further increase the convergence of the series. Except for the greater difficulty of calculating the numerical values of  $a_0$ , &c., it would perhaps have been better to develop the whole theory of the Z method on the basis of these more convergent series.



the smallness of the ratio of the third term to the second. It will be found that the presence of the negative element in the coefficient of the third term in (2) makes the ratio in question smaller in (2) than it is in (1); and it may therefore be expected that the Z method will give a mean value of P even more accurate than the mean value of  $a$  or  $\mathbf{a}$  corresponding to the same distribution.

3. This may be illustrated by examples. (A) Taking the values on the basis  $O^{M(5)}$  3 per-cent of  $a_{x:20}$  for  $x=20, 25, \dots 50$  (corresponding to  $M=40, 45, \dots 70$ ), the successive values of  $\Delta_5 \cdot a_{x+5:20} / \Delta_5 \cdot a_{x:20}$  are 1.54, 1.53, 1.52, 1.48, 1.44, showing a progressive falling away from the value of  $c^5=1.57$ . The corresponding values of  $\Delta_5 \cdot P_{x+5:21} / \Delta_5 \cdot P_{x:21}$  are 1.57, 1.57, 1.58, 1.56, 1.58, showing only small and casual variations from the value  $c^5=1.57$ . We should therefore expect to get by the Z mean age a very close approximation to the mean value of P, and this is confirmed by trial. Taking equal weights at ages 20, 25,  $\dots 50$ , the true mean values of  $a_{x:20}$  and  $P_{x:21}$  are respectively 13.252 and .04119; and the values of  $a$  and P corresponding to the Z mean age ( $39\frac{1}{6}$ ) are respectively 13.223 and .04118. (B) On the same basis taking  $\mu=30$ ,  $x=20, 25, \dots 40$ ,  $M=50, 55, \dots 70$ , the results may be briefly summarized as follows. Ratios of  $\Delta_5 a$ , 1.51, 1.49, 1.44; ratios of  $\Delta_5 P$ , 1.57, 1.56, 1.55. With equal weights at ages 20, 25,  $\dots 40$ , the true mean values of  $a$  and P are respectively 17.022 and .02643, while the values of  $a$  and P corresponding to the Z mean age ( $32\frac{1}{4}$ ) are respectively 16.995 and .02645. The results on the  $O^{[M]}$  Table would evidently be similar, since that table follows Makeham's Law with the same value of  $c$ .

4. In the important practical case of the  $O^M$  Table we should expect results less accurate—though accurate enough to supply a valuable check on gross errors—because that table does not follow Makeham's Law, and the application of the Z method involves the assumption that the mortality can be adequately represented by Makeham's Law with a uniform value of  $c$ , usually taken in practice as 1.08, for the ages with which we are principally concerned in Endowment Assurance business. Taking the same examples as before, but on the  $O^M$  basis, the results may be summarized as follows. (A) Ratios of  $\Delta_5 a$ , 1.29, 1.28, 1.35, 1.41, 1.42, steadily increasing but all less than  $c^5=1.47$ . Corresponding

values of  $\Delta_5 P$ , 1.29, 1.33, 1.41, 1.49, 1.55, steadily increasing, earlier values less but later values greater than  $c^5$ . True mean value of  $a_{x:20|}$  and  $P_{x:21|}$ , 13.366 and .04066 respectively. Values of  $a_{x:20|}$  and  $P_{x:21|}$  corresponding to the Z mean age ( $38\frac{5}{8}$ ), 13.324 and .04069 respectively. (B) Ratios of  $\Delta_5 a$ , 1.30, 1.31, 1.34, slightly increasing but all considerably less than  $c^5$ . Ratios of  $\Delta_5 P$ , 1.35, 1.39, 1.46, steadily increasing, practically up to  $c^5$ . True mean values of  $a_{x:30|}$  and  $P_{x:31|}$ , 17.237 and .02580 respectively. Values of  $a$  and  $P$  corresponding to the Z mean age ( $31\frac{6}{7}$ ), 17.185 and .02587.\*

5. The writer is not able to give the results of a comprehensive test on the  $O^M$  Table, but the above results encourage the hope that the test would be a valuable one, the accuracy of which could no doubt be improved by a simple system of small adjustments of the Z mean ages (*cf. J.I.A.*, vol. xxxviii, p. 49), such adjustments being based on the average distribution of the business of each particular Office.

6. In the writer's own Office the valuations are made on Offices'  $O^{M(5)}$  Table, and consequently only the  $O^{M(5)}$  Z's are available, while the net premiums are based on the  $O^M$  Table. Extensive trials of the test here described have been made for the whole of the Endowment Assurance business for some years. It has been found that, by making a small constant deduction from the  $O^{M(5)}$  mean ages, so close a check can be obtained on the  $O^M$  pure premiums as to detect quite small errors in individual policies. If, however, the results are found in practice to be less close the fineness of the test can of course be increased, with little additional trouble, by dividing each group into two or more sections—a course which may be otherwise convenient for balancing purposes.

7. It is to be observed that the method really supplies a concurrent test of the Z's and the pure premiums, since any considerable error in either would be disclosed by a corresponding discrepancy between the pure premiums separately calculated and those calculated by the mean age method.

8. The practical working of the test is of course very simple, because for the purpose of placing the new business on the books the business will almost certainly be divided into groups corresponding to the original terms of the assurances, that is in precisely the form required for the test.

\* The tests in paragraphs 4 and 5 are rather severe, since equal weights of maturity-ages ranging from 40 to 70 are hardly likely to arise in practice.



It will of course be necessary to exclude (1) cases "dated back" to a previous year, involving a disturbance of the original term, (2) cases of specially adjusted premiums, as for policies transferred and converted from other classes, (3) the sums assured and corresponding Z's for paid-up policies if these are included in the same classification with the annual premium policies. It may also be desirable to exclude any exceptionally large or otherwise "abnormal" case (*cf. J.I.A.*, vol. xxxviii, p. 32), though in practice this is but rarely found to be necessary.

9. The above remarks may throw some light on the method of classification of Industrial Endowment Assurances recently referred to by Mr. H. H. Austin, *vide supra*, p. 482. Mr. Austin "adopted" a classification by years of entry and original terms, the entry "age being found by dividing the premium by the sum assured." The mean entry age so found increased by the original term would give, as the above results show, a very close approximation to the mean maturity age, found by the Z method, if the *pure* net premiums were used; and assuming the loading to be fairly expressible as a constant plus a percentage of the pure premium, very similar results would be obtained by working on the *office* premiums. The method in question is therefore virtually another way of bringing out a Z valuation, basing the Z's on the premiums instead of on the sums assured; and as the two sets of mean ages are not found in practice to differ materially (*cf. J.I.A.*, vol. xxxviii, pp. 16-19), the method might be expected to give, as Mr. Austin finds that it gave, "results which compared quite favourably even with the Z method of weighting."

G. J. L.

## LEGAL NOTES.

By ROBERT ALLEN BATEMAN, B.Sc. (Econ.), *Barrister-at-Law*.

(1) *Is the difference between the amount paid for Treasury Bills and the amount received whether on a sale or at maturity a "profit on discount" within Case 3 of Schedule D of the Income Tax Act, 1842, and (2) may a person be charged with income tax in respect of such a difference in a year in which no such bills are held and no transactions take place in respect thereto?*

National  
Provident  
Institution  
v.  
Brown.  
Provident Mutual  
Life Assurance  
Association  
v.  
Ogston  
(1920) 3 K.B. 35.  
37 T.L.R. 804.

"Yes", said the House of Lords, in regard to (1) and "No" in regard to (2).

It will be recalled that Rowlatt, J., decided all points in

favour of the Crown as already reported in this *Journal* in vol. li, p. 352, where the full facts of the case will be found. On appeal the Court of Appeal materially varied the judgment of Rowlatt, J., by making a distinction between Bills sold during their currency and Bills held to maturity. In regard to the former by a majority the Court decided that it was necessary to send the case back to the Special Commissioners to decide in the case of each appeal how much represented actual discount and how much represented an accretion to capital. The Court unanimously reversed the verdict of Rowlatt, J., on the point as to whether an assessment could properly be made in a year where no bills were held and no transactions took place. On the point of Bills sold before maturity Lord Sterndale, M.R., said : "The case where the bill is sold before maturity is not so simple. If all other elements were eliminated the increased value of the bill would be regulated by the extent to which it had advanced towards maturity. But the other elements cannot be eliminated. The price of the bill in the market depends upon the state of the money market and the rise or fall in the value of money and any increased price attributable to these causes cannot be taxed as profit on a discount. In the case of a sale, therefore, I think the only amount that can be taxed is the amount by which the bill has increased in value by reason of its advance towards maturity and the consequent accrual of interest upon it. The amount of profit arising from the fluctuation in value of money does not arise from the discount, *i.e.*, the difference between the present value and the value at maturity, and does not therefore come within the words 'profit on a discount.' It might be taxed as profits arising from a business of discounting, but the Crown have deliberately elected not to assess the appellants under this head, no doubt because in that case account would have to be taken of losses. It follows from this that the appellants have not been assessed on a right principle in the case of sales for they have been assessed on the total profits made on the sales. and the case should go back to the Commissioners in order that the proper adjustment should be made in these cases." Scrutton, L.J., on the same point said : "The price, therefore, of a Treasury Bill would depend on two matters : (1) How much interest or discount had accrued by the progress of the bill towards maturity ; (2) how much the value of the promise to pay had altered by the rise or fall of the value of money.

“ While in taxation of a trade the latter element would be  
“ included in the profits of the trade, in my opinion in the  
“ taxation of interest or discount it is not included, for it is  
“ appreciation or depreciation of the capital sum. If the Crown  
“ taxed an insurance company, or a discount house, as a trade.  
“ then profits by the sale of bills during currency would be  
“ included, but then the taxpayer would deduct his working  
“ expenses and his losses. The representatives of the Crown  
“ think they do better by taxing an insurance company on the  
“ interest of its accumulated funds, and not as a trade. They  
“ may be wise to do this, but if they make this election they  
“ cannot, in my view, assess as interest or discount what is  
“ really a profit from buying and selling, additional to interest.  
“ The result in the present case appears to be that where the  
“ Institution has been taxed in respect of the year when Treasury  
“ Bills mature, on bills held to maturity, on the difference  
“ between amounts paid and amounts received, it is rightly  
“ taxed ; but that where the Institution is taxed on bills sold or  
“ discounted within the year on the difference between amounts  
“ paid and amounts received, it is wrongly taxed. for it is being  
“ taxed not only on interest or discount, but on an amount  
“ increased by appreciation or accretion, or decreased by loss. of  
“ capital.”

Dissenting from these judgments, Warrington, L.J., said :  
“ When a holder, whether the original purchaser or not, realizes  
“ during currency, he really receives a proportion of the total  
“ profit resulting from the fact that the bill was bought at a  
“ discount. It is true that that proportion may not bear an  
“ exact relation to the period of currency but may be determined  
“ by variations in the value of money, in the public credit and  
“ so forth. But it seems to me that the total of the profits  
“ received by the various sellers after deducting losses, if any.  
“ cannot exceed the difference between the price originally paid  
“ and the sum payable at maturity, and that the considerations  
“ I have referred to merely affect the distribution of that  
“ difference between the various holders. Profits made by  
“ discounting bills seem to me to rest on the same footing, and  
“ conversion into War Loan also.”

On the second point as to whether a person may be taxed under Case 3 in respect of a year in which there have been no transactions, Lord Sterndale, M.R., said : “ It seems to me to  
“ be a general principle of income tax law that a person in order

“ to be taxable in a particular year must have an income arising  
“ from a source existing in that year, and in order to justify this  
“ assessment the Crown must show some reason for departing  
“ from that general principle. It is admitted that if the taxation  
“ be in respect of a trade or business or an office or of property,  
“ the taxpayer must continue in the year of charge to carry on  
“ the trade or business or hold the office or the property. It  
“ was, however, contended for the Crown that the principle did  
“ not apply in this case because by the first rule of the Third  
“ Case the duty to be charged was computed according to the  
“ profits of the preceding year, and therefore if the last year  
“ was not taxed because there was no source one year escaped  
“ taxation altogether. I do not think the first rule has this  
“ effect. The provision as to computation of profits is the same  
“ as that in respect of trades ”; to which Warrington, L.J.,  
added: “ It was argued before us that the source of income  
“ was the possession by the Institution of funds for investment  
“ and that such source existed although no funds were invested  
“ in the particular manner in the year of assessment. I cannot  
“ accept this construction. I think the source to be looked for  
“ in each case is a separate source from which the profits to be  
“ charged are derived.”

When the case came before the House of Lords the decision of the Court of Appeal was varied to the extent of differing from the view of the majority of the Court of Appeal that there is any distinction between Bills sold during their currency and Bills held till maturity. On the other point the House (Viscount Cave dissenting) agreed with the judgment of the Court of Appeal. In regard to the first point Lord Haldane adopted the precise judgment of Warrington, L.J., quoted above which in his opinion was unanswerable. On the second point Lord Haldane said: “ It seems to me that the true meaning of the words the  
“ Legislature has used is that the tax is intended as matter of  
“ basic principle to be on profits and gains forming income in the  
“ year of assessment, though not measured by the income of  
“ that year. If a man carries on business by buying and  
“ discounting bills, this is, I think, as much a source of profit as  
“ any other for the purposes of the words employed. As in the  
“ case before us it is agreed that there was no such source I think  
“ that we have to assume that there was no income on which to  
“ base the tax. Reading the Income Tax Acts as a whole, it  
“ appears to me that the tax is one of a single kind based, speaking

“broadly, on a single principle. It is imposed on existing income, however the amount to be levied is to be computed in particular instances. This appears to be the scheme followed out in the Acts. and I think that if departure from it is alleged at any special point, the departure ought to be clearly demonstrated. The natural construction of the language of the third clause of Schedule D appears to me to be that the tax is imposed only where there are profits and gains arising within the year of assessment. but that the amount payable is to be measured by reference only to the profits and gains arising within the preceding year.”

Lord Cave in a dissenting judgment, agreed with the view held by Rowlatt, J., that “the Legislature has regarded the taxpayer himself as the only source of profit which need exist in the year of assessment and has taxed him upon any profits of this character which he may have made in the preceding year.”

Lord Cave seemed much impressed by the possibility of the decision of the Court of Appeal leading to the curious position that: “a taxpayer if he should think fit to confine his transactions of this character to alternate years would enjoy all his profits on such transactions without liability to tax.” This point was answered by Lord Sumner, who said that “this in-and-out attention to business is easier to state in theory than to carry out in practice.”

Lord Atkinson’s judgment was useful as containing a very exhaustive review of the Income Tax Acts and he made a point that in the present case the Income Tax Authorities are evidently aiming to read into the Income Tax Acts the special provisions of the Finance (1909—10) Act of 1910 in regard to super-tax. His Lordship was considerably impressed by the fact that the Crown were unable to produce a single authority in their favour in spite of the fact that they were proceeding under an Act passed 80 years ago.

Attorney-General  
v.  
The Public  
Trustee.  
(1920) 3 K. E. 675.

*Where, in contemplation of a marriage a policy of assurance is taken out by the intended husband in his own name on the understanding that an uncle of the intended wife shall pay the first and all subsequent premiums and such policy is brought into the Marriage Settlement, is succession duty payable in respect of the policy moneys as on a succession from the uncle?*

“Yes”, said Mr. Justice Rowlatt, on the facts which were



shortly as follows: The intended husband entered into an agreement which provided *inter alia* that he should effect a policy on his life for £2,000 upon which the husband of an aunt of the wife agreed to pay the first and all other premiums as they should become due during the joint lives of the husband and wife. The uncle paid the first premium before the Marriage Settlement was made and continued to pay the premiums down to the time of his death after which the premiums were paid by his executors until the death of the husband, who had survived his wife. The Crown claimed that on the death of the husband Succession Duty became payable by the Defendant, who, in the meantime, had been appointed Trustee of the Marriage Settlement, as on a succession from the uncle by virtue of the disposition made by the Settlement. In giving judgment for the Crown, Rowlatt, J., said: "A certain policy of insurance on the life of the Rev. J. Hart Davies, which had been effected by him in contemplation of his approaching marriage, and on which the first premium had been paid by Sir M. J. Cholmeley, an uncle of the intended wife, was, by the marriage settlement, assigned upon the trusts therein declared in favour of the husband, wife and children of the marriage, and all the subsequent premiums on the policy have been paid by the uncle or his executors; and, the husband having died, the question is whether the trustee of the settlement on behalf of the beneficiaries takes the policy moneys from the husband or from the uncle as 'predecessor' within the meaning of s. 2 of the Settlement Duty Act, 1853. That section enacts: 'the term "predecessor" shall denote the settlor, disponent, testator, obligor, ancestor, or other person from whom the interest of the successor is or shall be derived.' I apprehend that where the subject matter to which the successor becomes entitled is of a permanent and pre-existing kind such as land or funds, the words 'from whom the interest of the successor is or shall be derived' point to the person whose property it was before the settlement was made under which the interest of the successor arises, and who brought it into settlement. In the present case, the subject matter with which we are dealing is a policy of life assurance, which only comes into existence when it is taken out, and the value of which is reckoned on a calculation of the time that will elapse and the premiums that will be paid before the death happens, and is always fixed, subject to any profit that may be declared upon



“ the policy. In spite, however, of the fact that a policy of  
“ insurance is a subject-matter of a peculiar nature, if it can be  
“ found that it belonged to some person before the settlement,  
“ and that he settled it, I take it that without doubt the person  
“ would be the ‘ predecessor ’ contemplated by the section.

“ It has been argued on behalf of the Crown that Sir M. J.  
“ Cholmeley was the person whose property this policy was  
“ before the settlement, and who in effect settled it, inasmuch  
“ as he paid the first premium upon the policy before the  
“ settlement was made, and all the subsequent premiums.

“ On the other hand it has been argued on behalf of the  
“ defendant, that Mr. J. Hart Davies, the husband, was the  
“ person to whom the policy belonged and who brought it into  
“ settlement, inasmuch as he personally made the contract with  
“ the insurance company, and the policy was effected in his  
“ name, and formally assigned by him to the trustees of the  
“ settlement.

“ I think that it is quite clear that in cases of this kind one  
“ ought to look at the substance and not merely at the form of  
“ the transaction. One has to find out the person from whom  
“ the property really comes into the settlement; and the  
“ machinery by which, or the name under which, it is passed  
“ into, or through the course of, the settlement makes no  
“ difference.

“ The contention on behalf of the Crown—namely, that the  
“ policy, was the property of Sir M. J. Cholmeley, because he  
“ paid the first premium—appears to me to be attended by  
“ several difficulties. One of these is that it implies that  
“ Sir M. J. Cholmeley must have enjoyed, at least for the first  
“ period of its existence, an insurance upon a life in which he  
“ had no insurable interest. Another is that this policy was  
“ created not only by the money of Sir M. J. Cholmeley, which  
“ paid the premiums, but also by the acts of the husband in  
“ making the application for the policy and attending to be  
“ examined in respect of it. Nor do I think that the contention  
“ on behalf of the defendant, that the policy was the property  
“ of the husband before the settlement, is free from objection.  
“ If it was his property, then, supposing that he had died before  
“ the execution of the settlement, his executors would have  
“ taken the policy moneys simply as being property which he  
“ himself had provided without having derived it from any  
“ predecessor. I suppose, indeed, that the defendant would

“ have to go even further and maintain that immediately after  
“ Sir M. J. Cholmeley had paid the first premium on the taking  
“ out of the policy, the husband would have been entitled to go  
“ to the office of the insurance company, and, with or without  
“ consideration, to surrender the policy, or to assign it to another  
“ person. He could only be entitled to do that on the theory  
“ that he had received a gift of the first premium from Sir M. J.  
“ Cholmeley. It does not appear, however, that he did receive  
“ the first premium, or that, if he had immediately surrendered  
“ or assigned the policy, Sir M. J. Cholmeley could not have  
“ recovered back the amount of the first premium, not perhaps  
“ as money paid on a consideration which had failed, because the  
“ consideration had not failed, but at all events as money lent.

“ The true position appears to me to be this: Mr. Hart  
“ Davies applied for the policy and was examined with reference  
“ to it, but he only did so upon the footing that Sir M. J.  
“ Cholmeley was going to pay the premium; and Sir M. J.  
“ Cholmeley paid the premium, but he only did so upon the  
“ footing that Mr. Hart Davies had taken the policy out in his  
“ own name and was going to settle it for the benefit of his wife  
“ and the children of the marriage. It therefore seems to me  
“ that this policy never had a moment's existence as the free  
“ property of either of these gentlemen. Moreover the object  
“ for which the policy was brought into existence by the  
“ co-operation of these two persons was that it might be included  
“ in the marriage settlement, and whoever could legally deal  
“ with it as a matter of form, it was subject to a trust to go into  
“ that settlement. If, after the policy had been effected, the  
“ settlement had been made, but the marriage had not taken  
“ place owing to the previous death of either party, or from some  
“ other cause, the policy would no doubt have passed to the  
“ intended husband or his executors, but it would have so passed  
“ under the trusts of the settlement, and not by virtue of its  
“ being his independent property. If the settlement had never  
“ been made by reason, for example, of the previous death of  
“ the intended husband, the policy moneys would have gone to  
“ his executors, but that result would have followed because it  
“ would have been necessary to deal with these moneys as if  
“ the settlement had been made, and not because the policy  
“ had belonged absolutely to the intended husband. And if the  
“ settlement had never been made, not because of the death of  
“ the intended husband, but because the marriage went off, it

“ may be that in that case, as in the last-mentioned case, the husband would have got the policy, or it may be that Sir M. J. Cholmeley would have had the right to claim it. In no case, however, do I think that the policy after it was taken out and before the settlement was executed, would have been the free property of either party, because the difficulty of accepting either view appears to me to be insurmountable. I therefore think that I ought to regard the policy in question as having been created for the purpose of the marriage settlement *codem instanti* with the settlement, and as a thing which had no existence antecedent to the settlement. That is what it is in substance, and it is to its substantial nature that I ought to look.

“ In these circumstances I have to find out who is the ‘ predecessor ’ from whom the policy has been derived by the persons entitled to it under the settlement. I have come to the conclusion that I ought to hold that the policy has been derived from Sir M. J. Cholmeley. The essential fact is that the value of the policy is due to the money which he put into it, the amount having been based beforehand upon the prospect of life of the person insured. The policy moneys correspond to the premiums paid by him during the time of the insured’s life. In short, if I may so express it, the whole financial source of the policy and of the policy moneys was in this case Sir M. J. Cholmeley.”

Paddy  
v.  
Clutton  
(1920 2 Ch. 554.)

*Is the holder of a policy payable at the end of a term of years, who has mortgaged certain property to the insurance company, charging the policy by way of collateral security, the insurance company having subsequently sub-mortgaged the property to a third party and afterwards gone into liquidation, entitled to set off the value of the policy against the amount due to the sub-mortgagee in the event of his desire to redeem ?*

“ No”, said Russell, J., in a case, the circumstances of which were as follows : Paddy owned certain freehold property and in 1906 took out a policy of assurance with a company for £400 to be paid at the end of 28 years. In 1907 he conveyed the property by way of mortgage to the company to secure £320 and interest and charged the policy by way of collateral security. In 1907

by an indenture of sub-mortgage supplemental to the mortgage the company assigned to Clutton a sum of £320 then due under the mortgage and interest to become due, with proviso for redemption. In 1909 the business and assets of the company were taken over by the National Standard Life Assurance Corporation. On 25 May 1916 Clutton died. On 6 June 1916 an order was made for winding up the corporation and its assets were insufficient for payment of its debts in full. On 29 June 1916, Clutton's executors gave Paddy notice of the sub-mortgage and the latter thereafter paid the interest to the executors. On 17 January 1920, Paddy who desired to redeem the property tendered to the executors £200, being £325 due on the mortgage for principal and interest less £125 at which sum the policy had been valued in the winding-up under the Assurance Companies Act, 1909. The tender was refused and Paddy obtained leave to commence an action for a declaration that he was entitled to set off a sum of £125 against the amount due under the mortgage and for redemption. In giving judgment Russell, J., said that "at the  
" date of the winding up the policy had not matured or become a  
" claim, and, the value of the policy or the liability under it  
" having since been ascertained and not disputed, the plaintiff  
" was bound by the valuation and had become a creditor of the  
" corporation for £125 4s. The plaintiff's claim arises by virtue  
" of the provisions of the Bankruptcy Act, 1914, which by  
" s. 207 of the Companies (Consolidation) Act, 1908, are made  
" applicable to the winding up of an insolvent company." (His Lordship read s. 30, sub-ss. 1 and 3). "So in this case the  
" liability of the corporation, being a liability future or  
" contingent, is to be deemed to be a debt provable in the winding  
" up." (His Lordship read sub-ss. 4 and 8 (b) (c). "By having  
" resort to those provisions the £125. 4s., the value of the  
" corporation's liability, is arrived at and admitted. Then s. 31  
" provides for cases of mutual credit and set-off, so as to prevent,  
" where something is due on either side, one party from receiving  
" his amount in full without giving credit to the other." (His Lordship read s. 31 to 'respectively'). "It is under that  
" section that the plaintiff claims. He says that under that he  
" is entitled to set off his claim against the corporation's claim  
" against him. It was contended also, on his behalf, that,  
" apart from bankruptcy, he had a common law right to do this.  
" As to that I do not agree with him. In my opinion, his right  
" (if any) of set-off arises only under the administration in



“ bankruptcy. It is well settled that at common law set-off  
“ can only be made available as a plea in bar. Supposing the  
“ plaintiff had been sued for his mortgage debt, he could not  
“ have set off the £125. 4s., he would have had only the right  
“ to counter-claim for it, and could get nothing in respect of it  
“ till it should be ascertained in the action what the respective  
“ rights were. Till then he would have no right of set-off.

“ Whether the plaintiff has any right of set-off in the  
“ winding-up against the defendant corporation would be a  
“ difficult question to answer, having regard to the state of  
“ the authorities, in the absence of any decision expressly  
“ covering the case . . . . . I will refer first to *Lee &*  
“ *Chapman's Case* . . . . . So far, *Lee & Chapman's Case*  
“ shows that in order to be able to set off it is not necessary  
“ that the claim should be a legal debt, but it is sufficient if the  
“ claim which it is sought to set off is a right which could be  
“ proved in the winding-up. . . . . *Ex parte Price* is just  
“ like the present case. It dealt with a policy which was current  
“ at the crucial date. It was like the present case also in this,  
“ that the person claiming to set off was claiming in respect of  
“ a mortgage transaction. It was a case before James and  
“ Mellish, L.JJ. It has never, so far as I know, been questioned  
“ in any reported case. It has been cited without adverse  
“ comment by many text-writers, though one or two suggest  
“ that it might not be followed . . . . . In *Ex parte Price*  
“ Dr. Edwin Lankester was the holder of four policies of assurance  
“ on his own life in the English Widows' Fund, which was  
“ amalgamated with the European Life Assurance Society, for  
“ sums amounting altogether to £1,049. In November 1865,  
“ he borrowed £140 from the European Society, after the  
“ amalgamation, on the security of the policies. He thus  
“ became debtor to the assurance society, which held as security  
“ the policies on his life. On 12 January 1872 the two companies  
“ were ordered to be wound up, and their affairs were afterwards  
“ referred to arbitration under a private Act, and official  
“ liquidators were appointed by the arbitrator. By clause 6 of  
“ the private Act the arbitrator had power to value and estimate  
“ any liabilities or claims, whether present, future, liquidated,  
“ contingent, or otherwise, and ‘ liability ’ as defined by clause 2  
“ of the Act included ‘ policy . . . . . and any liability, present,  
“ ‘ future, liquidated, contingent, or other, on or in respect of  
“ ‘ a policy.’ Under that Dr. Lankester's policies were valued

“ at £446. 8s. and dividends amounting to £66. 19s. had been  
“ declared on this sum, being at the rate of 3s. in the pound.  
“ So Dr. Lankester would receive only £66. 19s. On 28 March  
“ 1872, Dr. Lankester filed a petition for liquidation, and on  
“ 1 May his creditors agreed to accept a composition, and a  
“ trustee was appointed. Dr. Lankester died in 1874, and his  
“ estate was insolvent and expected to pay 10s. in the pound.  
“ The official liquidators sought to prove against his estate  
“ under the composition for £140 and interest, but the trustee  
“ claimed a right to set off against the demand the sum of  
“ £446. 8s. at which the policies had been valued, and to prove  
“ for the balance against the estate of the European Society  
“ under the arbitration. The Registrar allowed the set-off and  
“ the official liquidators appealed from that decision. . . . .  
“ The Court of Appeal held there could be no set-off allowed.  
“ James, L.J., said: ‘ I am of opinion in this case that the  
“ ‘ official liquidators under winding up are clearly entitled to  
“ ‘ prove for the debt simpliciter against the estate of  
“ ‘ Dr. Lankester. There is no right of set-off under the  
“ ‘ Companies Act ’—that was true at that time—‘ except that  
“ ‘ of the common law right of set-off, which would be allowed  
“ ‘ in an ordinary action at law. That being so, according to  
“ ‘ my view there is no set-off against the claim of Dr. Lankester’s  
“ ‘ trustee under the winding up in favour of the liquidators.  
“ ‘ Nor, on the other hand, is the trustee entitled to a set-off in  
“ ‘ bankruptcy against the claim of the liquidators.’ He is not  
“ saying there is no set-off at all but he is contrasting the two  
“ cases. ‘ The mutual credit clause in the Bankruptcy Act, 1869,  
“ ‘ enlarged by these words, “ Mutual dealings,” still requires  
“ ‘ that there must be something of an account to be taken of  
“ ‘ what is due upon the one side and what is due upon the  
“ ‘ other ’—*i.e.*, there must be something of a nature on which  
“ an account can be taken. ‘ In that sense there never was  
“ ‘ anything due from the insurance company in which an  
“ ‘ account could be taken. It is impossible to say what was  
“ ‘ due upon the policy when the winding up commenced, any  
“ ‘ more than that on a subsequent occasion some valuation had  
“ ‘ been put upon the policy, and that a dividend had been  
“ ‘ ordered to be paid upon it under the winding up.’ That is  
“ exactly applicable to the present case. ‘ But that cannot  
“ ‘ make it a thing which was due by reason of mutual dealings  
“ ‘ and transactions between the parties, and you can by no



“ ‘possible construction of the clause bring this in as a debt  
 “ ‘due on the one side to be set off against a debt due on the  
 “ ‘other. It is simply that each party will prove for what he  
 “ ‘can.’ The judgment of Mellish, L.J., is to the same effect.  
 “ He says : ‘I am of the same opinion. It is quite clear that if  
 “ ‘it had been a solvent company that had lent money to the  
 “ ‘bankrupt, and the company had never stopped or been wound  
 “ ‘up, and the company had come to prove for the debt due,  
 “ ‘the trustee never could have claimed a set-off. There would  
 “ ‘have been no cause of action and no proof at all. It does  
 “ ‘not, in my opinion, make any difference that the company  
 “ ‘is being wound up.’ . . . . .

“ That case was a decision of the Court of Appeal. It is a  
 “ case which has caused difficulty in more ways than one, and it  
 “ may not be possible to justify, in view of later decisions, all  
 “ that is there said. But it is a decision which is exactly in  
 “ point. There it was decided that, when a policy holder in a  
 “ life assurance company had borrowed money from the company  
 “ on his policy, and before the death of the assured the company  
 “ was wound up and an estimated value put upon the policy,  
 “ and afterwards the policyholder filed a petition for liquidation,  
 “ the trustee in his bankruptcy was not entitled to set off the  
 “ value as ascertained of the policy against the amount advanced  
 “ by the company. Much as I should like to be bold enough  
 “ not to follow *Ex parte Price*, I feel bound to follow it. It has  
 “ never been adversely commented on judicially. I must hold  
 “ that the plaintiff has not any right of set-off as against the  
 “ defendant corporation, and the same result follows as against  
 “ the defendant corporation’s assignee, the sub-mortgagee.”

Greig  
 v.  
 Trustees of  
 Widows’ Fund,  
 &c., of Edinburgh.  
 1921 W C. and  
 Ins. Rep. 46.

*Is there a distinction between the presumption of life  
 in cases of contract and cases of succession ?*

“ No,” said the Court of Session in this case which it will be  
 recalled from the earlier report in this *Journal* which will be  
 found in vol. lii, p. 91, was for a declaration that one David  
 Greig, Junior, must be presumed to have died prior to 31 December  
 1900. In dismissing the action and refusing to allow a proof,  
 the Lord Ordinary (Blackburn) said : “ The presumption at  
 “ common law is that a person who has disappeared is presumed

“to continue in life until he would have reached the ordinary limit of human age, unless sufficient evidence is adduced to raise a presumption that he has died within that period.” From this the pursuer reclaimed and argued (see 1919 W.C. and Ins. Rep. 331) that in the circumstances the pursuer was entitled to a proof. The Court of Session allowed the proof. To the brief outline of the facts given in the earlier report in this *Journal* it is only necessary to add that evidence was adduced of intemperance on the part of Greig.

After completion of the proof Lord Blackburn assoilzied the defenders. “It is apparently the first time,” he said, “that an endeavour has been made to elide the presumption of life for the purpose of enforcing against a third party a contract conditional upon the missing man’s life. . . . In all cases dealing with succession . . . the ultimate loss, if any, in the event of the reappearance of the missing man falls upon the man himself who is the person primarily responsible for the position which has been created. In a case arising upon a contract with a third party the loss, if any, falls elsewhere.”

The pursuer reclaimed and argued that the defenders erred in maintaining the presumption of life was up to 100 years and that the distinction drawn by the Lord Ordinary between cases of succession and cases of contract was unsound. The Court of Session, by a majority, drew from the evidence the inference that Greig presumably died not later than 31 December 1910.

The Lord Justice Clerk said: “Treating the matter as a jury question, which it is, I think the most reasonable presumption is that Greig, who was 43 in 1900, and has not been heard of since about that date has been long since dead, and that the reasonable result from all the evidence is that he did not survive 1910 . . . I do not regard the distinction the Lord Ordinary takes as to the presumption of life or death between cases of contract and cases of succession as sound. In either case the question is one of fact, and the fact is the same in both classes.”

Lord Salvesen in the course of his judgment referred to the old rule of the law of Scotland which was that presumption of life of a missing person was so strong that unless he had attained a very ripe age, variously put at 80 or 100 years as at the date when declarator was sought that he must be presumed to have died, he would still be presumed to be alive

“apart from circumstances from which it could be inferred  
 “without any reasonable doubt that he had in fact ceased to  
 “live. That rule was established when the facilities of travel  
 “and postage, not to mention advertisement, were in a backward  
 “state compared with the times in which we now live. . . .  
 “The altered circumstances of life to which I have referred have  
 “a very direct bearing on the question which has to be  
 “answered.”

A claim for interest on the arrears found due to the pursuer was not allowed.

Pool  
 v.  
 Royal Exchange  
 Assurance  
 (1921) 1 A.C. 65.

*Is a Trustee chargeable with Income Tax on the ground of his being domiciled and resident in this country when he is Trustee of funds invested abroad, the dividends on which are paid direct to the beneficiary in a foreign country?*

“No,” said the House of Lords, and in giving judgment Viscount Cave said: “In this case the respondent company which has its principal place of business in the City of London, is the trustee of the will of Mr. J. P. Mellor (deceased); and the beneficial tenant for life under the will is Mrs. H. P. Munthe, a Swedish subject domiciled abroad. The will comprises foreign investments; and the whole income from such investments is paid directly to Mrs. Munthe abroad, no part of such income being remitted to this country. The District Commissioners of Taxes for the City of London made assessments upon the respondent company in respect of foreign possessions of £2,015 for the year ended 5 April 1915, and £2,018 for the year ended 5 April 1916, these sums representing the income of the foreign investments above referred to. But these assessments differed from those which are in question in *Williams v. Singer* in one respect—namely, that instead of being made (as in that case) upon the trustees by name without reference to any trust, they were made upon the respondent company ‘as trustees under the will of J. P. Mellor deceased for beneficiary Mrs. H. P. Munthe.’ The respondent company appealed to the Special Commissioners, who discharged the assessments; and this decision also has been affirmed by Sankey, J., and the Court of Appeal, and is the subject of Appeal to this House. It was decided in *Colquhoun v. Brooks* that the tax imposed by the Income Tax Acts, 1842 and

“ 1853 (Sch. D. Cases 4 and 5), upon the income from foreign securities and possessions was leviable upon so much only of that income as was remitted to the United Kingdom. But that limitation was to some extent abrogated by s. 5 of the Finance Act, 1914, which (so far as material in this appeal) is as follows: ‘Income tax in respect of income arising from securities, stocks, shares, or rents in any place out of the United Kingdom shall, notwithstanding anything in the rules under the fourth and fifth case in section 100 of the Income Tax Act, 1842, be computed on the full amount of the income, whether the income has been or will be received in the United Kingdom or not . . . . and the provisions of the Income Tax Acts (including those relating to returns) shall apply accordingly. . . . . Provided that this section shall not apply in the case of a person who satisfies the Commissioners of Inland Revenue that he is not domiciled in the United Kingdom or that, being a British subject, he is not ordinarily resident in the United Kingdom.’ ”

“ It is obvious that, having regard to the proviso to the above section Mrs. Munthe, who is domiciled abroad, could not have been assessed to income tax in respect of the foreign income above referred to. But the Revenue Authorities contend that they are entitled to levy tax upon that income by means of assessments upon the trustees, who are domiciled in this country. If this contention is upheld, the trustees will of course be entitled to retain the tax so paid out of the trust income payable to the beneficial life tenants, who will thus have to bear the burden of the tax from which the proviso appears to relieve them; but the appellants contend that this is the effect of the statutes. The question to be determined is whether they have that effect.

“ In support of the above contention counsel for the appellants relied principally upon the language of Sch. D. to the Income Tax Act, 1853, which provides that the duties thereby imposed are to be deemed to be granted and made payable ‘for and in respect of the annual profits or gains arising or accruing to any person residing in the United Kingdom from any kind of property whatever, whether situate in the United Kingdom or elsewhere,’ and upon the first general rule in s. 100 of the Income Tax Act, 1842, which provides that the duties upon profits imposed by Sch. D are to be charged on and paid by the persons ‘receiving or

“ ‘entitled unto’ such profits; and they contended that as the  
“ income in question in the cases under appeal ‘accrued’ to the  
“ trustees as the legal holders of the investments, and the trustees  
“ are the persons legally ‘entitled’ to receive it, they are the  
“ persons chargeable under the Act. Indeed, I understood  
“ Mr. Cunliffe to go so far as to say that, when funds are vested  
“ in trustees, the revenue authorities are entitled to look to  
“ those trustees for the tax, and are neither bound nor entitled  
“ to look beyond the legal ownership.

“ I think it clear that such a proposition cannot be  
“ maintained. It is contrary to the express words of s. 42 of  
“ the Income Tax Act, 1842, which provides that no trustee who  
“ shall have authorized the receipt of the profits arising from  
“ trust property by the person entitled thereto, and who shall  
“ have made a return of the name and residence of such person  
“ in manner required by the Act, shall be required to do any other  
“ act for the purpose of assessing such person. . . . .

“ The fact is that if the Income Tax Acts are examined, it  
“ will be found that the person charged with the tax is neither  
“ the trustee nor the beneficiary as such, but the person in  
“ actual receipt and control of the income which it is sought to  
“ reach. The object of the Acts is to secure for the State a  
“ proportion of the profits chargeable and this end is attained  
“ (speaking generally) by the simple and effective expedient of  
“ taxing the profits where they are found. If the beneficiary  
“ receives them he is liable to be assessed upon them. If the  
“ trustee receives and controls them, he is primarily so liable.  
“ If they are under the control of a guardian or committee for  
“ a person not *sui juris* or of an agent or receiver for persons  
“ resident abroad they are taxed in his hands. But in cases  
“ where a trustee or agent is made chargeable with the tax the  
“ statutes recognize the fact that he is a trustee or agent for  
“ others and he is taxed on behalf of and as representing his  
“ beneficiaries or principals. This is made clear by the language  
“ of many sections of the Act of 1842. For instance, s. 41  
“ provides that a person not resident in Great Britain shall be  
“ chargeable ‘in the name of’ his trustee or agent. Sect. 44  
“ refers to the trustee or agent of any person as being assessed  
“ ‘in respect of’ such person, and gives him a right to retain  
“ the tax out of any money of such person coming to his hands.  
“ Sect. 51, under which trustees and others are bound to make  
“ returns, refers to the event of the beneficiary being charged



“ either ‘ in the name of ’ the trustee or other person making the return, or in his own name. Sect. 53 refers to the trustee or agent as being charged ‘ on account ’ of the beneficiary ; and similar expressions are found in other sections. In short, the intention of the Acts appears to be that where a beneficiary is in possession and control of the trust income and is *sui juris*, he is the person to be taxed ; and that while a trustee may in certain cases be charged with the tax, he is in all such cases to be treated as charged on behalf or in respect of his beneficiaries, who will accordingly be entitled to any exemption or abatement which the Acts allow.

“ Applying the above conclusions to the present case, it follows in my opinion, first, that the respondent trustees, who have directed the trust income to be paid to the beneficial tenants for life and themselves receive no part of it, are not assessable to tax in respect of such income ; and secondly, that even if they were so assessable, they would be assessable as trustees on behalf of the life tenants, who would accordingly be entitled to the benefit of the exemption contained in the proviso in s. 5 of the Finance Act, 1914. The assessments in question in *Pool v. Royal Exchange Assurance Company*, which were made upon the respondents as trustees for the beneficiary Mrs. Munthe, and were probably so made with reference to ss. 41 and 108 of the Act of 1842, support this view of the Acts. . . . .

“ The above conclusion is supported by the consideration that under the express words of s. 5 of the Finance Act, 1914, a person thereby charged with tax is authorized to deduct from the taxable income ‘ any annuity or other annual payment payable out of the income to a person not resident in the ‘ United Kingdom.’ It is difficult to believe that it was the intention of the Legislature, while exempting from the tax any definite part of the income which is payable to a person abroad, to impose the tax upon the whole income when so payable.”

*Has the omission of the words “ at law or in equity ” from the Companies (Consolidation) Act, 1908, and the Bankruptcy Act, 1914, deprived the Assignee of a definite part of a debt of the right to present a winding up petition under the Companies Act ?*

*In re*  
Steel Wing Co.,  
Ltd.  
(1921) 1 Ch. 349.

“ No ”, said P. O. Lawrence, J. P held the controlling



interest in a company to which he had made advances in respect of which the company was largely indebted to him. P transferred the whole of his shares and assigned the whole of his debt to M. The assignment was effected by a document which set out particulars showing a balance of £4,632 and proceeded "and it is this balance that P hereby transfers to M for his own benefit." The assignment was brought to the attention of the company by M and part of the debt was repaid. At a later date M assigned to one Lord, "an equal half part of the sum of £4,632, &c." Notice in writing of this assignment was given to the company and at the same time the company was required to pay the one half of £4,632 and made default in payment. Giving judgment, P. O. Lawrence, J., said: "The main question arising on this petition is whether or not the petitioner, Frederick Thomas Lord, the assignee of a definite part of a debt owing by the company, is a creditor of the company entitled to present a winding up petition within the meaning of s. 137 of the Companies (Consolidation) Act, 1908. . . . . "The next point taken by the company is that the deed of 30 July 1920 did not operate to pass to the petitioner the legal right to one half of the debt thereby assigned on the ground that an assignment of part of a debt is not within sub-s. 6 of s. 25 of the Judicature Act, 1873. I am not sure whether in the circumstances of this case this point really arises. I can find no evidence that any notice in writing of the assignment of October 1917, from Pauling to Mooney, was ever given to the company. It would appear therefore that the legal right to the debt never passed to Mooney and that the latter was only an equitable assignee and could not confer upon the petitioner a legal right which he (Mooney) did not possess. As, however, this point was not taken by the company, I will deal shortly with the question, which was fully argued, whether an assignment of a definite part of a debt is within sub-s. 6 of s. 25 of the Judicature Act, 1873.

"Darling, J., in *Skipper v. Holloway* decided that such an assignment came within sub-s. 6. Bray, J., in *Forster v. Baker* dissented from this decision and held that it did not, and in the Irish case of *Conlan v. Carlow County Council*, Gibson and Boyd, J.J., concurred in the opinion of Bray, J. Although this point has been mentioned on at least two occasions by members of the Court of Appeal (see *Durham Brothers v. Robertson and Hughes v. Pump House Hotel Company*), there

“ is no decision of the Court of Appeal upon it. Having  
“ considered the arguments addressed to me in the present case,  
“ I concur in the opinions of Bray, J. and Gibson, J. In the  
“ result I hold that even if Mooney did acquire the legal right  
“ to the debt, the deed of 30 July 1920 did not operate to pass  
“ the legal right to one half of such debt to the petitioner,  
“ and that the petitioner is not a creditor of the company  
“ at law.

“ The next point taken by the company is that the petitioner  
“ is not even a creditor of the company in equity and that he  
“ stands in no better position than a garnishor of a debt due  
“ from the company and consequently cannot present a winding  
“ up petition (see *In re Combined Weighing and Advertising*  
“ *Machine Company*). In my judgment an assignee of part of  
“ a debt stands in an entirely different position from that of a  
“ garnishor. A garnishee order does not operate to transfer  
“ the debt and does not constitute the garnishor a creditor  
“ either at law or in equity of the garnishee. The assignment  
“ of part of a debt, however, operates in equity to transfer the  
“ part assigned, and consequently in my judgment constitutes  
“ the assignee a creditor in equity of the company in respect  
“ of that part. . . . .

“ It has been suggested that, by reason of the omission from  
“ s. 130 of the Companies (Consolidation) Act, 1908, of the  
“ words ‘ at law or in equity ’ which occurred in the corresponding  
“ s. 80 of the Companies Act, 1862, the word ‘ creditor ’ in  
“ s. 137 of the Act of 1908 ought to be confined to a creditor at  
“ law. I . . . . . hold that the omission was not intended to  
“ and did not in fact, effect any alteration in the law but was  
“ occasioned solely because the words ‘ at law or in equity ’ had  
“ become unnecessary since the passing of the Judicature Acts.  
“ . . . . .

“ I am therefore of opinion that the word ‘ creditor ’ in  
“ s. 137 of the Companies (Consolidation) Act, 1908, includes a  
“ creditor in equity as well as a creditor at law, and  
“ consequently that a creditor in equity can still petition for the  
“ winding up of a company.”

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Scottish Union  
and National  
Insurance Co.  
v.  
New Zealand  
and Australian  
Land Co.  
(1921) 1 A.C. 172.

*Where a company has paid both British and Colonial income taxes and a repayment has been obtained under section 43 of the Finance Act of 1916, are preference shareholders, who are entitled to a fixed rate of interest from which the full British rate of income tax has been deducted, entitled to share in such repayment?*

"No". said the House of Lords, dismissing an appeal from the Second Division of the Court of Session in Scotland, the judgment of which has already been reported in this *Journal* in vol. lii, p. 95, where the full facts of the case will be found.

In giving judgment. Lord Haldane said: "The question raised by the special case was whether the full dividend of 4 per-cent, to which, and to no more than which, the appellants as holders of preference shares in the respondent company were entitled, having been paid in full, the respondents were entitled to deduct the whole amount of 5s. in the £ as income tax for which they were said to be accountable under the Finance Act, 1916, to the Inland Revenue authority of the United Kingdom; or whether the appellants were entitled, in respect of the preference shares on which the whole of the possible dividend to which they were entitled had been paid, to a deduction or return to them, as between themselves and the respondent company and its ordinary shareholders, of the amount allowed by s. 43 of that Act, an amount which I may for convenience treat as being 1s. 6d. out of the 5s. Sect. 43 is in these terms: 'If any person who has paid, by deduction or otherwise, United Kingdom income tax for the current income tax year on any part of his income at a rate exceeding three shillings and sixpence proves to the satisfaction of the Special Commissioners that he has also paid any Colonial income tax in respect of the same part of his income, he shall be entitled to repayment of a part of the United Kingdom income tax paid by him equal to the difference between the amount so paid and the amount he would have paid if the tax had been charged at the rate of three shillings and sixpence or, if that difference exceeds the amount of tax on that part of his income at the rate of the Colonial income tax, equal to that amount.'

"In this section the expression 'United Kingdom income tax' means income tax charged under the Income Tax Acts; and the expression 'Colonial income tax' means income tax

“ ‘ charged under any law in force in any British possession or  
“ ‘ any tax so charged which appears to the Special Commissioners  
“ ‘ to correspond to United Kingdom income tax.

“ The respondent company carried on a large part of its  
“ business in New Zealand and Australia. It earned profits  
“ there on which Colonial income taxes were charged. The  
“ amounts of these Colonial taxes were deducted by the  
“ respondent company, just as were general expenses, and what  
“ remained as nett balance was remitted to Scotland, where  
“ the respondent company was domiciled, to be divided as  
“ profit earned along with other profit coming into general  
“ revenue account. There is no doubt about the liability of  
“ the respondent company to United Kingdom income tax in  
“ respect of these profits, nor is there any doubt about their  
“ right to claim a repayment of allowance under the terms of  
“ s. 43 which I have quoted. The question which arises is  
“ whether the preference shareholders, having received their  
“ dividends in full, are entitled to claim against the respondent  
“ company, and as between themselves and the company, to  
“ participate in the allowance made by the United Kingdom  
“ authorities to the latter under s. 43, in respect of the Colonial  
“ taxes which had to be paid in New Zealand and Australia  
“ before nett profits earned there could be ascertained and sent  
“ to Scotland. Other questions might conceivably emerge if  
“ the preference shareholder had not received the full amount  
“ to which by their contract they were entitled. But as they  
“ have received all that they could possibly claim under that  
“ contract, the only question that does arise is whether they  
“ are entitled as between themselves and the respondent company  
“ to any allowance under s. 43.

“ My Lords, the answer to this question depends on the  
“ language of the section. It will be observed that the words  
“ used confer a right to claim only on a person who can show  
“ that he has paid Colonial income tax on the part of his income  
“ on which he is taxed in the United Kingdom. Speaking for  
“ myself, I am wholly unable to understand how the appellants  
“ can be truly said themselves to have paid Colonial income  
“ tax on their preference dividends. They have received these  
“ dividends in full. No doubt the respondent company paid  
“ Colonial income tax on the profits out of the residue of which  
“ the preference dividends have been paid. But that did not  
“ diminish the income of the appellants. They neither paid

“ the tax themselves, nor were indirectly subjected to it in any  
“ way. It is the ordinary shareholders alone who have lost  
“ by it, and in a question between the various bodies of  
“ shareholders and the respondent company as to how the  
“ latter, who actually paid the tax, are to account for the amount  
“ allowed them over here for having paid it, I think that as  
“ against the ordinary shareholders the appellants have no  
“ title. Neither under the language of s. 43, nor indirectly,  
“ does it seem to me possible for them to substantiate a claim  
“ based on the words employed by the Legislature. The  
“ appellants as preference shareholders are outside the legislative  
“ jurisdiction of the Dominions, and are not taxed there. It is  
“ true that profits carried by the respondent company are  
“ within that jurisdiction, and are taxed. But the only income  
“ to which the appellants are entitled in respect of their preference  
“ shares has been left undiminished. It is taxed here only.  
“ So is taxed here the income of the respondent company out  
“ of which it is derived. But under the machinery of the Income  
“ Tax Acts the amount of the tax is exacted only once. The  
“ appellants have to bear the ultimate burden of the 5s., and  
“ they can claim no allowance under s. 43 ; because they have  
“ paid nothing, either directly or indirectly, in respect of which  
“ they can claim. Some one else paid, and that other person,  
“ the respondent corporation, has kept the appellants clear of  
“ all deduction from their income on that account.

“ The opinion which I have thus expressed is not consistent  
“ with the conclusion arrived at by Astbury, J., in *Rorer v.*  
“ *South African Breweries*. But after giving attentive  
“ consideration to the reasoning of the learned judge in that  
“ case, I am unable to satisfy myself with his construction of  
“ s. 43, or to follow him in his conclusion. I am therefore of  
“ opinion that the decision of the Court of Session in the case  
“ before us must prevail and should be affirmed.”

Viscount Finlay, in the course of his judgment, added : “ It  
“ cannot be contended that in paying the Colonial tax the  
“ company acted as agents for the preference stockholders.  
“ The Colonial income tax had to be paid in the Colony and the  
“ profits could not be remitted to the United Kingdom without  
“ paying it. It stands exactly on the same footing with any  
“ expenses necessarily incurred in the business of the company  
“ in the colonies. The company, in the conduct of its business,  
“ acts as a legal person, as a corporation. The different classes



“ of holders of stocks and shares are entitled to share in the  
 “ profits in terms of the articles, but the company carries on  
 “ its business itself as the principal. As such, it has paid the  
 “ Colonial taxes and the United Kingdom income tax, and it  
 “ is the only person entitled to claim and to receive from the  
 “ Commissioners the repayment provided for by s. 13.”

## CORRESPONDENCE.

### APPROXIMATE CALCULATION OF JOINT-LIFE ANNUITY-VALUES.

*To the Editors of the Journal of the Institute of Actuaries.*

SIRS,—In his interesting Note “On obtaining values of Life Annuities at isolated rates of interest”, by means of annuities-certain (*J.I.A.*, vol. lii, p. 171), Mr. E. H. Lever takes as his example No. 7 the calculation of the values of  $a_{[40]:[23]}^F$  and  $a_{[50]:[23]}^F$  at 5 per-cent, O<sup>[a]</sup> Table, having given the corresponding values at 3 per-cent and  $3\frac{1}{2}$  per-cent; and he states that the examples are chosen so that the method may be compared with that given in the memorandum at the end of the British Offices Life Tables (p. 229 *et seq.*). The values found by his method are closer to the values given by a Hardy summation formula than those found by the official method of interpolated equal ages—see the British Offices Life Tables, pp. 231–2; and on the assumption that the values by Hardy’s formula are unlikely to differ from the true values by more than a small third difference error, Mr. Lever regards his results as showing that his own new method gives results more accurate on the whole than the official method. In making the assumption in question he has, I think, overlooked the remarks made on this subject by the late T. G. Ackland (T.F.A., vol. iii, pp. 309–10), where he works out the same and other annuity-values by a method, due to the late G. F. Hardy, which is rigidly accurate except for very small errors in the last place due to first difference interpolations. Mr. Ackland remarks that the last mentioned method “gives results “which are practically identical with those deduced by Mr. Lidstone’s “approximate method” (*i.e.*, the method described in the memorandum at the end of the British Offices Life Tables); “and it “will, I think, also be found that the results by both methods are “practically exact.” Mr. Ackland points out that “the formula of “approximate summation employed in this comparison (probably “Mr. G. F. Hardy’s formula 39a) is not suitable for the calculation “of *select* annuity-values, because none of the terms of the formula



"(excepting  $a_0$ ) falls within the period of selection." Mr. Ackland tested the matter by calculating the annuity-values in question by G. F. Hardy's extended formula 38, employing 15 and 17 ordinates respectively, and obtained results very close to those found by the method he was describing and to those found by the official method. In order to test the matter still more closely I have had the annuity-values calculated at length by the formula  $\sum v' \cdot t p_{xy}$ , taking the calculations to 5 decimal places: and the following is a summary of the various results obtained:

Value	$a_{\overline{40} }^{\text{F.M.}}$	$a_{\overline{50} }^{\text{F.F.}}$
True ... ..	12.605	11.562
By official method (interpolated equal ages)	12.603	11.565
„ Mr. Lever's method ... ..	12.590	11.554
„ Hardy's method as discussed by Ackland	12.606	11.562
„ Hardy's extended summation formula 38, as given by Ackland ... ..	12.608	11.561

These results show that the official method, in the cases here tested, give results virtually correct in the 3rd decimal place, and do not support Mr. Lever's remark that the new method gives on the whole more accurate results. It seems desirable to correct this misunderstanding, as the method of interpolated equal ages, for which I was responsible, has been officially adopted in the British Offices Life Tables; but these remarks are not made by way of criticism of Mr. Lever's new method, which is ingenious, and no doubt quite sufficiently accurate for most practical purposes: while, as he points out, it has the general advantage that it does not depend for its application upon the existence of a complete table of joint life annuities at equal ages at the rate of interest at which values are required.

I am, Sirs,

Your obedient Servant,

G. J. LIDSTONE.

Edinburgh,

September 1921.

## THE INSTITUTE OF ACTUARIES.

## EXAMINATIONS, DECEMBER 1920.

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## EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE

## (PART I.—SECTION A).

*First Paper.*

1. A sum of money consists of coins of two different denominations. If coins of the higher denomination only had been used the total number would have been 12 fewer and if coins of the lower denomination only had been used the total number would have been  $2\frac{2}{3}$  times as many, while if the numbers of the coins of each denomination were interchanged the value would be diminished by one-eighth.

Find the number of coins of each denomination and the ratio of the values of the coins.

2. Prove the formula for the number of permutations of  $n$  different things  $r$  at a time.

Find the sum of all numbers between 100 and 1,000 in which no digit occurs more than once.

3. Expand  $\sqrt{\frac{a+x}{a-x}}$  in ascending powers of  $x$  as far as the term involving  $x^4$ .

Hence find  $\sqrt{1.1}$  to five places of decimals.

4. Prove that the product of any number of positive quantities whose sum is constant is greatest when the quantities are all equal.

5. If A and B are the first two terms of an Arithmetic Progression, Geometrical Progression and Harmonic Progression, and  $x, y$  and  $z$  are the third terms, prove that

$$4(zy - xz)(z - y) = (x - z)^2 y.$$

6. (a) If  $f(x) = x^{-2} + x^{-3} + x^{-4} + \dots$  to  $\infty$ , find the value of  $\sum_{x=2}^{x=\infty} f(x)$ .

(b) Find  $\sum_{x=1}^{x=n} (x^2 + 1)x$ .

7. A coin is tossed until both heads and tails have appeared twice. On an average how many times will the coin have to be tossed?

8. A purse contains five half-crowns and four shillings. A pays 5s. 3d. for the right to receive the value of three coins drawn at random. Criticize his bargain, and find the chance that after two attempts, the second on the same terms as the first, he will be a winner.

9. If a die whose faces are numbered from 1 to 6 is thrown four times, what is the probability that the sum of the four throws is 14?

### *Second Paper.*

1. Obtain a formula for  $u_n$  in terms of central differences of  $u_0$  as far as the fourth difference.

2. Prove that

$$\begin{aligned} (n-1)^m u_x + m(n-1)^{m-1} u_{x+1} + \frac{m(m-1)}{2} (n-1)^{m-2} u_{x+2} + \dots + u_{x+m} \\ = n^m u_x + mn^{m-1} \Delta u_x + \frac{m(m-1)}{2} n^{m-2} \Delta^2 u_x + \dots + \Delta^m u_x. \end{aligned}$$

Hence find the sum of the series

$$1^2 + m \cdot 2^2 + \frac{m(m-1)}{2} 3^2 + \dots + (m+1)^2.$$

3. Prove that, if third differences are constant,

$$u_{x+\frac{1}{2}} = \frac{1}{2} (u_x + u_{x+1}) - \frac{1}{16} (\Delta^2 u_{x-1} + \Delta^2 u_x)$$

and hence complete the series  $u_1 = 139$ ,  $u_3 = 375$ ,  $u_5 = 787$ ,  $u_7 = 1,423$ , and  $u_9 = 2,331$ .

4. Obtain an expression for the value of  $\frac{d}{dx}a^x$ .

Find the  $n$ th differential coefficient of  $e^{ax}(x-2)^2$ .

5. Show that there must be a minimum value between two maximum values of a rational algebraic function of one variable.

Find the minimum and maximum values of

$$x^4 - 4x^3 - 8x^2 + 48x - 48.$$

6. Evaluate (1)  $\int \frac{dx}{\sqrt{(x-1)(x-2)}}$

$$(2) \int \frac{x^2 dx}{(x^3 + a^3)^{\frac{1}{2}}}$$

$$(3) \int \frac{x^2 + 12x + 34}{(3x-1)(x+2)^3} dx.$$

7. Show that  $\int n \frac{dr}{dx} dx = nr - \int r \frac{dn}{dx} dx$ .

Find the value of  $\int_0^1 x^{2n-1} e^{x^n} dx$ .

8. Prove that

$$\int_0^1 u_x dx = \frac{1}{12} (5u_1 + 8u_0 - u_{-1}) \text{ approximately.}$$

If the speed of a train on a non-stop run is as shown in the table below, find the approximate mileage travelled between 12.0 and 12.30, using the above formula.

Time	Speed in miles per hour
11.50	24.2
12.0	35.0
12.10	41.3
12.20	42.8
12.30	39.2

## EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE

## (PART I.—SECTION B).

“A Short Collection of Actuarial Tables” will be supplied for use in answering this Paper.

1. If a sum be accumulated at an effective rate of interest of 5 per-cent per annum for the first 15 years,  $4\frac{1}{2}$  per-cent per annum for the next 10 years and 4 per-cent per annum for the last 5 years, find the equivalent force of interest, assumed constant, over the whole period of 30 years. Given  $\log_{10} e = .4343$ .

2. Find an expression for the value of an annuity for  $3n$  years of amount 1 at the end of each of the first  $n$  years, 2 at the end of each of the second  $n$  years, and 3 at the end of each of the third  $n$  years, and for the total amount of capital redeemed by the end of  $2n$  years.

3. A debenture for £10,000 redeemable at par on 1 January 1924, bearing interest at 7 per-cent per annum payable half-yearly on 1 January and 1 July, was bought on 1 September 1920 on a basis to pay 6 per-cent per annum convertible half-yearly. Draw up a schedule showing the amount to be written off capital out of each interest payment.

4. A 5 per-cent security, redeemable in 10 years at 105 and bearing interest (subject to income tax) half-yearly, is issued at par. If income tax be taken at 4s. in the £, at what price should a 4 per-cent security, bearing interest free of tax half-yearly, and redeemable in 10 years at par, be issued to give the same yield?

5. A corporation issues £100,000 in bonds of £100 each, bearing interest at 4 per-cent per annum payable yearly, redeemable by annual drawings, 50 bonds being drawn each year for 20 years, the bonds drawn in the  $n$ th year being repayable at a premium of  $n$  per-cent. If the bonds are issued at a price to yield 5 per-cent on the average find the yield to the holder of a bond drawn in the 17th year.

6. A Government makes an issue of £1,000,000 in £100 bonds, bearing interest at  $5\frac{1}{4}$  per-cent payable annually and redeemable at 105 by an annual sinking fund of  $\frac{1}{2}$  per-cent. What is the probability that the purchaser of a single bond will realize at least  $4\frac{1}{2}$  per-cent on his investment if he took up his bond at date of issue at a price of 110?

7. Tables are required of  $v^n$  and  $a_n$  to 4 places of decimals for values of  $n$  from 1 to 100 at  $3\frac{1}{3}$  per-cent interest. How would you construct them? State the number of decimals you would employ in working, and the checks you would apply to verify the accuracy of your results.

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART II).

*First Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. If  $l_x$  between the ages of 1 and 11 be such that it is equal to  $l_1(1 - \frac{1}{9} \log_{10} x)$  find

- (1) the complete expectation of life at age 1 during the next 10 years ;
- (2) the average age at death of those who die between ages 1 and 11 ;
- (3) the force of mortality and central death rate at age 6.

( $\log_e 10 = 2.3$ ).

2. The membership of an Old Boys' Association consists of (a) annual members who pay an annual subscription, and (b) life members who have commuted their subscriptions. Assuming

- (1) that 100 new members have joined each 1 January at exact age 18 :
- (2) that, of each year's entrants,
  - 10 become life members at the outset,
  - 10 become life members at exact age 23,
  - one-ninth of the surviving annual members at age 19 withdraw,
  - one-sixteenth of the surviving annual members at age 20 withdraw,
  - and there are no other transfers of membership or withdrawals :

(3) that the mortality follows the  $H^M$  Table ; and

(4) that the Association has been in existence 20 years give expressions, as at the end of the twentieth year, for the number, out of the 2,000 entrants, who have

- (a) died as annual members,
- (b) withdrawn           ,,
- (c) survived               ,,
- (d) died as life members.
- (e) survived               ,,

and prove, by addition, the accuracy of the expressions you give.



3. If  ${}_t p_x = 1 - ct$  for values of  $t$  from 0 to 10, where  $c$  is constant, find the probability that the second death amongst three persons now aged  $x$  will occur within 10 years from now and not more than 1 year after the first death.

4. If  ${}_t e_x = a + bc^x$  where  $a$ ,  $b$  and  $c$  are constants, find an expression for  $l_x$ .

5. Compensation granted to a widow with four children, aged exactly 13, 11, 9 and 7 respectively, includes the payment of

5s. a week to the eldest child until age 16 or previous death ;

4s.     "     "     second     "     "     "     "

3s.     "     "     third     "     "     "     "

2s.     "     "     youngest     "     "     "     "

On the termination of the weekly payment to any one child by death or attainment of age 16, the younger surviving children each receive 1s. a week in addition to their previous allowance and so on until the last child is in receipt of 5s. a week.

Find the value of the payments to the children in terms of single life and two-life temporary annuities.

6. The staff of a company is maintained at 1000 by a uniform number of entrants each year at exact age 20. There are no withdrawals except by death and superannuation at age 65. Each member of the staff on entrance effects with a staff assurance fund, an endowment assurance for £200 maturing at age 65. The premiums paid are the  $H^M$  3 per-cent net premiums. Using  $H^M$  mortality and 3 per-cent interest throughout, what will be the amount of the fund when it reaches a stationary condition ?

### *Second Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Prove that  $\dot{a}_x = a_x + \bar{A}_x \frac{i - \delta}{i\delta}$

and that  $\bar{A}_x = \mu_x \bar{a}_x - \frac{d\bar{a}_x}{dx}$ .

2. If  $\pi_{xt}$  is the net premium for an annuity of 1 per annum deferred  $t$  years with the return of the net premiums in the event of death before the annuity commences, prove that the value of the contract at the end of  $n$  years from the commencement ( $n < t$ ) on the same basis as that on which the premium is calculated is

$$n\pi_{xt}\bar{A}_{x+n, \overline{t-n}|} + \left(1 - \frac{\pi_{xt}}{\pi_{x+n, \overline{t-n}|}}\right) {}_{t-n}|a_{x+n}$$

3. Prove that

$${}_mV_x + {}_nV_{x+m}(1 - {}_mV_x) = {}_{m+n}V_x$$

and give a verbal explanation of the equation.

What is the corresponding relation for endowment assurances?

4. A continuous annuity-certain to commence on the death of  $x$  but in no case to continue after  $n$  years from the present time, is purchased by the payment of premiums payable momentarily during the lifetime of  $x$  within the next  $n$  years. Assuming  $x$  to be alive, give an expression from which the time when the numerical value of the net-premium reserve will be greatest can be ascertained approximately.

5. Express in the form of an integral the value of a curtate annuity of 1 per annum payable  $m$  times a year to  $x$ , to be entered upon at the death of  $y$ , and thence show that the value can be expressed approximately in the form

$$a_x - a_{xy} - \frac{1}{2m} A_{xy}.$$

Show also how this formula should be adjusted to give the value of a similar annuity

- (a) to be entered upon at the death of  $y$  and to be apportionable to the death of  $x$ ,
- (b) with the first payment due at the end of the year of death of  $y$ , the annuity to be curtate, and
- (c) with the first payment due at the end of the year of death of  $y$ , the annuity to be apportionable to the death of  $x$ .

6. Find the Carlisle 3 per-cent annual premium, payable until the risk determines, for an assurance on the life of A, aged 37, if he dies before B, aged 70, or within three years after the death of B.

7. A community of men experiences a rate of mortality equivalent at each age up to and including 59 to the  $O^{[NM]}$  rate two years after the date of assurance, and at and after age 60 to the ultimate  $O^{NM(5)}$  rate.

Show in detail how you would proceed to construct columns of  $D_x$  and  $C_x$  giving effect to this mortality. It may be assumed that complete  $O^{[NM]}$  and  $O^{NM(5)}$  Tables are available.

### Third Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Find by the  $O^{[NM]}$  3 per-cent Table with a loading of  $2\frac{1}{2}$  per-cent the single premium for a policy securing:

- (a) £100 at the end of 10 years, and

- (b) If a life now aged 30 die during the 10 years, interest payable half-yearly at the rate of 5 per-cent per annum on the amount of the single premium for the remainder of the term after the end of the year of his death.

2. Find the annual premium, on the basis of the O<sup>NM</sup> 3 per-cent table, with a loading of 9 per-cent on the gross premium for expenses, &c., for a double endowment assurance effected at age 30 for £200 payable at age 50,

- (a) if the sum payable at death before age 50 is £100 ;  
(b) if the sum payable at death before age 50 is £100 or the total amount paid in premiums, whichever is the greater.

3. Calculate by the H<sup>M</sup> 3 per-cent Table the net single and annual premiums for an annuity to a person aged 35, after the death of a person aged 50, for the remainder of a term of 25 years from the present time.

The annuity is to be payable yearly, a full year's payment being made at the end of the year of death of the person aged 50, and it is to be apportionable to the date of death of the person aged 35. The annual premium is to be payable until the end of a term of 10 years or the first death if it occurs previously.

What would be the reserve for the annual premium contract at the end of one year, if both persons are still alive ?

4. Find by the H<sup>M</sup> 3 per-cent Table the net single premium for an assurance of £100 payable on the death of the last survivor of two lives aged 30 and 35 within 20 years, the sum of £200 being payable at the end of that term, if both survive, or £100 if one only survives.

5. Find the net yearly premium, on the basis of the H<sup>M</sup> Table with 3 per-cent interest, for an endowment assurance effected at age 35 and maturing in 20 years, with compound reversionary bonuses allotted quinquennially from the date of effecting the policy at the rate of 2 per-cent per annum on the sum assured and on previous quinquennial bonuses, and with simple reversionary interim bonuses in the event of death during a quinquennium at the rate of 1 per-cent on the sum assured in respect of each yearly premium paid during the quinquennium.

6. Find the value by the H<sup>M</sup> 3 per-cent Table of an annuity of £100 per annum payable by half-yearly instalments until the end of 5 years or the end of the year following that in which a life now aged 80 dies, whichever is the later.

EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW  
(PART III.—SECTION A).

*First Paper.*

1. It is desired to investigate the experience in regard to sickness, mortality and secession of a large friendly society with branches.

Describe carefully and fully the methods by which the data should be collected and analyzed. State, giving reasons, the nature of the investigations which you would make into the experience.

2. What life tables were prepared in connection with the 1911 Census of England and Wales? How did the sectional life tables differ from those constructed at previous censuses?

Discuss the desirability of preparing tables of the various types mentioned and suggest any alterations which you consider advisable.

3. Describe a method by which an aggregate mortality table may be readily and continuously obtained from year to year by means of the valuation class books of a large assurance company in respect of whole life policies.

State the formula you would adopt for the exposed to risk.

*Second Paper.*

1. Given the following extract from the experience of a provident fund, calculate the unadjusted central rates of withdrawal and mortality at age 20.

The members of the fund are entitled to sickness benefits after 6 months' membership, and the total number of weeks' sickness experienced at age 20 was 1,035. Calculate the unadjusted central rate of sickness per member per annum at that age.

Age <i>x</i>	Members at beginning of period	Entered during period	Withdrew during period	Died during period	Members at end of period	Age <i>x</i>
14	26	1,175	34	1	29	14
15	162	919	77	1	261	15
16	307	737	177	5	409	16
17	426	352	533	2	471	17
18	421	145	806	6	349	18
19	309	75	626	5	114	19
20	353	52	414	7	202	20

N.B.—The experience was investigated in calendar years and the ages were calculated in the following manner:

Members at beginning and end of period—ages nearest birthday.

„ entered, died, withdrew or sick—ages nearest birthday at beginning of calendar year of entry, death, withdrawal or sickness respectively.

2. Given the following adjusted rates of mortality and central rates of withdrawal :

Age	$q_x$	$mw_x$
16	·0023	·230
17	·0024	·182
18	·0025	·151

Compute the life and withdrawal table for ages 16 to 18, both inclusive.

3. Contrast the two following summation formulas from the point of view of (a) accuracy, and (b) smoothing power :

$$\frac{[5][5]}{25}[-2u_{-1} + 5u_0 - 2u_1]$$

$$\frac{[4][5][6]}{120}[-u_{-2} + u_{-1} + u_0 + u_1 - u_2]$$

Which formula would you use in practice ?

4. The following data (Table I) were obtained from a large benefit fund which was valued on the last occasion by assuming the rate of mortality indicated in Table II. Work out figures to show whether the recent mortality experience confirms the past and discuss special divergencies (if any) :

TABLE I.

AGES		NO. OF SUBSCRIBERS		Deaths 1909-13
Not under	But under	1 Jan. 1909	31 Dec. 1913	
20	25	220	200	10
25	30	340	330	20
30	35	290	300	14
35	40	300	320	16
40	45	270	300	14
45	50	200	220	14
50	55	140	160	14
55	60	100	120	10
60	65	60	80	12
65	70	20	30	6
70	75	2	8	0

TABLE II.

Age $x$	10,000 $q_x$
20	101
25	93
30	90
35	92
40	99
45	116
50	142
55	189
60	259
65	373
70	550
75	907

5. An office granting annuities desires to examine its mortality experience over a period of 10 years. It is in possession of the following facts :

- (a) Exact date of purchase ;
- (b) Age last birthday at date of purchase ;
- (c) Office year of death.

There are no withdrawals. The office year ends on 30 June. Show how the experience may be used for the calculation of select mortality tables.

6. What general features in regard to mis-statement of age were revealed by the 1911 Census of England and Wales ? State how the irregularities at individual ages were dealt with in constructing the English Life Table No. 8 and describe the process of graduation.

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EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW  
(PART III.—SECTION B).

*First Paper.*

“ A Short Collection of Actuarial Tables ” will be supplied for use in answering this Paper.

Table I gives the total business on the books of an office as at 1 January 1919 ; Table II gives the new business ; and Table III the cancelments for the year 1919. Table IV gives the items for the Revenue Account for the year 1919 which cannot be obtained from Tables I, II and III. The net premiums are calculated by the  $H^M$  3 per-cent Table for the age next birthday at entry and policies are valued at the nearest age on 31 December.

Make a valuation of the business as at 31 December 1919 by the  $H^M$  3 per-cent Table ; ascertain from the completed Revenue Account the surplus ; trace this surplus to its various sources and state what rate of simple reversionary bonus could be declared.

The whole of the business consists of whole-life policies.

N.B.—The work should be done throughout in tabular form and all working should be shown.



TABLE I.

Nearest Age on 31 Dec. 1918	Sum Assured	Reversionary Bonus	Annual Office Premium	Net Premium HM 3 %	Annual Extra Premium
	£	£	£	£	£
51	2,000	100	30	24	...
52	3,000	200	60	45	5
53	3,000	150	75	60	...
54	4,000	300	80	63	15
55	1,000	60	15	12	...
...	13,000	810	260	204	20

TABLE II.

Date of Birth	Date of Entry	Sum Assured	Annual Office Premium	Renewal Date
		£	£	
20 Aug. 1866	28 Feb. 1919	300	15	28 Feb.
15 Feb. 1867	10 Mar. 1919	400	20·5	10 Mar. and 10 Sept.
29 Nov. 1865	15 July 1919	300	15·8	15 July
20 July 1865	16 Dec. 1919	1,000	55·0	16 Dec. and 16 June
...	...	2,000	106·3	...

N.B.—Office premiums are charged according to age next birthday.

TABLE III.

Cause of Cancelment	Date of Cancelment	Nearest Age on 31/12/18	Sum Assured	Rever- sionary Bonus	Annual Office Premium	Renewal Date	Net Premium HM 3 %
			£	£	£		£
Death	28 Feb. 1919	51	100	5	2·0	14 May	1·5
Death	15 Aug. 1919	52	100	...	4·0 {	15 March and 15 Sept. }	2·5
Surrendered for £48 }	14 June 1919	52	300	...	9·0	30 Aug.	7·5
Lapse Bonus }	1 Feb. 1919	54	250	...	12·0	1 Feb.	10·0
Surrendered for £5 }	2 Dec. 1919	55	...	10	...	...	...
...	...	...	750	15	27·0	...	21·5

TABLE IV.

	£			£
Fund at beginning of year	5700·0	Commission	... ..	25
Interest (less tax)...	228·2	Expenses	... ..	20
		Depreciation	... ..	250

*Second Paper.*

1. An office distributes its surplus upon the tontine plan under which the reversionary bonuses vest when the premiums paid accumulated at 4 per-cent compound interest have equalled the sum assured.

Discuss this method of distribution in the cases of a recently established office having a surplus for the first time and an old-established office with a large number of policies in force with durations greater than the accumulation period.

Mention particularly the basis upon which you would allocate the surplus to

- (a) Those policies with durations in excess of the accumulation period;
- (b) Those policies with durations less than the accumulation period.

What reserve would you make in a valuation in respect of previous distributions ?

2. Under the provisions of the Courts (Emergency Powers) Act, 1914, industrial assurance companies could not enforce the lapse of any weekly life or endowment policy for an amount not exceeding £25, which had been two years in force prior to 4 August 1914.

State what reserves you consider should be made in respect of such policies in arrears at a valuation prior to the termination of the Act.

On the termination of the Act what general scheme of re-instatement of such policies would you suggest, subject to evidence of health but not involving payment of arrears or increase of premium ?

3. An office issues a large number of participating policies under the double endowment plan for a common period of 20 years and subject to uniform monthly instalment premiums for all ages at entry from 20 to 50 inclusive.

A separate fund is kept and 80 per-cent of the divisible surplus is distributable to the policyholders.

Describe in detail what you would consider to be the simplest method of classifying and valuing such policies and upon what basis you would advise that the surplus be distributed by way of reversionary bonus to the policyholders and by way of interim bonus on claims during the following quinquennium.

4. What are "negative values"? How do they arise in practice (a) in the case of an ordinary life office, (b) in the case of an industrial life office? State how you would treat them in a valuation and give your reasons.

5. An old-established employers' liability insurance company has got into financial difficulties. How would you value the contracts (a) for a solvency test, and (b) for the purposes of winding-up?

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EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW  
(PART IV.—SECTION A).

*First Paper.*

Write an essay on the *two* following subjects :

- (1) The Collecting Societies and Industrial Assurance Companies Act, 1896, as regards its financial and actuarial aspects, dealing more particularly with :

*The amendments which are suggested in the report of the departmental committee of enquiry.*

*The standards of solvency which should be adopted.*

*The method of determining paid-up policies and surrender values and the desirability of paying the latter in cash.*

*The treatment of policies in case of liquidation.*

- (2) The principles which should guide an office in framing tables of surrender values and paid-up policies for use in the case of whole life and endowment assurances, including :

*The advisability of guaranteeing such surrender values and paid-up policies in the case of with-profit and non-profit policies respectively.*

*The necessity for making provision for adverse selection.*

*The practical calculation and checks to be employed in quoting and the intervals at which increases would be granted.*

*The desirability of using surrender values in payment of overdue premiums.*

*Second Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Question 5 of the Fourth Schedule of the Assurance Companies Act, 1909, asks for a statement as to "the actual proportion of the annual premium income, if any, reserved as a provision for future expenses and profits." An office gives the following reply:

"The proportion of annual premium income reserved as a provision for future expenses was as follows: 15 per-cent was deducted from the office premiums to obtain the premiums actually valued, but after elimination of all negative values the total proportion of the office premiums reserved was 35 per-cent."

Criticize this answer and state what amendments (if any) you would suggest in order to meet the requirements of the Act.

2. £5,000 5 per-cent War Loan (1929-47) is set aside for the purpose of providing annuities of £100 each to A, a male aged 55, and B, a female aged 75. C, who is entitled absolutely to a moiety of the fund subject to the annuities, desires to sell his interest. How would you arrive at the value? It may be assumed that the trustees will not divide the fund until both the annuitants have died.

3. A, a male aged 35, is possessed of whole life policies for £8,000, with £700 added in bonuses, at annual premiums of £200. A has become practically uninsurable, but wishes to raise as much as possible by giving a charge upon his reversionary life interest in settled estates of ample value, using his policies to cover the charge in the event of his predeceasing B, the present tenant for life, a male aged 75. Calculate the utmost that can be given for A's interest assuming that, in view of possible future bonuses, the purchaser is willing to proceed upon a 6 per-cent basis throughout. Explain numerically how the transaction would be closed (a) upon the death of A during B's lifetime; and (b) upon the death of A after B's death.

4. What forms of option to repurchase are granted by buyers of reversions, and how is the value adjusted in each case? What legal provision has to be considered in framing the terms of such an option?

5. Write an essay on the following subject:

The various methods which have been proposed for valuing reversions and life interests and their underlying principles, explaining:

*Which formula you think most satisfactory for practical purposes in the case of absolute and contingent reversions being purchased.*

*The margins required to allow for fluctuation and depreciation in the case of all kinds of Stock Exchange and mortgage securities, freeholds, ground rents, and house property, and the considerations which would influence you in dealing with such property as regards site and occupation.*

*The manner of arriving at the loan value and the remedies in the event of the non-payment of interest upon such loans.*

*The best way of dealing with life interests to ensure the permanence of the income and the proof of the existence of the life tenant.*

*The effect of the present high rate of income tax in the case of the purchase of life interests and reversionary life interests (a) by an insurance company and (b) by an investment company.*

## EXAMINATIONS, JUNE 1921.

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## EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE (PART I.—SECTION A).

### *First Paper.*

1. On a double line of rails between A and B two down trains start from A at 12.0 and 1.0. An up train starts from B at 2.0. It meets the first down train at 3.12, is equidistant from the two down trains at 3.46 $\frac{2}{3}$ , and meets the second down train at 4.30.

Find how long each train takes to travel between the two towns assuming each travels at a uniform rate.

2. Solve the equation

$$225x^4 - 930x^3 + 1321x^2 - 744x + 140 = 0$$

being given that the sum of two roots equals the sum of the other two.

3. Calculate correct to 5 places of decimals

$$\begin{aligned} & \cdot 3 \left[ 1 + \frac{2}{3} \cdot \frac{1}{10} + \frac{2}{3} \cdot \frac{4}{5} \cdot \left(\frac{1}{10}\right)^2 + \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \left(\frac{1}{10}\right)^3 + \dots \right] \\ & + \cdot 07 \left[ 1 + \frac{2}{3} \cdot \frac{2}{100} + \frac{2}{3} \cdot \frac{4}{5} \cdot \left(\frac{2}{100}\right)^2 + \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \left(\frac{2}{100}\right)^3 + \dots \right] \end{aligned}$$

4. Show that the series  $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots$  is convergent

when  $n$  is greater than unity and is divergent when  $n$  is equal to unity or less than unity.

5. Find the sum of the series

$$(1) \frac{3}{1 \cdot 2 \cdot 4} - \frac{4}{2 \cdot 3 \cdot 5} + \frac{5}{3 \cdot 4 \cdot 6} - \frac{6}{4 \cdot 5 \cdot 7} + \dots \text{ to } \infty.$$

$$(2) \frac{1}{6 \cdot 12} + \frac{7}{6 \cdot 12 \cdot 18} + \frac{7 \cdot 9}{6 \cdot 12 \cdot 18 \cdot 24} + \frac{7 \cdot 9 \cdot 11}{6 \cdot 12 \cdot 18 \cdot 24 \cdot 30} + \dots \text{ to } \infty.$$

6. Show that when  $x$  is small compared with  $a$  or  $b$

$$\begin{aligned} 4 \frac{\sqrt{1-ax} - \sqrt{1-bx}}{\sqrt{1-ax} + \sqrt{1-bx}} &= (b-a)x + (b^2-a^2) \frac{x^2}{2} \\ &+ (b-a)(5a^2+6ab+5b^2) \frac{x^3}{16} \text{ approx.} \end{aligned}$$

7. If two of the first hundred numbers are chosen at random what is the probability that their difference exceeds 10?

8. A bag contains  $m$  white balls and two red balls. The balls are drawn from the bag one at a time without being replaced, until a red ball is drawn. If 1, 2, 3 . . . white balls are drawn, A is to receive  $1^2$ ,  $2^2$ ,  $3^2$  . . . shillings respectively. What is the value of his expectation?



9. A and B play a match to be decided as soon as either has won two games. The chance of either winning a game is  $\frac{1}{20}$  and of its being drawn  $\frac{9}{10}$ . What is the chance that the match is finished in 10 or less games?

*Second Paper.*

$$\begin{array}{lll} 1. & u_0 = 0 & u_3 = 204 \\ & u_1 = 79 & u_5 = 301 \\ & u_2 = 146 & u_8 = 415 \end{array}$$

Find  $u_4$ ,  $u_6$ , and  $u_7$ .

2. Prove that  $f(E)a^x u_x = a^x f(aE)u_x$ .

Hence or otherwise prove that

$$\Sigma a^x u_x = \frac{a^x}{a-1} \left[ u_x - \frac{a}{a-1} \Delta u_x + \left( \frac{a}{a-1} \right)^2 \Delta^2 u_x - \dots \right] + c.$$

3. Prove that if fourth and higher differences are ignored the sum of  $n$  successive terms of a function of which  $u_0$  is the central term is

$$n u_0 + \frac{n^3 - n}{24} \Delta^2 u_{-1}$$

where  $n$  is an odd number.

4. Differentiate  $\left( \frac{x-2}{x-3} \right)^x$ .

Find the  $n$ th differential coefficient of  $\frac{ax+b}{x^2-c^2}$  and the second differential coefficient of  $x^{e^x}$ .

5. Expand  $\log(x + \sqrt{a^2 + x^2})$  in ascending powers of  $x$ .

6. A window is in shape a rectangle with a semicircle covering the top. If the perimeter of the window be a fixed length  $l$ , find what is its maximum area. (The area of a circle of radius  $r$  is  $\pi r^2$  and the length of the circumference  $2\pi r$ .)

7. Prove that

$$\text{Lt}_{h=0} h [\phi(a) + \phi(a+h) + \phi(a+2h) \dots + \phi(b)] = \psi(b) - \psi(a)$$

where  $\frac{d\psi(x)}{dx} = \phi(x)$

assuming that the function  $\phi(x)$  is finite and continuous between the limits  $a$  and  $b$ .

8. Given  $u_{-2}$ ,  $u_0$ , and  $u_2$ , find an approximate value for  $\int_{-2}^3 u_x dx$ .

9. In a line AB of length  $3a$ , a point P is taken at random and then in AP a point Q is taken at random. What is the probability that PQ exceeds  $a$ ?

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART I.—SECTION B).

“A Short Collection of Actuarial Tables” and the Supplement will be supplied for use in answering this Paper.

1. Prove that the usual approximation for the equated time for a number of sums due at various dates is in favour of the debtor.

2. Given that the modulus of common logarithms is  $\cdot43429$ , find the value at 4 per-cent per annum effective of a perpetuity payable half-yearly whose successive payments are 1,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ , &c.

3. A loan of £4,000 was made on 1 January 1915 repayable in 25 years by means of equal half-yearly annuity payments on the basis of interest at the rate of 5 per-cent per annum convertible half-yearly. On 1 January 1920 an additional repayment of £1,000 was made with the half-yearly payment then due. Calculate the new annuity payment commencing 1 July 1920 on the assumption that the term outstanding is to remain unaltered and draw up a schedule for the first four new payments, showing the amount of principal and interest in each and the principal outstanding at each half-year.

Show clearly what checks you would employ to ensure the correctness of the schedule as far as you have worked.

4. In consideration for a loan of £2,000, A agrees to make the lender 14 yearly payments of £189. 6s. 9d., the first payment being made in twelve months from the date of the loan.

After 10 payments A finds himself unable to keep his contract, but offers instead to pay £30 every three months. For how long on the basis of the same rate of interest should he continue this reduced payment and what will be the amount of the last quarterly payment? The first of the quarterly payments is to be made at the end of  $10\frac{1}{4}$  years from the commencement of the contract.

5. An issue is made of debentures bearing interest at 6 per-cent per annum, payable half-yearly, redeemable in four equal amounts at the end of 5, 10, 15 and 20 years, at 102, 104, 106 and 108 respectively. What should be the price of issue to yield 8 per-cent per annum, convertible half-yearly.

6. A corporation issues a loan of £500,000, bearing interest at the rate of 4 per-cent per annum, redeemable at par by means of a cumulative sinking fund of amount sufficient to repay the loan in 50 years. After 15 years sinking fund payments are discontinued, interest only being paid for the next ten years. Sinking fund operations are then resumed, the annual charge being fixed at an amount sufficient to redeem the loan by the date originally fixed. Find the average yield realized if the issue price be 70.

7. An annuity certain of £100 per annum for 20 years, deferred 10 years, the first payment to be made 11 and the last 30 years hence, is offered for sale. What price should be offered by an intending purchaser who wishes to realize 5 per-cent on his outlay and to replace his capital by means of a capital redemption assurance for 30 years, at an annual premium, assuming that the latter can be obtained on a net  $3\frac{1}{2}$  per-cent basis? For what sum should the capital redemption assurance be effected?

EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE  
(PART II).

*First Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Given that the probability that of three lives aged  $x$  one, and one only, will survive  $n$  years is 27 times the probability that all will die within  $n$  years.

Find the probability that

- (a) At least two will survive  $n$  years ;
- (b) At least one will die within  $n$  years.

2. Give an expression in terms of joint expectations of life (two lives) and probabilities of survival (single life) for the probability that two men aged 30 and 40 respectively will die within ten years of each other.

3. Assuming the law of mortality between ages 40 and 50 to be that indicated by the following values of  $l_x$ , find  $\mu_{42}, {}_{10}p_{40}$  and the average age at death of those who die between ages 40 and 50 :

$x$	$l_x$
40	76240
41	75477
42	74691
43	73880
44	73042

4. A society in a stationary condition is supported by 500 entrants at exact age 20. The withdrawals from the society are so distributed over the first few years after entry as to be equivalent to the withdrawal of 5 per-cent of those still in the society at each of the ages 23 and 26, and all those attaining age 60 retire at that age with a pension.

The members of the society (excluding pensioners) are placed in three groups, 60 per-cent of the total number being in Group A, 30 per-cent in Group B, and 10 per-cent in Group C. On entry a member is placed in Group A, and transfer from Group A to Group B and from Group B to Group C is by seniority alone.

Find, on the basis of the  $H^M$  Mortality Table,

- (a) The number of members of the society (excluding pensioners) ;
- (b) The ages at which a member will, on the average, be transferred to Groups B and C.

5. On 1 January each year a staff is brought up to a total of 10,000 by a uniform number of entrants. It consists of two classes :

Class A, maintained by entrants at exact age 25 and transfers from Class B at exact age 40 ; and

Class B, maintained by entrants at exact age 20.

5 per-cent of the survivors at age 40 in Class B are transferred to Class A each year.

On attaining age 60 members of both classes are retired. There are no other withdrawals. The proportion of men in Class B to men in Class A after recruitment on 1 January in each year is 10 to 1.

Assuming that  $H^M$  Mortality is experienced, find the annual number of entrants in each class.

6. Find the values of  $\frac{dL_x}{dx}$  and  $\frac{dT_x}{dx}$ .

Show that  $\bar{a}_x$  can be expressed in the form

$$\bar{e}_x - \delta \int_0^{\infty} v^t {}_t p_x \bar{e}_{x+t} dt.$$

*Second Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Given the following data :

$\log l_{50} = 4.8621$	$\log D_{52} = 4.1804$
„ $l_{51} = 4.8552$	„ $D_{53} = 4.1598$
„ $l_{52} = 4.8479$	„ $D_{54} = 4.1388$
„ $l_{53} = 4.8402$	„ $D_{55} = 4.1172$
„ $l_{54} = 4.8320$	„ $D_{56} = 4.0950$
„ $l_{55} = 4.8232$	„ $D_{57} = 4.0723$
„ $l_{56} = 4.8139$	

and the logarithm tables supplied, find at 3 per-cent interest the net annual premium for a term assurance payable at the first death of three lives aged 50, 51 and 52, if it occur during a term of 5 years, and demonstrate the formula you adopt.

2. Find by the  $H^M$  Table at 3 per-cent interest the value of a curtate annuity of £100 per annum payable half-yearly until the death of the survivor of two lives aged 30 and 35 respectively, the annuity to cease on both lives, or the survivor, attaining age 50.

3. Prove that, in the case of a mortality table following Makeham's law,

$$\bar{A}_{xyz}^1 = e^x \frac{1}{3} \bar{A}_{wvw} + \log e^{87} wvw - \log e^8 \cdot \bar{A}_{wvw}$$

$c^w$

where  $3c^w = c^x + c^y + c^z$ .

4. A life aged 30 desires to purchase a deferred annuity of £100 per annum, payable yearly, to be entered upon on attainment of age 60, subject to the following conditions :

- (a) The annuity-payments to continue in any event for ten years certain ;
- (b) In the event of a life now aged 25 surviving until (30) attains age 60, the annuity is to be continued for ten years after the death of (30) should (25) live so long.

Find the net value of this annuity on the basis of the  $H^M$  Table with 3 per-cent interest.

5. Find by the  $H^M$  Table at 3 per-cent the value of an assurance of £100 payable at the second death among three lives aged 30, 35 and 40 years respectively, with the condition that if the life aged 40 die first, £50 will be payable at the first death and £50 at the second death.

It may be assumed that in each case the sum assured is payable at the end of the year of death.

6. A reversionary annuity of £100 is to be payable so long as at least two of three lives aged 30, 35 and 55 respectively survive after the death of a life aged 60. The annuity is to be payable annually, the first payment to be made at the end of the year of death of (60).

The whole annuity is to be paid to (55) so long as he survives with either or both of (30) and (35), but if (55) die before these two, the annuity payments are to be divided equally between them.

Obtain expressions in terms of joint life annuities for the present values of the interests of (30), (35) and (55) respectively. Prove your answers.

7. Show in detail how you would construct, by a continued process, a table showing the amount of the free policy, payable at death, that can be granted for the surrender of a whole-life non-profit assurance subject to annual premiums, of any age at entry and duration.

State what checks you would employ to ensure accuracy.

### *Third Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Find the annual premium by the  $O^{[NM]}$  Table at 3 per-cent for an endowment assurance for £100 on a life aged 30, for a term of 20 years, under which

- (a) In the event of death within the first 10 years the premiums paid are to be returned without interest ;
- (b) In the event of death during the second half of the term, the sum assured of £100 is payable ;
- (c) At maturity, the sum assured is to be paid by ten annual instalments of £10 each, first payment at the end of one year from the date of maturity, and interest at the rate of 6 per-cent is to be allowed at the end of each year on the amount outstanding at the beginning of the year.

The gross premium is equal to the net premium loaded with a constant of 5s. and 5 per-cent of the gross premium.

Also give expressions for the value of the policy at the end of 4 and 14 years respectively.

2. Find the purchase price, including a loading of 10 per-cent of the gross price, of a curtate annuity of 1 per annum payable yearly during the life of ( $x$ ), to be entered upon at age  $x+n$ , with the condition that in the event of death before age  $x+n$  two-



thirds of the purchase price shall be returned, but if death occurs after the annuity is entered upon and before the payments amount to the purchase price, the balance of the full purchase price shall be returned.

3. A whole-life assurance for £1,250, subject to annual premiums, was effected 15 years ago on a life then aged 30. The assured now desires to reduce the sum assured to £1,000, and, in addition, to pay £200 in cash in order to convert the contract into an endowment assurance for £1,000, the annual premium to remain the same.

Find, on the basis of the  $H^M$  Table at 3 per-cent, the age at which the endowment assurance will mature, and ascertain what cash sum would be required in order that the endowment assurance shall mature at age 55.

Assume net functions throughout.

4. A whole-life assurance is effected on a life aged  $x$ , under which the net annual premium for an ordinary whole-life assurance is payable for the first 5 years. At the end of 5 years the assured has the option of converting the policy into an endowment assurance for the same sum assured, payable on his attaining age  $x+n$  or previous death.

- (a) Find an expression for the net annual premium to be paid for the remaining term of the assurance if the assurance is so converted.
- (b) Show that the reserve by the prospective method at the end of 5 years after such conversion, immediately before payment of the 6th increased premium, is equal to the reserve which would have subsisted had the policy not been converted, together with the difference in the net premiums paid, accumulated with benefit of survivorship.

Assume that the reserve is calculated on the same basis of mortality and interest as the net premiums.

5. An assurance is to be paid on the death of a life now aged  $x$  if he survive a life now aged  $y$ . The sum assured is to be equal to the number of years (integral or fractional) elapsed from the present time to the date of the death of  $(y)$ .

- (i) Assuming different mortality tables for  $(x)$  and  $(y)$  respectively, show how you would calculate the net single premium for the assurance by means of the following formula :

$$\int_0^{\infty} u_x dr = n \{ .28u_0 + 1.62u_n + 2.2u_{2n} + 1.62u_{3n} + .56u_{6n} + 1.62u_{7n} \}$$

Give a skeleton schedule.

- (ii) If you were supplied with the values of  $D_x$ ,  $D_{xy}$ ,  $M_x$ ,  $M_{xy}^1$ ,  $R_x$  and  $R_{xy}^1$ , show how you would use the result deduced in (i) to find the net single premium for an assurance of 100 payable at the death of ( $x$ ) if he survive ( $y$ ), with a simple annual reversionary bonus of 1 in respect of each complete year after the death of ( $y$ ).

6. An endowment assurance for £100 effected on a life aged 30 for a term of 20 years, with the condition that in the event of death before maturity, in addition to the sum assured being payable all premiums paid are to be returned without interest, has been 10 years in force.

Find, on the basis of the  $H^M$  Table at 3 per-cent, the amount of the free policy, payable at age 50 or predecease, which can be granted for the surrender of the assurance,

- (a) If the premiums paid are still to be returned in the event of death before age 50 ;  
 (b) If the premiums paid are not so returnable.

Use net functions throughout.

### EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW (PART III.—SECTION A).

#### *First Paper.*

1. Give an account of the method of graduation by a summation formula, dealing particularly and fully with :

- (i) The theoretical basis of the method ;  
 (ii) The reasons why the method produces a graduated series ;  
 (iii) The method of application ;  
 (iv) The circumstances in which it can be used most advantageously ;  
 (v) Its disadvantages.

Illustrate your remarks by constructing a formula true to third differences, and apply it to find a graduated value of the middle term of the following series :

12, 14, 17, 18, 18, 21, 22, 24, 24, 26, 29, 32,  
 34, 37, 41, 44, 49, 51, 56, 59, 63.

2. For practical reasons, you are desirous of graduating an experience of male annuitants by Makeham's formula.

An experimental graduation of the ultimate table by Hardy's method, taking  $\log_{10} c = .039$ , gives good results for ages over 50, but gives values of  $q_x$  which are impossibly low at the younger ages.

What causes may have tended to give this result and how would you proceed in the matter?

3. What are the *objects* of graduation? How would you test whether a graduation was satisfactory?

What special points arise when the data are in "select" form?

4. Given the following figures of population at 31 December 1910, and of births and deaths for the following five years, estimate as closely as possible the population at each year of age from 0 to 5 on 31 December 1915. You may assume, if necessary, that 70 per-cent of the deaths in the first year of life occur in the first six months of age:

Age	Number living at 31 Dec. 1910	NUMBER DYING IN YEAR					Year	Number born in Year
		1911	1912	1913	1914	1915		
0-1	790	125	106	112	97	95	1911	890
1-2	740	28	30	29	27	26	1912	900
2-3	750	10	8	11	10	9	1913	930
3-4	760	6	7	5	7	6	1914	920
4-5	770	4	4	3	5	4	1915	910

### *Second Paper.*

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. What is meant respectively by the "policy", "calendar", and "life" year methods of construction of mortality and sickness tables?

In what circumstances can each be used with special advantage?

2. Describe fully how you would construct select and aggregate mortality tables from the experience of whole-life policies of a very large insurance company, the information available in respect of each policy being

Office year of entry.

Age next birthday at entry.

Office year of exit.

Number of premiums paid.

3. What do you understand by the term "exposed to risk" as applied to rate of mortality, central rate of withdrawal and central rate of sickness?

4. In what way and for what reasons do rates of mortality differ in tables constructed from :

- (a) Population Statistics.
- (b) Statistics of Assured lives.
- (c) Statistics of Annuitants.

Illustrate your remarks by reference to tables in current use.

What special points have to be borne in mind in drawing conclusions from comparisons of such tables?

5. The following table shows the experience, for a calendar year, of a railway friendly society which grants sickness benefits for a period of 52 weeks and death benefits, subject to six months' membership :

Age Groups	NUMBER OF MEMBERS		Actual Deaths (nearest ages at beginning of year)	Actual weeks of Sickness (nearest ages at beginning of year)
	Beginning of year (nearest ages)	End of year (nearest ages)		
18-22	4677	4997	11	5778
23-27	5204	5220	15	5904
28-32	6827	6726	24	6920
33-37	7010	7180	30	7855
38-42	6355	6366	38	7428
43-47	4901	5108	35	6500
48-52	3772	3962	40	5804
53-57	2640	2802	48	5405
58-62	1777	1879	56	5301

Test the experience by means of the Manchester Unity Whole-Society Tables and state what conclusions (if any) you would draw from the result.

What further information would you require in order to make your test more accurate?

#### EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW (PART III.—SECTION B).

##### *First Paper.*

1. A mutual office issues only whole-life with-profit policies and distributes its surplus quinquennially by the simple reversionary bonus method. It divides its business into two sections (1) abstainers, and, (2) non-abstainers, but no separate funds are kept.

The rates of premium are the same in both sections, but in allotting bonuses it is desired to give effect to the mortality profit actually earned in the respective sections. Explain the investigation you would make and the methods you would employ, at a division of profits, so that such mortality profit may be equitably apportioned between the two sections.

2. A life office issues endowment assurance policies with tontine bonuses; no bonuses are actually allotted to an individual policy until the quinquennial valuation immediately preceding maturity, and then in the form of a uniform bonus for each year the policy has been in force.

The premiums charged under this table are calculated so as to provide an immediate simple reversionary bonus at half the rate of that on full profit endowment assurances.

State clearly the principles you would adopt in allotting to the tontine endowment class at any division its share of the surplus. How would you calculate the balance of the surplus in hand from the previous valuation, and decide on the annual rate of bonus to be allotted to the policies maturing before the next valuation?

3. Describe in detail how you would analyze the surplus of a large annuity fund arising at a quinquennial valuation.

The fund contains a large number of deferred annuity policies with return of premiums (without interest) in the event of death before commencement of the annuity, and these policies are entitled during the period of deferment to the whole of the surplus on the fund, immediate annuities not being entitled to share in the profits. Upon what basis would you distribute the surplus?

4. A, an insurance company transacting all classes of insurance business, including life, desires to extend its connections and approaches B, a mutual company, with a view to establishing some form of working agreement between the two companies. Discuss the various forms this agreement might take.

In the case of B being actually taken over by A, state the nature of the investigations you would make in advising A as to the terms on which B should be taken over.

5. Discuss generally the question of allotting interim bonuses. Deal specially with the case of an office that, through depreciation in securities, is unable to declare a bonus, but has paid interim bonuses on claims during the previous five years.

### *Second Paper.*

A large life assurance company, valuing as at 31 December, and distributing profits in the form of a simple reversionary bonus, has the following classes of policies on its books, both with and without profits, namely:

- (1) Whole-life,
- (2) Endowment assurances (some policies mature on the birthday and some on the anniversary), and
- (3) Limited premium policies (whole-life only).

There are no cases where an ascending or descending scale of premiums is payable, but some premiums are permanently reduced or extinguished by bonus or commutation.

In the past all policies have been valued individually by a gross premium method. It has been decided in future to make an  $O^M$  3 per-cent net premium valuation, valuing in groups (endowment assurances and limited premium policies according to Lidstone's "Z" method), and to adopt a continuous method of tabulating the data so as to enable the valuation to be made annually without delay.

- (1) Draft specimens of the valuation cards you would use.
- (2) Draw up a full list of the instructions which you would issue for writing up the cards from the office records, and explain clearly the assumptions you make and the reasons for those assumptions as regards office years of birth, ages at entry, maturity ages, &c.
- (3) Describe the system of books and sheets you would use for keeping up a continuous classification.
- (4) State the usual checks you would have with the office records and any other methods you would employ to enable you to be satisfied with the accuracy of your valuation data.

Having obtained your facts for the valuation, explain for each class of policy, where applicable,

- (a) The method of finding the valuation ages,
- (b) The valuation functions you would use,
- (c) Your treatment of half-yearly and quarterly cases,
- (d) The additional loading reserves you would set up,
- (e) How you would provide for claims being payable immediately on proof of death and title.

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#### EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW (PART IV.—SECTION A).

##### *First Paper.*

"A Short Collection of Actuarial Tables" and the Supplement will be supplied for use in answering this Paper.

1. (a) An office established 10 years ago transacting a large and increasing new business at a present expense rate of about 30 per-cent of the gross premium income, shows a small surplus



after valuation upon an  $O^{M(5)}$  4 per-cent basis, using modified net premiums to allow for an initial new business expense of £2. 10s. 0d. per-cent, negative values being excluded.

The office now wishes to grant guaranteed minimum surrender values of  $33\frac{1}{3}$  per-cent of the gross premiums paid, after three years' premiums have been paid under whole life assurances with uniform premiums or limited premiums for periods exceeding 20 years, and after two years' premiums have been paid for whole life assurances subject to limited payments for 20 years or less and for all endowment assurances.

State fully your opinion on this proposal.

(b) State generally upon what basis you would grant surrender values and paid up policies upon double endowment assurances.

2. What steps must be taken by a registered friendly society desiring to convert itself into a company under the Companies Acts? Discuss the question as to whether such a society, having converted itself into a company, can afterwards alter or extend its memorandum of association.

3. A provisional agreement has been made whereby an insurance company undertakes, subject to acceptance by the requisite majority of the shareholders of a reversionary company, to buy their shares at a price to be determined by an actuarial valuation of the entire assets and liabilities of the reversionary company.

Write a report to the directors of the insurance company stating what steps are necessary and dealing with :

- (a) The effect of the Assurance Companies Act of 1909 ;
- (b) The mortality tables recommended ;
- (c) The rate of interest to be assumed in the calculations ;
- (d) The titles to and the adequacy of security for loans upon reversions and life interests ;
- (e) Expenses of conducting the business and compensation to the outgoing directors and to the staff of the reversionary company ; and
- (f) The value (if any) attaching to an issue of £200,000 debentures bearing interest at  $3\frac{1}{2}$  per-cent and redeemable in 1960, which is treated as a liability in the balance sheet of the reversionary company.

4. A lady, aged 40, has a rent-charge of £500 payable out of a large estate owned by her father, aged 65, during their joint lives, and an office has accepted their lives for a whole-term joint life policy of adequate amount at an annual premium at the rate of £9 per-cent without profits. Calculate the maximum market value.

*Second Paper.*

"A Short Collection of Actuarial Tables" and the Supplement will be supplied for use in answering this Paper.

1. An insurance company transacting all classes of business has been ordered to be wound up. Describe the procedure from the point of view of the actuary engaged, dealing with the following points more particularly, namely :

- (i) Allocation of deposits.
- (ii) Rules under 6th Schedule of the Assurance Companies Act 1909, for valuing (a) life assurance : annuity and policy ; (b) employer's liability : weekly payment and policy.
- (iii) The bases, as regards mortality and interest, which you consider suitable, and why ?
- (iv) Principles to be followed in fixing the amounts entitled to rank in respect of policies which have (a) become claims, (b) been borrowed upon, (c) guaranteed surrender values, (d) guaranteed bonuses, (e) premiums in arrear, (f) extra premiums for occupation, climate, illhealth or cessation during incapacity.
- (v) Practical methods of obtaining and verifying particulars of individual policies and valuing them.

2. On the death of a lady of 73, A will be entitled to the whole of the following fund provided A's father (who is 75 with a wife of 70) has no further issue by his present or any future marriage, but such issue would share the fund with A, namely :

£1,300 Great Western Railway 5 per-cent Rent-charge Stock,  
£1,700 London & South-Western 4 per-cent Preference Stock, and  
£40 per annum Great Indian Peninsula "B" Annuity.

How much can be lent to A upon the security of his reversionary interest and what policy should be effected ?

3. Value, for purchase by an insurance company, the absolute reversion expectant upon the death of a male aged 67 to an appointed sum of £1500 (free of duty) ranking after £3,000 (free of duty) raisable out of :

£2,000 4 per-cent National War Bonds, 1928.  
£5,000 Port of London 4 per-cent "B" Stock.

What special legal precautions are necessary in dealing with such sums ?

4. An army officer, aged 40 next birthday, is entitled, contingently upon survival, upon the death (or re-marriage) of his mother, aged 65, to one-third of the capital and, during their joint lives, to one-fourth of the income from

£3,000 New Zealand 4 per-cent Inscribed Stock, 1929,

£3,500 Argentine Government  $4\frac{1}{2}$  per-cent Stock.

Would you advise that a loan of £500 be granted to the officer, whose life is assurable at the ordinary rate for his profession?

5. Subject to the power of a widow of 70 to appoint the income of the following fund to a future husband if she re-marry, X is absolutely entitled upon her death to one-third of:

£5,000 Dominion of Canada C.P.R.  $3\frac{1}{2}$  per-cent Land Grant Bonds,

£4,000 Central Argentine Railway 4 per-cent Debenture Stock,

£4,000 Bass Ratcliffe & Gretton, Limited,  $4\frac{1}{2}$  per-cent Debenture Stock, and

£7,500 Alliance Investment Company, Ltd., 4 per-cent Debenture Stock.

What offer would you recommend should be made for the purchase of X's reversion? What loan could be granted if the power were released by the tenant for life?

# EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW (PART IV.—SECTION B).

## *First Paper.*

1. Write a *brief* essay on the subject of insurable interest.

2. Discuss the present position regarding income tax abatements in respect of life assurance premiums.

3. An own life assurance which has never been assigned or charged becomes a claim through the death of the life assured. What evidence in proof of death and title will be required from the claimant? Deal with the matter fully, stating, for example, the requirements if the assured died testate or intestate, solvent or insolvent.

4. Discuss the provisions of the Courts (Emergency Powers) Act, 1914, in so far as they relate to policies of assurance. How are these provisions affected by the War Emergency Laws (Continuance) Act, 1920?

5. Enumerate under headings the whole of the essential items to be incorporated in a life assurance policy, including all usual conditions. Actual policy drafting is not desired.

A life office issues joint-life partnership assurances, and states in its prospectus that it is prepared upon receipt of an application to alter the policy into two separate assurances, each for one half of the sum assured under the joint contract, at the rates of premium appropriate to the respective ages at entry of the lives assured. Draft

- (1) A clause embodying this offer as a condition of the joint-life contract, and
- (2) An endorsement to be placed upon the usual single life policy form when the option is exercised.

6. A mutual life office at its quinquennial valuation in 1915, as a matter of war-time precaution, made no general distribution of bonus but announced that an interim bonus at the rate of £1 per-cent of the sum assured would be paid on policies becoming claims before the 1920 valuation in respect of each premium paid since the 1915 valuation. The 1920 valuation has just been completed on the same basis as on previous occasions and, after meeting depreciation, the surplus is only sufficient to provide a simple reversionary bonus at the rate of 15s. per-cent.

Draft a report to the Directors dealing with the situation generally, and making recommendations as to the desirability or otherwise of

- (1) Only partially covering the depreciation,
- (2) Varying the valuation basis,
- (3) Making a general declaration of bonus, and
- (4) Declaring an interim bonus only.

### *Second Paper.*

1. Draft a report to your directors setting forth the reasons or and against making an increase in the rates of premium for new assurances under some or all of the tables in the prospectus, and indicate your recommendation as to any action you may consider desirable. Assume that your present rates are the same as those in existence before 1914.

2. Give a general formula which you would consider suitable upon which to base rates for immediate annuities at the present time. State clearly what bases of mortality, interest and loading you would employ, giving reasons for your selection.

3. Indicate the method you would adopt for the calculation of the annual premium for an endowment assurance policy under which the benefit is to take the form of the transfer of a 4 per-cent Victory Bond for £100 to the policyholder upon the happening of the first of the following events, namely (1) the expiration of the original term of the policy, (2) the drawing of the Bond for repayment by the Government and (3) the death of the assured.

4 per-cent Victory Bonds were issued at 85 per-cent and an annual charge of  $4\frac{1}{2}$  per-cent of the original nominal amount was set aside for service of the loan. Assume that the premium calculation is made at the date of issue of the Loan.

4. Give in schedule form the details you would submit to the Inland Revenue Authorities in putting forward a claim for refund of Income Tax on expenses of management.

Assume that you are acting for a life office where the gross interest is considerably in excess of the profits.

5. Extra premiums for occupation and foreign residence may be payable :

- (1) For a term certain agreed when the policy is issued,
- (2) For the period in which the assured remains in the hazardous occupation or in the unhealthy area, or
- (3) For the whole duration of the policy.

What are the arguments for and against each of these methods of charging extra premiums? Which method do you prefer? Do you consider it advisable to offer alternatives in the case of an individual proposal?

6. It has been customary for extra premiums to be charged in respect of persons engaged in the trade in intoxicating liquors. What statistical warrant has there been for this practice? Do you consider that all classes of persons connected with the trade should be charged the same rate of extra premium? If not, distinguish between the classes and suggest appropriate rates of extra premium.

Some offices accept such lives under double endowment assurances without charging extra premiums. Do you consider that this practice is justified?

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EXAMINATION FOR ADMISSION TO THE CLASS OF FELLOW  
(PART IV.—SECTION C).

*First Paper.*

1. An old-established railway sick and insurance fund society grants the following benefits subject to weekly contributions varying according to age at entry :

*Sickness Allowances*:—Off period 12 months.

Full-pay for the first 26 weeks.

Half-pay for the second 26 weeks.

Quarter-pay for the second 52 weeks.

*Insurance Benefits*:

Lump sum on death of member.

„ „ member's wife in husband's lifetime.

No benefits are payable until 26 weeks' contributions have been paid.

On sick quarter pay being exhausted a member may continue his membership for funeral benefits subject to the payment of a reduced contribution.

A further rule of the society states that when a member retires from the service of the railway company he may continue as an "Out" member, but the sickness benefits are reduced by 10 per-cent.

A quinquennial valuation is due, and you are asked to take out the experience of the society and to make the valuation.

State precisely the principles upon which you would proceed.

2. A large industrial concern desires to establish a pension fund for its employees on the following basis:

*Pensions*.—On retirement on or after the attainment of age 60 a pension of  $1\frac{1}{2}$  per-cent of the total salary or wages on which the employee has paid (or is deemed to have paid) contributions to the fund.

*Incapacity*.—Subject to 10 years' service and to attainment of age 30 on retirement in consequence of total incapacity for work a reduced rate of pension increasing from 50 per-cent of the ordinary pension as calculated on the above basis at age 30 to 97 per-cent of the ordinary pension at age 59.

*On death in the service or withdrawal before retirement*.—Return of employee's actual contributions with  $2\frac{1}{2}$  per-cent compound interest.

*On death after retirement*.—Return of excess amount (if any) of employee's actual contributions accumulated at  $2\frac{1}{2}$  per-cent compound interest over actual pension payments made.

*Employee's Contributions*.—A varying rate of contribution according to age at entry into service increasing from  $2\frac{1}{2}$  per-cent at age 16 to  $5\frac{1}{2}$  per-cent at age 50 of salary or wages.

In respect of employees in receipt of salary or wages in excess of £250 per annum in addition to the ordinary contribution on the whole salary a further contribution as follows:



In respect of the first £50 of excess,  $\frac{1}{2}$  per-cent per ann.

„ „ second £50 „ 1 „ „ „  
 „ „ third £50 „  $1\frac{1}{2}$  „ „ „ and so on.

*Company's Contributions.*—The company to contribute in each year a sum equal to the total amount contributed by the employees.

At the outset the company is prepared to make a donation to the fund sufficient to provide pensions to the existing employees on the basis that the employees of the company shall have been deemed to have contributed to the fund for the whole period of their past service.

You are consulted by the company to ascertain the amount of the donation.

State in detail on what lines you would proceed and how you would evaluate the benefits and contributions.

3. Describe the basis of reserve values under the National Insurance Acts as regards sickness, mortality, &c.

In what manner does the basis of transfer values differ from that of the reserve values, and what were the reasons for the alterations?

### *Second Paper.*

1. What alterations were made in the questions in the Householder's Schedule for the 1921 Census as compared with that used for the 1911 Census?

Criticize these alterations.

State briefly your views in regard to the classification of the population according to (a) occupation and (b) industry.

2. At the desire of their members the executive of a large friendly society order consults you with regard to the establishment of a centralized widows' and orphans' fund upon a voluntary basis.

The benefits proposed are as follows :

(a) Maintenance grant to the widow of a contributing member with a child under 14 years of age of 5s. per week.

(b) Maintenance grant of the same amount for each child under 14. Three children to be the maximum number receiving grants at any one time.

(c) Cash payment of £20 to a childless widow or a widow with no children under 14.

The maintenance grants to the widows are payable until the last surviving child attains age 14 or until earlier death or marriage of the widow.

The maintenance grants to the children cease on the attainment of age 14 or at previous death. In the latter event the grant may be transferred to any other child under 14 in respect of whom a grant is not already being paid, provided the maximum family grant is not exceeded.

What are the essential points to be considered and upon what basis would you calculate the necessary weekly contributions?

What information would you recommend be obtained in the proposal form for membership?

3. What do you know of the various investigations which have been made into the fertility of married persons in Great Britain?

Criticize the statistics which were obtained in this connection at the 1911 Census.

4. Calculate the mean age, median age and standard deviation in terms of year of age for the following data :

Age	No. of Cases	Age	No. of Cases
5-	20	40-	60
10-	300	45-	35
15-	860	50-	20
20-	435	55-	10
25-	230	60-	7
30-	150	65-	3
35-	70	...	...
		Total	2,200

Give the probable error of the mean and state generally the limitations of the expression you use giving a general explanation of those limitations.

#### PROCEEDINGS OF THE INSTITUTE.—SESSION 1920-1921.

##### *First Ordinary Meeting, 29 November 1920.*

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

The President delivered an Inaugural Address.

##### *Second Ordinary Meeting, 17 January 1921.*

Sir JOSEPH BURN, K.B.E. (Vice-President), in the Chair.

Messrs. James Davie, F.F.A., John Jamieson, F.F.A., Alexander Bertram Johnston, F.F.A., Robert McDougall, F.F.A., George Wilson Melville, F.F.A., and John Muirhead Ross, F.F.A., were elected Associates of the Institute.

A paper entitled "On a Short Method of Constructing Select Mortality Tables" was submitted by the Author, Mr. George King.

The following gentlemen took part in the discussion:—Messrs. C. W. Kenchington, A. Henry, W. P. Elderton, and the Chairman.

*Third Ordinary Meeting, 14 February 1921.*

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

Mr. George Steel Campbell, F.F.A., was elected an Associate of the Institute.

A paper entitled "The Census of 1921. Some remarks on Tabulation" was submitted by the Author, Mr. F. A. A. Menzler, B.Sc.

The following gentlemen took part in the discussion:—Messrs. S. J. Gunningham, E. H. Brown, S. P. Vivian (a visitor), W. P. Elderton, H. H. Wolfenden, and the President.

*Fourth Ordinary Meeting, 21 March 1921.*

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

A paper entitled "Superannuation Funds. Notes on some post-war problems, together with an account of a Pensioners' Mortality Experience (Civil Service Pensioners, 1904-1911)" was submitted by the Author, Mr. G. S. W. Epps, B.A.

The following gentlemen took part in the discussion:—Messrs. R. G. Maudling, D. C. Fraser, J. Bacon, A. S. Holness, S. G. Warner, L. E. Clinton, and the President.

*Fifth Ordinary Meeting, 25 April 1921.*

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

Mr. Albert Gibson Barron, F.F.A., was elected an Associate of the Institute.

A paper entitled "Modern developments in the methods of Industrial Assurance Valuations", was submitted by the Author, Mr. C. W. Kenchington.

The following gentlemen took part in the discussion:—Sir Joseph Burn, Messrs. R. Levey, C. F. Peters, A. Henry, H. H. Austin, H. L. Trachtenberg, S. G. Warner, and the President.

*The Seventy-fourth Annual General Meeting, 6 June 1921.*

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

The proceedings at the Annual General Meeting will be found on page 561.

REPORT, 1920-1921.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1920-1921, the seventy-third year of its existence.

There has been an *increase* of 23 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of

Members was 434; twenty-five years later, at 31 March 1910, it was 963. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1911	267	339	308	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896
1918	295	332	215	18	860
1919	288	330	205	18	841
1920	305	345	197	18	865
1921	324	345	201	18	888

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

*Schedule of Membership, 31 March 1921.*

	Fellows	Associates	Students	Corres- ponding Members	Total
i. Number of Members in each class on 31 March 1920 .	305	345	197	18	865
ii. Withdrawals by					
(1) Death . . .	6	3	2	...	34
(2) Resignation or otherwise . . .	3	4	16	...	
	296	338	179	18	831
iii. Additions to Membership					
(1) By Election . .	...	7	...	...	57
(2) By Examination .	...	...	46	...	
(3) By Re-instatement	...	1	3	...	
	296	346	228	18	888
iv. Transfers					
(1) By Examination: from Associates to Fellows . . .	...	27	...	...	...
	27	...	...	...	...
	323	319	228	18	888
(2) By Examination: from Students to Fellows . . .	...	...	1	...	...
	1	...	...	...	...
	324	319	227	18	888
(3) By Examination: from Students to Associates . .	...	...	26	...	...
	...	26	...	...	...
v. Number of Members in each class on 31 March 1921 .	324	345	201	18	888

Dr.				Cr.			
1920.				1921.			
£	s.	d.	Amount	£	s.	d.	£ s. d.
8,242	10	7	Get	480	16	2	
				43	2	6	
460	6	8	Me	523	18	8	
395	10	2	Bro	247	0	2	
987	12	10	G. ar				276 18 6
10,086	0	3		53	16	8	
			Subscrip. &c.	...			
947	2	0	Fellg Bookcases				53 16 8
727	13	0	Ass				66 13 2
222	12	0	Stu	761	19	11	
154	7	0	Pro	356	9	6	
2,051	14	0	Candidates (1920)				405 10 5
10	10	0	Fou	707	2	0	
2,062	4	0	Entrance and II	521	17	0	
10	10	0	Ass Students				185 5 0
44	12	6	Stu	600	0	0	
84	0	0	Pro	669	7	5	
139	2	6	Expenses	96	3	5	
206	7	1	Balance other Insurance	42	4	10	
			Dividend re and Fittings	...			
290	10	2	Genery and Printing	194	0	3	
16	15	2	Mes and Telegrams	59	4	2	
14	8	0	Bro	15	17	8	
53	14	6	G. F				1,676 17 9
375	7	10	Statistics				19 6 4
			atriotic Fund Corporation (out of pocket				26 19 6
			uge Securities—	1,006	7	10	
				81	16	0	
							1,088 3 10
			of the year as per Balance Sheet				8,986 0 2
			nd found correct, 4 May 1921.				

## Dr.

1920		1921	
£	s. d.	£	s. d.
Amount of Funds at the beginning of the year—			
General Fund (including Stock of Publications, other than <i>Journal</i> )		8,004	17
Messenger Legacy Fund		177	10
Brown Prize Fund		469	18
U. F. Handy Memorial Fund		1,041	7
		9,593	14
Subscriptions—			
Fellow		954	13
Associates		7	0
Students		201	4
Publication		142	6
		2,306	0
Four Annual Subscriptions Compounded for		77	4
		2,383	14
Residence—			
Associates		14	10
Students		62	9
Publication		43	11
		120	15
Balance of Publications Account		180	12
Includings and Interest—			
General Fund		367	12
Messenger Legacy Fund		23	16
Brown Prize Fund		20	9
U. F. Handy Memorial Fund		55	6
		467	5

£12,609    7    5                                  £12,785    11    4

## Publications Account for the year ending 31 March 1921.

1844	£	s	d.
29 4 5	10	14	12 0
29 19	10	14	12 0
51	46	0	0
56	75	12	10
58	180	12	0
1845	£145	16	10

*Balance Sheet, 31 March 1921*

ASSETS		LIABILITIES	
£	s. d.	£	s. d.
1917	100,000 0 0	1917	7,139 8 10
1918	100,000 0 0	1918	207 9 6
1919	100,000 0 0	1919	509 18 8
1920	100,000 0 0	1920	24 16 10
1921	100,000 0 0	1921	477 1 10
1922	100,000 0 0	1922	249 0 0
1923	100,000 0 0	1923	299 7 11
1924	100,000 0 0	1924	480 7 11
1925	100,000 0 0	1925	20 9 9
1926	100,000 0 0	1926	409 18 2
1927	100,000 0 0	1927	903 7 0
1928	100,000 0 0	1928	138 6 11
1929	100,000 0 0	1929	1,096 13 11
1930	100,000 0 0	1930	181 6 7
1931	100,000 0 0	1931	137 2 7
1932	100,000 0 0	1932	959 11 4
1933	100,000 0 0	1933	8,286 0 2
1934	100,000 0 0	1934	116 9 5
1935	100,000 0 0	1935	50 0 5
1936	100,000 0 0	1936	13 6 6
1937	100,000 0 0	1937	£9,185 12 2

## Cr.

1920.		1921.
42	<i>E. J. J. Journal—</i>	
578 10 3	Printing of Nos 273, 274	£ s d
43 3 6	Editorial Expenses	480 16 2
567 1 8		43 2 6
206 9 3	<i>Less Subs during the year</i>	523 18 2
355 19 3		237 11 8
43 16 8	Library— <i>Books, Purchases, Ac.</i>	
82 15 7	<i>Borrowing and refilling Books, &amp;c.</i>	53 16 8
127 14 3		276 19 4
49 10 1	Meetings	
608 1 2	Examination Charges	23 16 0
293 1 0	<i>Less Fees received from Candidates (1920)</i>	60 19 2
408 0 2		
750 3 0	Tutors for Classes in Parts I and II	
591 3 0	<i>Less Fees received from Students</i>	767 2 0
150 0 0		21 17 0
600 0 0	Office Expenditure—Rent	185 5 0
101 16 6	Salaries	
600 14 3	Honorary Expenses	600 0 0
32 17 10	Fuel and other Insurance	660 7 5
10 17 3	Stationery and Printing	96 3 5
286 6 6	Furniture and Fittings	42 14 0
49 4 11	Postage and Telegrams	174 0 3
6 6 11	Supplies	50 4 2
1,692 2 11		15 17 8
18 2 6	Lectures on Finance and Statistics	1,676 17 9
70 8 0	Valuations for the Royal Patriotic Fund Corporation (out of pocket expenses)	19 6 4
	Loss on Sale of Stock Exchange Securities—	
	General Fund	1,696 7 10
	G F Hardy Fund	81 16 0
		26 19 6

*Examined and found correct, 1 May 1921.*

£12,869 1 8	STANLEY HAZELL, D. M. CARMENT, Auditors	£12,785 11 4
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## Examined and found correct, 4 May 1921

STANLEY HAZELL, D. M. CARMENT,	<i>Analysts</i>	£416.16.10
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## ASSETS

1,875	6	£2,800 <i>Native &amp; per cent Consolidated Stock, 1929-49</i>		
706	8	£1,000 <i>Domestic of Canada, 3 per cent Registered 1920-30 Stock</i>		
307	13	£2,000 <i>New South Wales 4 per cent Registered 1920-30 Stock</i>		
307	13	£2,000 <i>Belgian Government &amp; per cent Sterling Loan of 1911</i>		
353	11	£1,300 <i>5 per cent War Stock, 1919-26</i>		
353	11	£2,000 <i>National War Loan, 4 per cent, 1927</i>		
-	-	£2,570 <i>5 per cent Registered 5 per cent National War Bonds, 1922</i>	4,501	12
-	-	£2,000 <i>5 per cent Registered 5 per cent National War Bonds, 1922</i>	2,621	7
144	12	Stock of Publications (except Journals) in hand	720	9
628	10	6 Cash on Current Account and in hand	683	6
628	10	Subsidiary (see A10)	18	0
909	7	£292 of 5 per cent War Stock, 1929-37 (cf. F. Hardy Jones, 1958: 174-82. Registered-5 per cent National War Bonds, 1922)	839	14
		(cf. F. Hardy Jones, 1958: 174-82. Registered-5 per cent National War Bonds, 1922)	107	2
		Interest accrued but not received	107	2

*Examined and found correct, 4 May 1921.*

£10,003 4 6	STANLEY HAZELL, D. M. CARMENT, 1927	<i>Analysts.</i>	11,165 12 6
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There are also 224 candidates admitted as Probationers, and 110 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes at 31 March 1915 was 188 and 72. Since that date the numbers have been as follows :

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1916	172	73	1919	169	73
1917	173	67	1920	230	79
1918	156	70	1921	224	110

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of seven Fellows, Messrs. L. G. Atkins, M. Besso, W. P. Clirehugh, W. G. Cooper, W. O. Nash, Dr. T. B. Sprague, and E. W. Townley; three Associates, Messrs. J. A. Greig, Andrew Robinson, and E. Ryley; and two Students, Messrs. F. M. Bowles and H. E. Vineberg.

A Memoir of Dr. T. B. Sprague, by Mr. George King, setting out his great services to the Institute and to the profession generally, appeared in the last issue of the *Journal*. Mr. E. W. Townley, though one of the youngest Fellows, served the Institute well both as a member of the Board of Examiners and recently as a member of the Council. Mr. W. O. Nash had also served on the Council, and as an Examiner.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £2,204. 9s. 6d., as compared with £2,201. 6s. 6d. in the previous year. The Income and Expenditure for the year were £2,852. 7s. 0d. and £2,711. 7s. 4d. respectively.

After careful consideration the Railway and other Stocks were sold at the beginning of the financial year, and the proceeds reinvested in short-dated securities. The loss involved amounted to £1,088. 3s. 10d., as shown in the Revenue Account. The £5,000 Treasury Bill appearing in the present Balance Sheet has been discounted since the 31 March last and invested in 3 per-cent Local Loans and 4 per-cent Funding Loan.

The Examinations were held in June and December last year with the following results :\*

Part	Entered	Sat	Passed
I, Sec. A . . . . .	216	195	58
I, Sec. B . . . . .	193	163	57
II . . . . .	74	68	29
III, Sec. A . . . . .	70	68	24
III, Sec. B . . . . .	50	47	27
IV, Sec. A . . . . .	42	39	29
IV, Sec. C . . . . .	1	1	1

\* For the names of the successful Candidates see pp. 228 and 557.

Notice has been given that the Examinations following those to be held next month will take place in April 1922, and thereafter annually.

The question of a revised Syllabus is now receiving the consideration of the Council.

A Supplement to the Short Collection of Actuarial Tables has been published during the year. The Supplement was prepared mainly for use in connection with the Examinations, but it is believed that the contents will be found of general utility to the profession.

The Council regret to report that Mr. W. C. Sharman has been obliged to relinquish his valuable contribution of Legal Notes to the *Journal*. The

Council have arranged with Mr. R. A. Bateman, Barrister-at-Law, to contribute similar Legal Notes for the current year.

The Council have had under consideration for some time the question of a new *Text-Book*, Part II. The existing volume was first published in 1887 and revised in 1902 and the actuarial profession, throughout the world, is deeply indebted to its Author, Mr. George King, for a work which has been invaluable in the training of the actuary. Educational conditions have, however, changed and the Council concur in the view that a re-arrangement of the *Text-Book* is necessary in order that students entering upon their actuarial studies may take full advantage of their previous training. It has been decided that the new *Text-Book* shall be in two volumes, one relating to the mathematical basis of actuarial science and the other to the theory of life contingencies. The preparation of these volumes has been entrusted respectively to Mr. A. Henry and Mr. E. F. Spurgeon.

The Council have also to report that preliminary arrangements are being considered with a view to a fresh investigation into the mortality of Life Annuitants.

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
30,496 . . . .	Parts of <i>Journal</i> .
715 . . . .	Index to Vols. 1 to 40.
1,148 . . . .	<i>Text-Book</i> , Part I (Revised Edition).
140 . . . .	<i>Text-Book</i> , Part II (Second Edition).
627 . . . .	Government Joint-Life Annuity Tables.
724 . . . .	Select Life Tables.
191 . . . .	A Short Collection of Actuarial Tables (New Edition).
475 . . . .	Supplement to the above.
610 . . . .	Frequency-Curves and Correlation, with Addendum and Errata (W. P. Elderton).
36 in cloth } . . . .	Lectures on Finance and Law (Clare and Wood Hill).
2,270 in paper } . . . .	
1,514 . . . .	Lectures on the Companies Acts (A. C. Clanson).
1,157 . . . .	Lectures on the Law of Mortgage (W. G. Hayter).
666 . . . .	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,269 . . . .	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
735 . . . .	Lectures on Stock Exchange Investments (Sir J. Burn, K.B.E.).
455 . . . .	Messenger Prize Essay on Friendly Societies (Sir G. F. Hardy, K.C.B.).
1,451 . . . .	Lectures on Friendly Society Finance (Sir A. W. Watson, K.C.B.).
309 . . . .	South African War Mortality (F. Schooling and E. A. Rusher).
203 . . . .	Life Assurance Law (A. R. Barrand).
48 . . . .	Newton's Interpolation Formulæ (D. C. Fraser).
606 . . . .	British Offices' Valuation Tables.
607 . . . .	British Offices' 2½ per-cent Temporary Annuity Values.
127 . . . .	Transactions of the Second International Congress of Actuaries.
765 . . . .	Index to Transactions of Seven International Actuarial Congresses.
179 . . . .	Pamphlet of Examination Questions.

## EXAMINATIONS. DECEMBER 1920.

Examinations were held on the 13th, 14th and 15th December 1920, at London, Liverpool, Birmingham, Cardiff, Norwich, Edinburgh, Dublin, Melbourne, Sydney, Wellington, Montreal, Toronto, Ottawa, Winnipeg, Bombay, and Calcutta, with the following results:

## PART I.—SECTION A.

One hundred and five candidates sent in their names, of whom ninety-four presented themselves, and forty-two passed, namely:

Ascott, G. R.	Hawes, A. P. M.	Pollock, W.
Bain, A. W.	Hickox, W. E. H.	Powell, H. R.
Blacknell, E. H.	Jennings, G.	Ratcliff, E. W. N.
Briscoe, H. J.	Kingham, C. E.	Scott, A. W.
Brown, F. S.	Lancashire, J. K.	Smyth, H. G.
Callum, J. A.	Lane, H. A.	Tate, J.
Cope, H. J. B.	Lavery, R. W.	Tulloch, Miss M. M. N.
Cullinan, C. B.	Legge, Miss P. C.	Tyler, A. T.
Cunningham, P. G.	Lockwood, B.	Usherwood, K. A.
Elrick, W.	Matthews, H.	van Nickerk, J.
Fagg, R. J.	Melville, C. A.	Whitwell, T.
Flew, G. H.	Melville, L. G.	Wright, J. I. W.
Gardner, W. F.	Mitchell, H. H.	
Geddes, G. W.	Mogg, H. B.	
Guest, C. F.	Plant, N.	

## PART I.—SECTION B.

Eighty-eight candidates sent in their names, of whom seventy-five presented themselves, and thirty passed, namely:

Allen, P. A.	Hornsby, S. J.	Pearson, A. M.
Briscoe, H. J.	Jennings, E. E.	Powell, H. R.
Cullinan, C. B.	Kemp, W. B.	Subramonyam, A. V.
Davis, C. M.	King, G. C. H.	Tyler, A. T.
Fenner, H. A.	Kitton, J. H.	Usherwood, K. A.
Flew, G. H.	Maggs, C. G.	Williamson, W. H. R.
Gardner, C. G.	Melville, C. A.	Willows, C. E.
Gill, D. H.	Melville, L. G.	Yetton, J. L.
Gregory, Miss G. C.	Mitchell, H. H.	
Hawes, A. P. M.	Pal, A. T.	
Hickox, W. E. H.	Paul, H. C.	

## PART II.

Thirty-eight candidates sent in their names, of whom thirty-six presented themselves, and ten passed, namely :

Cale, C. E.	Grimshaw, W. E.	Scott, Miss H. M.
Clough, W. H.	Murrell, R.	Walker, R. B.
Davis, C. M.	Perks, W.	Williams, G. T.
Davis, Miss D. B.		

## PART III.—SECTION A.

Thirty-four candidates sent in their names, of whom thirty-two presented themselves, and fourteen passed, namely :

Andras, J. B.	Marriott, A.	Spurgeon, C. B.
Freeman, H.	O'Brien, H.	Walker, J. R.
Hocking, W. S.	Reynolds, P. C.	Waller, F. W.
Holgate, B.	Schwehr, W. E. P.	Woodrow, G.
Johnston, W. N.	Shrewsbury, A. H.	

## PART III.—SECTION B.

Twenty-six candidates sent in their names, of whom twenty-four presented themselves, and fourteen passed, namely :

Coleman, H. D.	Morton, F. W.	Thorpe, A. H.
Goodfellow, P.	Perryman, F. S.	Warren, C. F.
Hooker, P. F.	Priestman, B.	Welsh, W.
Innes, F. F.	Rowell, A. H.	Woodrow, G.
Milnes, H. L.	Thomas, J. H.	

## PART IV.—SECTION A.

Eighteen candidates sent in their names, of whom sixteen presented themselves, and eleven passed, namely :

†Atkins, F. C.	Knowles, M.B.	†Ruddle, F.
†Baker, S. H.	†Johnson, A. N.	†Stockman, G. D.
†Brown, S. P.	†Jones, R. McNair	†Wickens, C. H.
†Clegg, C.	†Preston, J. E.	

Those marked (†) have now completed the Examination for the Class of Fellow.

By Order of the Council,

C. R. V. COUTTS,

*Chairman of Board of Examiners.*

H. M. TROUNCER,

A. C. THORNE,

*Joint Honorary Secretaries.*

24 February 1921.

## EXAMINATIONS, JUNE 1921.

Examinations were held on the 20th, 21st and 22nd June 1921, at London, Liverpool, Birmingham, Cardiff, Norwich, Edinburgh, Dublin, Melbourne, Sydney, Wellington, Montreal, Toronto, Ottawa, Winnipeg, Bombay, Calcutta, Singapore, and Shanghai, with the following results :

## PART I.—SECTION A.

One hundred and two candidates sent in their names, of whom ninety-five presented themselves, and twenty passed, namely :

Aiyangar, M. N.	Joshi, K. Y.	Puckridge, C. E.
East, G. W.	Kitton, J. H.	Read, E. K.
Eedle, A. J.	Lynch, L.	Subramonyam, A. V.
Gardner, C. G.	Madhava, K. B.	Turnbull, J.
Grealy, L. J.	Patterson, J. R.	Turner, G. C.
Gupta, R.	Pearson, A. M.	Wenborn, N. S.
Hislehurst, G.	Pradhan, G. N.	

## PART I.—SECTION B.

Ninety-three candidates sent in their names, of whom eighty-one presented themselves, and twenty-seven passed, namely :

Ascott, G. R.	Goodfellow, F.	Madhava, K. B.
Bain, A. W.	Holmes, G. L.	Matthews, H.
Brown, F. S.	Honey, F. J. C.	Mogg, H. B.
Cooper, F. R.	Horsely, N. A.	Plant, N.
Cope, H. J. B.	Isaac, L. F.	Ratcliff, E. W. N.
Daubney, T.	Joshi, K. Y.	Smyth, H. G.
East, G. W.	Kempe, F. E.	Strachan, W. B.
Eedle, A. J.	Lane, H. A.	Tate, J.
Gardiner, W. H.	Lavery, R. W.	Whitwell, T.

## PART II.

Fifty-three candidates sent in their names, of whom fifty-two presented themselves, and twenty-four passed, namely :

Balmford, W. C.	Guest, C. F.	Polden, L. S.
Care, H. C.	Haynes, T. W.	Rowan, H. R.
Culliman, C. B.	Hornsby, S. J.	Scott, A. W.
Cutler, D. G. S.	Jones, H. G.	Smyth, H. G.
Dodwell, Miss M. E.	Keady, P. J.	Taylor, J. A. G.
Dowsett, R. E.	Mitchell, H. H.	Usherwood, K. A.
Gardner, W. F.	Parker, R. O.	Vaidyanathan, L. S.
Gregory, Miss G. C.	Patterson, J. R.	Whitwell, T.

## PART III.—SECTION A.

Forty candidates sent in their names, all of whom presented themselves, and sixteen passed, namely :

Brown, E. A. L.	Hooker, P. F.	Robinson, A.
Clough, W. H.	Morton, F. W.	Scott, Miss H. M.
Cope, A.	Murrell, R.	Taylor, F. G.
Davis, C. M.	Perks, W.	Walker, R. B.
Forster W. A.	Perryman, F. S.	
Gostelow, C.	Priestman, B.	

## PART III.—SECTION B.

Twenty-nine candidates sent in their names, of whom twenty-eight presented themselves, and fifteen passed, namely :

Andras, J. B.	Garland, W. E.	Sanger, C. W.
Cale, C. E.	Lochhead, R. K.	Shepherdson, H. J.
Davie, J.	Millward, G. D.	Taylor, F. G.
Davis, C. M.	Pedoe, A.	Waller, F. W.
Davis, Miss D. B.	Reynolds, P. C.	Wilson, A. B. S.

## PART IV.—SECTION A.

Nineteen candidates sent in their names, all of whom presented themselves, and thirteen passed, namely :

†Bradshaw, F. L.	†Holgate, B.	Rowell, A. H.
†Carey, N. L.	Innes, F. F.	†Rutherford, C. D.
†Carpmael, C.	Johnston, W. N.	†Spurgeon, C. B.
Freeman, H.	Mabon, J. B.	
Hocking, W. S.	Marriott, A.	

## PART IV.—SECTION B.

One candidate presented himself for this Section and passed, namely :

†Rutherford, C. D.

## PART IV.—SECTION C.

Six candidates sent in their names, all of whom presented themselves, and one passed, namely :

†Knowles, M. B.

Those marked (†) have now completed the Examination for the Class of Fellow.

By Order of the Council,

C. R. V. COUTTS,

*Chairman of Board of Examiners.*

A. C. THORNE,

H. J. BAKER,

*Joint Honorary Secretaries.*

13 September 1921.



## PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventy-Fourth Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 6 June 1921, Sir A. W. Watson, K.C.B. (President), in the Chair.

The PRESIDENT said that before proceeding with the ordinary business of the meeting he proposed to have the ballot papers for the election of new members of the Council distributed, so that they might be collected and counted during the meeting. On the present occasion the form of the ballot paper had been somewhat altered to bring it into conformity with that usually adopted in other similar professional bodies. The names of the seven Fellows nominated for the five vacancies were set out, and the members were invited to strike out two names. If more names than two were struck out, the ballot paper would not thereby be invalidated; there was nothing in the Bye-laws to compel members to vote for the full number of five to fill the vacancies, and a member was at liberty to strike out more than two names if he felt disposed to do so. At the same time he would like to bring before the members some reflections which had occurred to him and others during the past twelve months and which he thought were deserving of attention. The system of election by ballot had now been in existence for two years, and a tendency had been observed on the part of some members to vote for only one or two candidates. That was clearly and absolutely within their right, but he was very much afraid that if this tendency continued and developed it might be found in the course of a few years that the Council, instead of being representative of the whole of the Institute, was individually representative of particular sections. He thought that that would be a transformation of the present conditions which everyone would deplore. The Council had duties cast upon it by public authorities and others from time to time of a peculiarly onerous and responsible character, and personally he would like it understood, not merely by the Institute but by the world outside, that there were no sections amongst the members of the Institute, that the professional interests of all members were the care of every one, and that the Council was not elected as it were by sections but was representative of the whole of the actuarial community. He hoped he was not trespassing beyond the liberty which should be allowed to the President in bringing the matter forward. It was entirely within the option of the members to vote for as many candidates as they liked not exceeding five, but he would put it to their personal judgment as to whether it was not desirable that, as far as possible, each should endeavour to vote for five candidates.

The ballot papers were then distributed and it was announced at the conclusion of the meeting, that Mr. C. R. V. Coutts, Mr. John Spencer, Mr. A. D. Besant, Sir G. E. May, K.B.E., and Mr. J. M. Laing had been elected, and that the Council and Officers for the ensuing year would accordingly be as follows:

## ELECTION OF OFFICERS.

*President.*

SIR ALFRED WILLIAM WATSON, K.C.B.

*Vice-Presidents.*JAMES DOUGLAS WATSON.  
ABRAHAM LEVINE, M.A.| WILLIAM PALIN ELDERTON, C.B.E.  
| HAROLD MOLTKE TROUNCER, M.A.

*Council.*

HERBERT HENRY AUSTIN.	REGINALD GEORGE MAUDLING.
HENRY JAMES BAKER.	*SIR GEORGE ERNEST MAY, K.B.E.
ARTHUR RHYS BARRAND, M.P.	HENRY EDWARD MELVILLE.
*ARTHUR DIGBY BESANT, B.A.	HENRY JOHN PERCY OAKLEY, M.C.
LOUIS ERNEST CLINTON.	WILLIAM PENMAN.
*CHARLES RONALD VAWDREY	WILLIAM PEYTON PHELPS, M.A.
CLUTTS.	JOHN SPENCER
WILLIAM PALIN ELDERTON, C.B.E.	FRANK PERCY SYMMONS.
ALFRED HENRY.	ALFRED CHARLES THORNE.
LEWIS FREDERICK HOVIL.	HAROLD MOLTKE TROUNCER, M.A.
CHARLES WILLIAM KENCHINGTON.	SAMUEL GEORGE WARNER.
*JAMES MURRAY LAING.	SIR ALFRED WILLIAM WATSON.
ABRAHAM LEVINE, M.A.	K.C.B.
GEORGE JAMES LIDSTONE, F.R.S.E.	JAMES DOUGLAS WATSON.
HAROLD EDWARD WILLIAM LUTT.	ARTHUR THOMAS WINTER.
GEOFFREY MARKS, C.B.E.	

\* Not Members of the last Council.

*Treasurer.*

LEWIS FREDERICK HOVIL.

*Honorary Secretaries.*

ALFRED CHARLES THORNE.

| HENRY JAMES BAKER.

In moving the adoption of the report and accounts, the PRESIDENT referring to the paragraph which recorded the losses sustained by death since the last annual meeting, expressed regret that they had again, as was the case last year, to mourn the loss of a member of the Council, who had died since the report was prepared. He referred to the lamented death of their colleague, Mr. W. C. Sharman. Mr. Sharman became a Fellow of the Institute in 1901, and had served for several years on the Council. He had been a member of the Board of Examiners, and for some years had contributed the Legal Notes to the *Journal*. It was to the Council a matter of great personal regret to have lost so excellent and engaging a colleague as Mr. Sharman. Conclusive testimony to his many admirable qualities was afforded by the high esteem in which he was held by his colleagues in the great institution in whose service he had risen to a high place. He was sure that the feeling of regret at the untimely loss of Mr. Sharman would be unanimous.

The report set out in fairly complete detail the activities of the Council during the last twelve months. He would refer to one or two matters that were not mentioned in the report, but which he thought should be known to the members of the Institute generally. Twelve months ago, at the desire of the Government, the Council appointed a Committee to assist in the very important work of preparing the pensions part of the claim against Germany for reparations. That work was now finished, and so far as the preparation of the pension claim was concerned it was, he thought, due to those members of the Council who assisted in the work that he should say how adequate and how substantial their assistance had been. He thought he was betraying no official confidence when he said that the claim sent in was eminently satisfactory in the sense that it was not seriously challenged, and for that he was sure no little part of the credit was due to those members of the Institute who were consulted in its preparation.

One other matter arising out of the war was also omitted from the present annual report. On the table was a plan of the proposed memorial to be erected in the Hall, as a tribute to the memory of those who

sacrificed their lives for their country in the great struggle of 1914-18. The Council had been extremely anxious that whatever memorial was put up in the Hall should be in keeping with the character of the Hall itself. It was believed that at length a satisfactory solution had been arrived at and a form of memorial obtained which would be a credit to the Institute, which would be fitting in view of the services it was intended to commemorate, and which would be in keeping with the general characteristics of the ancient Hall. Had it been thought proper, it would have been possible to erect a memorial at the cost of the Institute without appealing to the members to contribute. That course received due consideration from the Council, but they felt upon reflection that it would be a poor and unsatisfactory thing were they merely to provide a memorial and charge it to the existing funds of the Institute. They felt it would be the desire of every member to have some personal share in the tribute to their departed colleagues which the memorial was to represent, and they therefore decided to give every member an opportunity to subscribe directly an amount not exceeding one guinea towards the cost. Those gallant members of the Institute who served and happily came back would, he was sure, be glad to contribute their share to the memorial to their departed colleagues, while those who stayed behind could only in that way directly testify to the deep and unflinching gratitude they felt to those who had sacrificed their lives. He hoped that when the members reassembled in the autumn the memorial would be erected and ready for unveiling. The only other matter he need mention was the interesting activity of the International Congress of Actuaries. Their colleagues on the Continent who belonged to the allied nations seemed to have kept the desirability of maintaining the Congress in their minds during the long years of war, and now that peace had arrived and reconstruction was in progress, they were keenly desirous of reviving the activities of the Congress. It was, of course, a rather delicate matter to know precisely how to bring into harmonious relations in a scientific and social gathering the actuaries of not only the Allied and associated countries but also those of the countries which had been opposed to us. Mr. Woods and he had had the great pleasure of attending the annual meeting of the Permanent Committee of the International Congress a few days ago, and from what he could gather of the views of the Allies and associates some time would have to elapse before their former enemies could be again received as friends. There was a strong desire on the part of some members of the profession abroad to have a Congress at as early a date as possible, and he thought it quite likely that it would be called in 1923. The English, the Scottish, and, he believed, the French actuaries had stipulated that the Congress should not be earlier than 1923, but when it did come he sincerely hoped that the Institute would take the same prominent share in the work and in the preparation of papers to be submitted that it had taken in respect of all previous Congresses.

Sir JOSEPH BURN, in seconding the motion, said that he had felt the death of their late colleague, Mr. Sharman, more than he could express, and he knew that there were many members of the Institute who shared his feelings. With regard to the report itself, he felt sure that most members would appreciate the reference to the suggested investigation into the mortality of life annuitants. The Council was very anxious to obtain an experience which should be for the first time up to date, and that meant getting the work done quickly so that the results might be obtained within quite a short time after the closing date of the experience. He thought the proposed rearrangement of the *Text-Book* was a matter for congratulation, and with regard to the alteration in the syllabus, he knew that some members thought it might be an advantage to include again some knowledge in reference to finance. That was a matter which was to be further considered.

On the motion of Mr. TUTILL, seconded by Mr. COPE, Messrs. D. M.

Carment and Alex. S. Sellar were re-elected and Mr. E. A. Woodall elected as auditors for the coming year.

Mr. CROSS, in proposing a vote of thanks to the President, Vice-Presidents, Council and Officers for their services during the past year, said that but for the fact that the President was taking office for another year he would like to have expressed their great indebtedness to him, but, as things were, that pleasing duty could be left to the next annual meeting. He would like to include in the vote of thanks Mr. Jarvis, the Assistant Secretary.

Mr. MORAN seconded the motion, which was carried unanimously.

Mr. COLLINS moved a vote of thanks to the auditors—Messrs. Stanley Hazell, D. M. Carment and A. S. Sellar—for their services during the past year, and, Mr. SHERIFF having seconded, the motion was unanimously carried.

Mr. HAZELL, on behalf of his co-auditors and himself, briefly thanked the meeting for the kind recognition of their services.

The PRESIDENT, on behalf of the Officers and the Council, tendered very sincere thanks for the vote of thanks that had been given them. It had been a very arduous session. The Council had disposed of what he thought was probably a record amount of business. Speaking for himself, he would be doing less than justice were he not to acknowledge the splendid assistance he had received from the Vice-Presidents and most especially from the Honorary Secretaries. He would like to say how much the Institute owed to the zeal, industry and discretion of its Assistant Secretary, Mr. Jarvis.

### *Additions to the Library.*

The following works have been added to the Library since the publication of the *Journal* for October 1920 :

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Accountants, Institute of Chartered, in England and Wales. List of Members, &c., 1921.	<i>The Institute.</i>
Accountants and Auditors, Society of. List of Members, &c., 1921.	<i>The Society.</i>
Actuarial Society of America. Transactions, 1920.	<i>The Society.</i>
Containing, <i>inter alia</i> —	
“Influence of Occupation on Mortality”, by A. Hunter and Dr. O. H. Rogers.	
“Notes on Problems of Small Pension Funds”, by J. B. Maclean.	
“Mortality among American Annuitants and Premiums based thereon”, by A. Hunter.	
“Joint and Survivor Annuity. Calculation of Values by a continuous method”, by E. G. Fassel.	
“Actual Deaths in the Mutual Life Insurance Co. of New York compared with the Expected by three standards”, by W. A. Hutcheson.	
“A practical system of calculating expected mortality and its relation to valuation and dividend apportionment”, by M. L. Johnson.	

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JAMES MILES LANGSTAFF, Fellow of the Institute, Major, Canadian Expeditionary Force.

*Killed in Action, 1 March 1917.*

ERNEST MILTON MARTIN, Probationer of the Institute, Lieutenant, 24th Battalion, 2nd Division, Australian Imperial Force.

*Killed in Action, 31 March 1918.*

JOSEPH JEFFERSON TWENTYMAN, D.C.M., Probationer of the Institute, 2nd Lieutenant, Highland Light Infantry.

*Died on Service, 11 February 1919.*

## ERRATA.

Vol. xlviii, p. 126, 6th line from bottom:

for "Reversionary Bonnses" read "Sums Assured."

.. lii, p. 103, Equation (1):  $\frac{1}{7}$  should be  $\frac{1}{7}$ .

.. .. line 13: 1 12 .. ..  $1/12$ .

.. .. p. 105, 4th line of § 7: for  $e^{-1/12n}$  read  $e^{1/12n}$ .

.. .. 6th .. .. 1600 .. 1200.

.. .. 12th .. .. A.S.E. .. R.S.E.

.. .. 3rd line from bottom:

for  $\log_{10} 100!$  read  $\log_{10} 100!$

.. .. p. 106, 9th line: for  $e^{-1/12n}$  read  $e^{1/12n}$ .

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